On the Design of Multi-rate Tracking Controllers: An Application to Rotorcraft Guidance and Control

Duarte Antunes* and Carlos Silvestre† and Rita Cunha‡

Instituto Superior Técnico, Institute for Systems and Robotics, 1049-001 Lisbon, Portugal

This paper presents a new methodology for the design and implementation of gain-scheduled controllers for multi-rate systems. The proposed methodology allows to cast the integrated guidance and control problem for autonomous vehicles with outputs sampled at different instants of time in a natural way. A controller structure is first proposed for the regulation of non-square multi-rate systems with a greater number of available measured outputs than inputs. Based on this structure, an implementation for the gain-scheduled controller is obtained which verifies an important property known as the linearization property. The method is applied to the path following problem of steering an autonomous rotorcraft along a predefined path. By defining a convenient path-depend error-space which includes dynamic and kinematic vehicle variables, the path-following problem is reduced to that of regulating the new error variables to zero. Due to the periodic multi-rate characteristics of the sensors, the controller synthesis is dealt with under the scope of linear periodic systems theory. Simulation results obtained with a full non-linear rotorcraft dynamic model are presented and discussed.

I. Introduction

A. Introduction

Over the last few decades there has been a surge of interest in the development of efficient and reliable algorithms for the guidance and control of unmanned vehicles. An endless number of application can be found in the literature, ranging from underwater geological surveillance to spacecraft missions. Increasing advances in sensor technology, demands for the development and design of control systems capable of taking full advantage of the sensors characteristics. Moreover the designer is often faced with interesting control challenges due to the choice of cost-effective sensor solutions.

Traditionally, the guidance and control problem for autonomous vehicles involves the design of an inner and an outer loop. The inner loop is designed to stabilize the vehicle dynamics and usually requires a high sampling rate, whereas the outer loop design relies essentially on the vehicle’s kinematic model, converting tracking errors into inner loop commands, and is amenable to a lower sampling rate. As explained in Ref. 2, since the two systems, kinematics and dynamics, are effectively coupled, stability and adequate performance of the combined systems are not guaranteed. A good feature of this technique is that, since normally the kinematic variables are available at lower rates (consider for example the case of a GPS receiver), the design of the different loops at different rates often handles naturally the multi-rate characteristic of the sensors.

Another stream of work is the integrated guidance and control presented in Ref. 3, 2 and 4. This solution amounts to defining a convenient non-linear path-dependent transformation involving both kinematic and dynamic variables of the vehicle. In this new variables, referred to as error-space variables, the problems of trajectory tracking or path-following reduces to the problem of driving this newly-defined error to zero. The family of error transformations presented in Ref. 3, 2 and 4 have the notably property that the linearization of the error dynamics is time-invariant along trimming trajectories which comprise arbitrary straight lines and z-aligned helices. Due to the wide range of dynamical behaviors through the set of conditions under

*PhD Student, Department of Electrical Engineering and Computer Science (DEEC), Av. Rovisco Pais 1; dantunes@isr.ist.utl.pt.
†Assistant Professor, DEEC, Av. Rovisco Pais 1; cjs@isr.ist.utl.pt. Member AIAA.
‡PhD Student, DEEC, Av. Rovisco Pais 1; rita@isr.ist.utl.pt.
which the autonomous vehicle is expected to operate, gain-scheduling control theory is normally used. So far, the multi-rate characteristics of the sensor have been treated in the scope of the navigation system design, although the problem can be posed as a control design problem.

This paper follows the line of work of the integrated guidance and control approaches and proposes a novel method to take into account the multi-rate characteristics of the sensors in the controller design. In order to do so, theoretical results are first derived which make use of the existing background for multi-rate and gain-scheduling controller theory. The theory for multi-rate systems is intimately related with the theory of periodically time-varying systems. See Ref. 6 and the references therein for early work on the subject and Ref. 7 for a more recent work where the $H_\infty$ problem for multi-rate sampled-data systems is solved. Noteworthy, is the bulk of work by Bittanti, Colaneri and co-authors, which have set theoretical foundations, for example with the definitions of stabilizability and detectability for periodic systems, and solved a great amount of problems, such as the regulation problem for square multi-rate systems, the output stabilization for periodic systems and the $LQG$ optimal control problem for multi-rate systems, just to cite a few which are related with the work herein presented. In the field of gain-scheduling an excellent survey can be founded in Ref. 12. The development of gain-scheduled controllers usually involves the following steps:

1. Obtain a family of parameter dependent linear models, usually by Jacobian linearization of the plant about a finite number of representative operating points characterized by parameter values. The parameters correspond to fixed values of scheduling variables which are functions of internal state variables and exogenous signals.

2. Design a family of linear controllers for the parameter dependent linear models. If the family of linear models is obtained by linearization at a finite set of parameter values then linear controllers are designed for each fixed parameter value and their coefficients are interpolated.

3. Implement the gain-scheduled controller on the nonlinear plant. The controller coefficients (gains) computed for time-frozen parameters are on-line varied according to the current value of the scheduling variables.

4. Extensive simulate the resulting control system to access its performance characteristics.

Since, for a given equilibrium operation point, the controllers are designed for time-frozen parameters and when implemented the parameters are allowed to vary, the linearization of the nonlinear gain-scheduled controller does not in general match the designed time-frozen linear controller. This mismatch is commonly known as the hidden coupling and might lead to performance degradation or even instability. Therefore the following property is required to hold in the correct implementation of the controllers:

*Linearization property:* At each equilibrium point, the linearization of the nonlinear gain-scheduled controller must be the same as that of the designed controller.

A technique known as the velocity implementation, presented in Refs. 13, 14 and discussed in Refs. 12, 15 and that is related with the work herein presented, provides a simple solution for the implementation of controllers with integral action which verifies the linearization property.

Based on the foundations of gain-scheduling and multi-rate control theory and in the problem at hand, the contributions of this paper are two fold: i) a new structure is proposed for the regulation of non-square multi-rate systems with a greater number of outputs than inputs and based on this structure ii) a straightforward method is obtained for the implementation of gain-scheduled controllers which verify the linearization property. Using these results, and casting the integrated guidance and control problem as a regulation problem, we are able to solve the guidance and control problem for autonomous systems equipped with multi-rate sensors in a systematic manner. An application is presented for the path-following problem of steering an autonomous rotorcraft along a pre-defined path. The performance of the resulting non-linear multi-rate guidance and control law is compared to that obtained using a standard single-rate compensator designed using equivalent weighting matrices.

The paper is organized as follows. First we present a theoretical framework for the design of multi-rate gain scheduled controls valid for a wide class of non-linear systems. Section II presents the problem formulation and Sections III and IV present the main results: the regulator structure in Section III and gain-scheduling implementation in Section IV. Secondly, we apply the methodology to the path-following problem in Section V.
B. Notation

The space of $n$-dimensional continuous-time signals, $x(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^n$, will be denoted by $\mathcal{L}(\mathbb{R}^+, \mathbb{R}^n)$ or simply $\mathcal{L}(\mathbb{R}^+)$ and the space of $n$-dimensional discrete-time signals, $x_k : \mathbb{Z}^+ \rightarrow \mathbb{R}^n$, will be denoted by $\mathcal{I}(\mathbb{Z}^+, \mathbb{R}^n)$ or simply $\mathcal{I}(\mathbb{Z}^+)$. The notation diag($[a_1 \ a_2 \ \ldots \ \ a_n]$) indicates a block diagonal matrix where its entries $a_i$ can be either scalar or matrices. Whenever the matrices dimensions are clear, identity and zero matrices are denoted by $I$, $0$ and $1$ denotes the row vector $1 = [1 \ 1 \ \ldots \ 1]$. Otherwise dimension indication is added, for example, $f_{3 \times 3}$, $0_{3 \times 2}$ or $1_4$. Further notation will be introduced when necessary.

II. Problem Formulation

Consider the nonlinear system

$$ G := \begin{cases} \dot{x}(t) = f(x(t), u(t), w(t)) \\ y(t) = h(x(t), w(t)) \\ z(t) = g(x(t), u(t), w(t)) \end{cases} \quad (1) $$

where as usual $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the control input. The vector $w(t) = [r(t)^T \ d(t)^T]^T \in \mathbb{R}^n$ is an exogenous signal where $r(t)$ represents references to be tracked and $d(t)$ represents both external disturbances and output noise. The vector $y(t) = [y_m(t)^T \ y_r(t)^T]^T = [h_m(x(t), w(t))^T \ h_r(x(t), w(t))^T]^T \in \mathbb{R}^p$ contains a vector of measured outputs $y_m(t) \in \mathbb{R}^{n_{ym}}$, $n_{ym} = p - m$, and a vector of outputs $y_r(t) \in \mathbb{R}^{n_{yr}}$, $n_{yr} = m$, which we assume to have the same dimensions as the control input and that should track the reference $r(t)$ with zero steady state error. Some of the components of $y_r(t)$ may be included in $y_m(t)$ as well. We denominate the error variable that we wish to regulate to zero at steady state by $e(t)$$

$$ e(t) = y_r(t) - r(t) $$

Finally, the vector $z(t) \in \mathbb{R}^{n_z}$ is a performance output.

A. Linearization family

We assume that there exists a family of equilibrium points for $G$ of the form

$$ \Sigma := \left\{ (x_0, u_0, w_0) : f(x_0, u_0, w_0) = 0, \ y_{m0} = h_m(x_0, w_0), \ y_{r0} = h_r(x_0, w_0) = r_0 \right\} $$

which can be parameterized by a vector $\alpha_0 \in \Xi \subseteq \mathbb{R}^s$, such that,

$$ \Sigma = \Sigma(\alpha_0) := \left\{ (x_0, u_0, w_0) = (x_0(\alpha_0), u_0(\alpha_0), w_0(\alpha_0)) : a(\alpha_0), \alpha_0 \in \Xi \right\} \quad (2) $$

and that there exists a function $v \in C^1$ s.t. $\alpha_0 = v(y_0, w_0)$. By applying the function $v$ to the measured values of $y$ and $w$, we obtain the variable

$$ \alpha = v(y, w) \quad (3) $$

which is usually referred to as the scheduling variable.

Linearizing the nonlinear system $G$ about the equilibrium manifold $\Sigma$ parameterized by $\alpha_0$ yields the family of linear systems

$$ G_1(\alpha_0) := \begin{bmatrix} \dot{x}_i(t) \\ z_i(t) \\ y_i(t) \end{bmatrix} = \begin{bmatrix} A(\alpha_0) & B_1(\alpha_0) & B_2(\alpha_0) \\ C_1(\alpha_0) & D_{11}(\alpha_0) & D_{12}(\alpha_0) \\ C_2(\alpha_0) & D_{21}(\alpha_0) & 0 \end{bmatrix} \begin{bmatrix} x_i(t) \\ w_i(t) \\ u_i(t) \end{bmatrix} \quad (4) $$

where, for example, $A(\alpha_0) = \frac{\partial f}{\partial x}(a(\alpha_0))$ and $x_i(t) = x(t) - x_0(\alpha_0)$.

B. Multi-rate sensors and actuators

We consider that the sample-hold devices that interface the discrete-time controller and the continuous-time plant operate at different rates. The setup is shown in Figure 1, where a generic compensator $K$ is also added.
For some positive integer $h$, the multi-rate sample and hold operations can then be written in compact form as

$$\sigma_j^i \in \{j_i \tau, \ldots, j_i \tau + 1\}.$$  

We assume that each sample and each hold operations are periodic, that is, $h_{r_i}, h_{r_j} : \tau^i_j - \tau^i_{j+h_{r_i}}$, $\sigma^i_j - \sigma^i_{j+h_{r_j}}$ are constant and such that their periods $h_{r_i}$, $h_{r_j}$ are related by rational number. Thus, there exist equally spaced time instants $t_k$, $t_k + 1 - t_k = t_s$, such that for each $j$, $i$, there exists a $k$ for which $\sigma^i_j = t_k$ or $\tau^i_j = t_k$.

Each mapping $S_i : y_i(t) \in \mathcal{L}(\mathbb{R}^+) \to y_{ik} \in l(\mathbb{Z}^+)$ is given by

$$(S_i y_i)_{k} := g_i(k) y_i(t_k) = y_{ik}, \quad g_i(k) = \begin{cases} 1 & \text{if } \sigma^i_j = t_k \text{ for some } j \\ 0 & \text{otherwise} \end{cases}$$

where the output of the sampler $S_i$ is defined to be zero if channel $i$ is not sampled at time $t_k$, and each mapping $H_i : u_{ik} \in l(\mathbb{Z}^+) \to u_i(t) \in \mathcal{L}(\mathbb{R}^+)$ is given by

$$\xi_{ik+1} = (1 - r_i(k)) \xi_{ik} + r_i(k) u_{ik}, \quad \xi_{i0} = 0$$

$$\tilde{u}_{ik} = (1 - r_i(k)) \xi_{ik} + r_i(k) u_{ik}, \quad r_i(k) = \begin{cases} 1 & \text{if } \tau^i_j = t_k \text{ for some } j \\ 0 & \text{otherwise} \end{cases}$$

$$u_i(t) = \tilde{u}_{ik}, \quad t \in [t_k, t_{k+1}]$$

Defining the matrices

$$\Gamma_k := \text{diag}(g_1(k), \ldots, g_p(k))$$

and

$$\Omega_k := \text{diag}(r_1(k), \ldots, r_m(k))$$

the multi-rate sample and hold operations can then be written in compact form as

$$\begin{align*}
S : \quad y(t) \in \mathcal{L}(\mathbb{R}^+) & \to y_k \in l(\mathbb{Z}^+) \\
y_k &= \Gamma_k y(t_k)
\end{align*}$$

$$\begin{align*}
H : \quad u_k \in l(\mathbb{Z}^+) & \to u(t) \in \mathcal{L}(\mathbb{R}^+) \\
x_{k+1} &= (I - \Omega_k) x_k + \Omega_k u_k, \quad x_0 = 0 \\
\tilde{u}_{k} &= (I - \Omega_k) x_k + \Omega_k u_k \\
u(t) &= \tilde{u}_{k}, \quad t \in [t_k, t_{k+1}] 
\end{align*}$$

where due to the periodic nature of the sample and hold devices, for some positive integer $h$ which denotes the period, we have

$$\Gamma_k = \Gamma_{k+h}, \quad \Omega_k = \Omega_{k+h}$$
and this set of \( h \)-periodic matrices completely characterize the multi-rate set-up.

The operators \( S : \mathcal{L}(\mathbb{R}^+) \to l(\mathbb{Z}^+) \) and \( H : l(\mathbb{Z}^+) \to \mathcal{L}(\mathbb{R}^+) \) can be decomposed into \( S = \Gamma_d S_{t_s} \) and \( H = H_{t_s} \Omega_d \), with

\[
\begin{align*}
\Omega_d : u_k &\in l(\mathbb{Z}^+) \to \tilde{u}_k \in l(\mathbb{Z}^+) \\
\xi_{k+1} &\equiv (I - \Omega_k)\xi_k + \Omega_k u_k \quad \xi_0 = 0 \\
\tilde{u}_k &\equiv (I - \Omega_k)\xi_k + \Omega_k u_k
\end{align*}
\]

\[
\begin{align*}
H_{t_s} : \tilde{u}_k &\in l(\mathbb{Z}^+) \to u(t) \in \mathcal{L}(\mathbb{R}^+) \\
u(t) &\equiv \tilde{u}_k \quad t \in [t_k, t_{k+1}]
\end{align*}
\]

\[
\begin{align*}
S_{t_s} : y(t) &\in \mathcal{L}(\mathbb{R}^+) \to \tilde{y}_k \in l(\mathbb{Z}^+) \\
\tilde{y}_k &= \begin{bmatrix} \tilde{y}_{mk} \\ \tilde{y}_{rk} \end{bmatrix} = y(t_k)
\end{align*}
\]

\[
\begin{align*}
\Gamma_d : \tilde{y}_k &\in l(\mathbb{Z}^+) \to y_k \in l(\mathbb{Z}^+) \\
y_k &= \Gamma_{mk} \tilde{y}_k = \begin{bmatrix} \Gamma_{mk} \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{y}_{mk} \\ \tilde{y}_{rk} \end{bmatrix}
\end{align*}
\]

where \( \Gamma_k = \begin{bmatrix} \Gamma_{mk} & 0 \\ 0 & \Gamma_{rk} \end{bmatrix} \) has been partitioned according to the output decomposition \( y^T = [y_m^T \ y_c^T] \).

The following definitions will also be useful. Given the set of matrices \( \Gamma_k \), \( k = 1, \ldots, h \), we define \( \Gamma \) as

\[
\Gamma = \text{diag}(\Gamma_1, \Gamma_2, \ldots, \Gamma_h)
\]

which is of the form \( \Gamma = \text{diag}([\gamma_1 \ \gamma_2 \ \ldots \ \gamma_h \times p]) \), \( \gamma_i = \begin{cases} 1 & \text{if } i \in \{i_1, \ldots, i_n\} \\ 0 & \text{otherwise} \end{cases} \)

and we also define a projection matrix \( \Pi_{\Gamma} \) which extracts the nonzero components of \( \Gamma v \), for some vector \( v \)

\[
\Pi_{\Gamma} \Gamma v = [v_{i_1} \ v_{i_2} \ \ldots \ v_{i_n}]^T
\]

and verifies

\[
\Pi_{\Gamma}^T \Pi_{\Gamma} = \Gamma \quad \Pi_{\Gamma} \Pi_{\Gamma}^T = I_{n \times n} \quad \Pi_{\Gamma} = \Pi_{\Gamma} \Gamma
\]

Matrices \( \Gamma_m, \Gamma_r, \Omega \) and \( \Pi_{\Gamma_m}, \Pi_{\Gamma_r}, \Pi_{\Gamma} \) are defined in the same manner.

Similarly to \( y(t) \), the values of \( r(t) \) at sampling instants \( t_k \) will be denoted by \( \tilde{r}_k \) and to take into account the multi-rate nature of the outputs we define \( r_k = \Gamma_{rk} \tilde{r}_k \). We can then define the error variables \( \tilde{e}_k = \tilde{y}_k - \tilde{r}_k \) and \( e_k = y_k - r_k \).

C. Problem Statement

Given this setup the problem addressed in this paper can be stated as follows:

**Problem statement.** 1. For each operating point \( \alpha_0 \), find a possibly time-varying discrete-time linear controller \( C(\alpha_0) : [y_{mk}, e_{sk}] \to u_{sk} \)

\[
C(\alpha_0) = \begin{bmatrix} x_{sk+1}^e \\ u_{sk} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{k}^e(\alpha_0) & B_{k}^e(\alpha_0) D_{k}^e(\alpha_0) \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_{sk}^e \\ y_{mk}^e \\ e_{sk}^e \end{bmatrix}
\]

for the linearization of the nonlinear plant (Eq. (4)) with interface \( SH \) (Eq. (5)), that stabilizes and achieves zero steady-state output error for \( e_k \), where \( u_{sk} = u_k - a_0, y_{mk} = y_k - \Gamma_{mk} a_0 \) and \( e_{sk} = e_k \).

2. Based on the family of linear controllers \( C(\alpha_0) \), implement a discrete-time controller, \( K \), possibly nonlinear and time-varying

\[
K = \begin{bmatrix} x_{k+1}^e = f_e(x_k^e, y_{mk}, e_k, \alpha_k, k) \\ u_k = h_e(x_k^e, y_{mk}, e_k, \alpha_k, k) \end{bmatrix}
\]

that verifies the linearization property, where \( \alpha_k \) is the scheduling variable at sampling time \( t_k \), Eq. (3), and for depending on this variable, \( K \) is called a gain-scheduled controller. By linearization property we formally mean that if we consider a family of equilibrium points for the controller compatible with \( \Sigma \), Eq. (2)

\[
\Sigma^e(\alpha_0) := \{x_{0}^e(\alpha_0) : x_{0}^e = f_e(x_{0}^e, y_{0}, \alpha_0, k), \ u_{0} = h_e(x_{0}^e, y_{0}, \alpha_0, k) \) and \( (x_0, u_0, w_0) = (a(\alpha_0), \alpha_0 \in \Xi) \}
\]
the linearization of $K$ about this equilibrium family is the same as that of the designed controller $C(a_0)$, that is,

\[
\begin{align*}
\frac{\partial f_c}{\partial x_c}(a_0^c(k), k) &= A_k^c(a_0) \quad \frac{\partial f_c}{\partial y_m}(a_0^c(k), k) = B_{1k}^c(a_0) \\
\frac{\partial h_c}{\partial x_c}(a_0^c(k), k) &= C_k^c(a_0) \quad \frac{\partial h_c}{\partial y_m}(a_0^c(k), k) = D_{1k}^c(a_0)
\end{align*}
\]

In Section III we propose a controller structure which solves part 1 of the problem statement. This structure renders simple the controller implementation which is given in Section IV where we prove that the linearization property is verified for this implementation (part 2 of the problem statement).

III. Regulator Structure

In this section we will focus on the existence of a linear controller for $G_l$, Eq. (4), with interface $SH$, Eq. (5), that achieves closed loop stability and zero-error regulation for the desired outputs. To simplify the notation, the dependency on $a_0$, which here is assumed a constant parameter, is dropped as well as the notation $\delta$ which indicates that the linearization, Eq. (4), is done locally, for example, we will use $x(t)$ instead of $x_i(t)$. We will come back to this local viewpoint at the end of the section. Furthermore, since $w(t)$, $z(t)$ and the associated matrices play no role in stability and regulation issues, they will also be temporarily disregarded. The equations for the plant are then simply given by

\[
G_l = \begin{cases}
\dot{x}(t) = Ax(t) + B_2u(t) \\
y(t) = C_2x(t)
\end{cases}
\]

Due to the periodic nature of the multi-rate sample and hold devices some of the systems involved in this paper will be linear periodically time-varying, and therefore we first review some of the results regarding periodic systems. Then we present an existing solution for the regulation for multi-rate square linear systems and propose a solution for the regulation of non-square systems.

A. Periodic systems theoretical background

Consider the discrete-time linear periodic system

\[
P := \begin{cases}
x_{k+1} = A_kx_k + B_ku_k \\
y_k = C_kx_k + D_ku_k
\end{cases}
\]

with initial time $k = 0$ and initial condition $x_0$ and where $A_k$, $B_k$, $C_k$, $D_k$ are $h$-periodic matrices, for example, $A_k = A_{k+h}$. Associated with this system, the lifted time-invariant system $\tilde{P}$ is defined as

\[
\tilde{P} := \begin{cases}
\tilde{x}_{l+1} = \bar{A}\tilde{x}_l + \bar{B}\tilde{u}_l \\
\tilde{y}_l = \bar{C}\tilde{x}_l + \bar{D}\tilde{u}_l
\end{cases}
\]

with $\bar{x}_l = x_{ih}$, $\bar{u}_l = [u_{ih}^T \quad u_{ih+1}^T \quad \ldots \quad u_{ih+h-1}^T]^T$, $\bar{y}_l = [y_{ih}^T \quad y_{ih+1}^T \quad \ldots \quad y_{ih+h-1}^T]^T$ and $\bar{A} = \Phi(h, 0)$, $\bar{B} = [\Phi(h, 1)B_0 \quad \Phi(h, 2)B_1 \quad \ldots \quad B_{h-1}]$, $\bar{C} = [C_0^T \quad C_1^T \Phi(1,0)^T \quad \ldots \quad (C_{h-1}^T \Phi(h-1,0))^T]^T$, $\bar{D} = d_{ij}$

\[
d_{ij} = \begin{cases}
C_{i-1}\Phi(i-1,j)B_{j-1} & i > j \\
0 & i \leq j
\end{cases}
\]

System $P$ is stable, if and only if $\tilde{P}$ is stable, which is equivalent to the eigenvalues of matrix $\bar{A}$, $\lambda_i(\bar{A})$, having norm less that one $\|\lambda_i(\bar{A})\| < 1$, $\forall i$. These eigenvalues are called the characteristic multipliers of system $P$. Detectability and stabilizability are defined in the following way

Definition III.1. The periodic system $P$, Eq. (11), is said to be stabilizable if there exists a set of periodic matrices $F_k$, $F_k = F_{k+h}$, such that $x_{k+1} = (A_k + B_kF_k)x_k$ is stable and it is said to be detectable if there exists a set of periodic matrices $G_k$, $G_k = G_{k+h}$, such that $x_{k+1} = (A_k + G_kC_k)x_k$ is stable.

As for stability, these conditions can be verified using the lifted LTI system Eq. (12)
Theorem III.1. The periodic system $P$, Eq. (11) is detectable and stabilizable if and only if $\overline{P}$, Eq. (12), is respectively detectable and stabilizable.

Proof. See Ref. 8

The following is also true for periodic systems

Theorem III.2. There exists a periodic linear controller $K_p : y_k \rightarrow u_k$ such that the closed loop is asymptotically stable if and only if the periodic system Eq. (11) is detectable and stabilizable.

Proof. See Ref. 10

B. Regulation of multi-rate systems for square plants

For continuous and single-rate discrete systems, it is well-known that zero-error regulation of a number of outputs no greater than the number of inputs can be achieved by incorporating in the controller system a number of integrators equal to the number of tracking variables. Zero-error regulation is not tied to linearity and is achieved even in the presence of model uncertainty that do destroy stability. These integrators are usually placed after the system, integrating the error outputs. For the discrete multi-rate output regulation problem the structure in Figure 2 is proposed in Ref. 9 for square systems, which means $n_{ym} = 0$, $n_{yr} = p$, $m = p$, $C_r = C_2$. The block $C_I$ represents a periodically time-varying system whose state space description is given by

$$
C_I = \begin{cases} 
    x_{k+1}^I &= x_k^I + \Omega_k u_k^I \\
    y_k^I &= x_k^I + \Omega_k u_k^I
\end{cases}
$$

and that integrates its input at the sampling instants at which the hold operation is active. The block $C_K$ represents a linear controller which, in general, is required to be periodically time-varying. Notice that in this case integrators are placed at the input of the plant, because otherwise directly integrating the error would produce a non-constant signal at the input of the plant, due to the fact that $C_K$ is generally required to be time-varying.

As usual, integrators play the key role for achieving zero-output regulation. In Ref. 9 it is proved that, under mild assumptions, the augmented system seen by the controller $C_K$ is detectable and stabilizable. By Theorem III.2 there exists a controller $C_K$ that asymptotically stabilizes the closed loop. The system trajectories will therefore tend to the unique equilibrium point that due to the integrators, and under the assumption of no transmission zeros at $z = 1$ of the discretization of $G_l$, Eq (10), is characterized by $u_k = u_0$ and $y_k = \Gamma_k r_0$, $y(t) = r_0$.

C. Regulation for non-square plants

For a large class of control problem the number of available outputs is larger than the number of actuators. This is generally the case for guidance and control problems for autonomous vehicles. It is well-known that we can only impose zero-error regulation for a number of outputs no greater than the number of inputs. Some small modifications have been made, namely notation and considering a slightly different system for integral action, $C_I$. These changes however do not impact on the ideas and results presented in Ref. 9.
However we would like to take advantage of all the available outputs for control and not just the ones for which we require zero-error regulation. For the robust linear regulation problem, i.e., assuming uncertain on the coefficients of the matrices describing the plant, the remaining outputs of the plant, \( y_{mk} \), will have constant but unknown values. Therefore, in general, we cannot assure \( e_k = y_k - r_k \in \mathbb{R}^p = 0 \) for the error components corresponding to \( y_{mk} \), which would be necessary in the scheme presented in Figure 2 to force the input of the integrator system \( C_I \) to be zero.

Motivated by this discussion, the structure in Figure 3 is proposed. The periodic system \( C_D \) has state space realization given by

\[
C_D = \begin{cases} 
  x^D_{k+1} = (I - \Gamma_{mk})x^D_k + \Gamma_{mk}u^D_k \\
  y^D_k = -\Gamma_{mk}x^D_k + \Gamma_{mk}u^D_k
\end{cases}
\]

(14)

![Figure 3. Regulator structure for non-square systems](image)

This system has the role of differentiating the output \( y_m \) at sampling instants at which the sampling operation is active. At steady state the constant unknown values of \( y_m \) are differentiated bypassing the previously stated problem. In the context of gain-scheduling, a method to guarantee the linearization property named velocity implementation also performs a differentiation of the non-tracking variables. The relation between the proposed structure and the velocity implementation will be given in the next Section (IV. B).

The system seen by the controller \( C_K \) to be synthesized (Figure 3), denoted by \( G_a \), consists of \( i) \Omega_I: \) The multi-rate input system \( \Omega_d \) and periodic integrator \( C_I \) \( ii) \) \( G_d: \) The linear system with sample-hold interface and \( iii) \) \( \Gamma_D: \) The multi-rate output system \( \Gamma_D \) and differentiator \( C_D \).

The operations of systems \( \Omega_d \) and \( C_I \) can be put together to yield the simple description of \( \Omega_I \), which is the same as \( C_I \)

\[
\Omega_I = \begin{cases} 
  x^I_{k+1} = x^I_k + \Omega_k y^d_k \\
  \tilde{u}_k = x^I_k + \Omega_k y^d_k
\end{cases}
\]

As usual the equations for \( G_d = S_tG_IH_{ts} \) can be written as

\[
G_d = \begin{cases} 
  x_{k+1} = A_dx_k + B_d\tilde{u}_k \\
  \tilde{y}_k = \begin{bmatrix} y_{mk} \\ \tilde{y}_{rk} \end{bmatrix} = C_d x_k = \begin{bmatrix} C_{dm} \\ C_{dr} \end{bmatrix} x_k
\end{cases}
\]

(15)

where \( A_d = e^{At} \), \( B_d = \int_0^t e^{At}d\tau B_2 \) and \( C_d = C_2 \). Finally the equations for \( \Gamma_D \) are

\[
\Gamma_D = \begin{cases} 
  x^D_{k+1} = (I - \Gamma_{mk})x^D_k + \Gamma_{mk}e_k \\
  \tilde{y}_k = \begin{bmatrix} y_{mk} \\ \tilde{y}_{mk} \end{bmatrix} = \begin{bmatrix} \Gamma_{mk} \\ 0 \end{bmatrix} x^D_k + \begin{bmatrix} 0 \\ \Gamma_{rk} \end{bmatrix} e_k
\end{cases}
\]

Computing the series of these three systems we obtain
Suppose that the equations for an asymptotic stabilizing controller
\[ D. \]
Solution to part I of the Problem Statement
asymptotically stabilizes the closed loop, i.e., the feedback connection between
\[ Proof. \]
Theorem III.4.
that only depend on the original discrete-time plant \( G_d \), Eq. (15), obtained form the discretization of \( G_i \), Eq. (10).

Theorem III.3. Assume that the following conditions hold

1. \((A_d, B_d)\) is stabilizable and \((A_d, C_d)\) is detectable.
2. Each output is sampled and each input is updated at least once in a period \( T \), i.e., \( \forall i \exists k \in \{1, \ldots, n\} : (\Gamma_k)_{ii} = 1 \) and \( \forall i \exists k \in \{1, \ldots, k\} : (\Omega_k)_{ii} = 1 \).
3. For all distinct eigenvalues \( \lambda_j(A_d), \lambda_i(A_d) \) of \( A_d \), \((\lambda_j(A_d))^b \neq (\lambda_i(A_d))^b \) and \( \exists \lambda_k(A_d) \neq 1 : \| \lambda_k(A_d) \| = 1 \) and \( (\lambda_k(A_d))^b = 1 \).
4. There are no transmission zeros at \( z = 1 \) from the input of \( G_d \), Eq. (15), to the regulated output, i.e.,
\[
\begin{bmatrix}
A_d - I & B_d \\
C_d r & 0
\end{bmatrix}
\]
is full rank.

Then, the augmented periodic system \( G_a \), Eq. (16), is detectable and stabilizable.

\[ Proof. \] See Appendix.

By Theorem III.2 detectability and stabilizability guarantees the existence of an asymptotic stabilizing controller for the augmented system \( G_a \). Due to the previous discussion, the following theorem comes with no surprise

Theorem III.4. Consider the regulator structure presented in Figure (3) and suppose the controller \( C_K \) asymptotically stabilizes the closed loop, i.e., the feedback connection between \( C_K \) and \( G_a \), Eq. (16). Then zero-error output regulation for \( y_{rk} \) is achieved even in the presence of model uncertainty that do not destroy the closed loop stability.

\[ Proof. \] See Appendix.

D. Solution to part I of the Problem Statement

Suppose that the equations for an asymptotic stabilizing controller \( C_K \), for plant \( G_a \), are given by
\[
C_K = \begin{bmatrix}
x_{k+1}^K \\
y_k^K
\end{bmatrix} = \begin{bmatrix}
A^K_k & B^K_{1k} & B^K_{2k} \\
C^K_k & D^K_{1k} & D^K_{2k}
\end{bmatrix}\begin{bmatrix}
x_k^K \\
u_1^K \\
u_2^K
\end{bmatrix}
\]

Returning to the original problem, stated in part I of the problem statement, of finding a local controller, with the properties therein referred, we rewrite the equations for \( C_I, C_K, C_D \) in the form
where we have added the dependency on $\alpha_0$ and the notation indicating that it is a local controller, for example, $x_{skk}$ instead of $x_k$. These equations can be combined together to yield

$$C(\alpha_0) = \begin{bmatrix} x_{skk+1} \\ y_{skk} \\ x_{sk} \\ y_{sk} \\ u_{sk} \end{bmatrix} = \begin{bmatrix} A_k^C(\alpha_0) & B_{1k}^C(\alpha_0) & B_{2k}^C(\alpha_0) \\ C_k^C(\alpha_0) & D_{1k}^C(\alpha_0) & D_{2k}^C(\alpha_0) \end{bmatrix} \begin{bmatrix} x_k^C \\ y_{sk} \\ e_{sk} \end{bmatrix}$$

(19)

where $x_{sk}^C = [(x_{sk})^T (y_{sk})^T (u_{sk})^T]^T$

$$A_k^C(\alpha_0) = \begin{bmatrix} A_k^C(\alpha_0) & -B_{1k}^C(\alpha_0)\Gamma_{mk} & 0 \\ 0 & I - \Gamma_{mk} & 0 \\ \Omega_k C_k^C(\alpha_0) & -\Omega_k D_{1k}^C(\alpha_0)\Gamma_{mk} & I \end{bmatrix} B_{1k}^C(\alpha_0) = \begin{bmatrix} B_{1k}^C(\alpha_0)\Gamma_{mk} \\ 0 \end{bmatrix} B_{2k}^C(\alpha_0) = \begin{bmatrix} 0 \\ \Omega_k D_{2k}^C(\alpha_0) \end{bmatrix} C_k^C(\alpha_0) = \begin{bmatrix} \Omega_k C_k^C(\alpha_0) & -\Omega_k D_{1k}^C(\alpha_0)\Gamma_{mk} & I \end{bmatrix} D_{1k}^C(\alpha_0) = \begin{bmatrix} 0 \\ \Omega_k D_{2k}^C(\alpha_0) \end{bmatrix}$$

(20)

which constitutes a solution for Part 1 of the problem statement.

### IV. Gain-scheduling implementation

#### A. Gain-scheduling implementation

Having designed the parameterized family of linear controllers $C(\alpha_0)$, Eq. (19), suppose we implement the gain-scheduled non-linear controller, $K$, as follows

$$K(\alpha_k, k) = \begin{bmatrix} x_{sk}^+ \\ y_{sk}^+ \\ x_{sk}^b \\ y_{sk}^b \\ u_k \end{bmatrix} = \begin{bmatrix} A_k^C(\alpha_k) & B_{1k}^C(\alpha_k) & B_{2k}^C(\alpha_k) \\ C_k^C(\alpha_k) & D_{1k}^C(\alpha_k) & D_{2k}^C(\alpha_k) \end{bmatrix} \begin{bmatrix} x_k^C \\ y_{sk} \\ e_{sk} \end{bmatrix}$$

(21)

$$A_k^C(\alpha_k) = \begin{bmatrix} A_k^C(\alpha_k) & I - \Gamma_{mk} & 0 \\ \Gamma_{mk} & 0 & 0 \\ \Omega_k C_k^C(\alpha_k) & -\Omega_k D_{1k}^C(\alpha_k) & I \end{bmatrix} B_{1k}^C(\alpha_k) = \begin{bmatrix} B_{1k}^C(\alpha_k)\Gamma_{mk} \\ 0 \end{bmatrix} B_{2k}^C(\alpha_k) = \begin{bmatrix} 0 \\ \Omega_k D_{2k}^C(\alpha_k) \end{bmatrix} C_k^C(\alpha_k) = \begin{bmatrix} \Omega_k C_k^C(\alpha_k) & -\Omega_k D_{1k}^C(\alpha_k)\Gamma_{mk} & I \end{bmatrix} D_{1k}^C(\alpha_k) = \begin{bmatrix} 0 \\ \Omega_k D_{2k}^C(\alpha_k) \end{bmatrix}$$

(22)

Notice that $\alpha_k$, which was considered to be a constant design parameter during the design process, now becomes a scheduling variable computed on-line from the plant available outputs. Due to the multi-rate nature of the output, system described by Eq. (22.5) is used to perform a hold operation so that the scheduling variable is computed, at each iteration, according to the last sampled value of the output. The exogenous vector is assumed available at each sampling instant, $w_k = w(t_k)$. Notice that $K$ is a nonlinear controller of the form Eq. (8), where $x_k = [(x_{k+1})^T (x_{k+1})^T (x_{k+1})^T]^T$,
\[ f_c(x_k^e, y_{mk}, e_k, \alpha_k, k) = \begin{bmatrix}
A_k^e(\alpha_k) & -B_{1k}^e(\alpha_k)\Gamma_{mk} & 0 \\
0 & I - \Gamma_{mk} & 0 \\
\Omega_k C_k^e(\alpha_k) & -\Omega_k D_{1k}^e(\alpha_k)\Gamma_{mk} & I
\end{bmatrix}
\begin{bmatrix}
x_k^e \\
x_k^d \\
x_k^l
\end{bmatrix}
+ \begin{bmatrix}
B_{1k}^e(\alpha_k)\Gamma_{mk} & B_{2k}^e(\alpha_k) \\
\Omega_k D_{1k}^e(\alpha_k)\Gamma_{mk} & \Omega_k D_{2k}^e(\alpha_k)
\end{bmatrix}
\begin{bmatrix}
y_{mk} \\
e_k
\end{bmatrix} \tag{23}\]

\[ h_c(x_k^e, y_{mk}, e_k, \alpha_k, k) = \begin{bmatrix}
\Omega_k C_k^e(\alpha_k) & -\Omega_k D_{1k}^e(\alpha_k)\Gamma_{mk} & I
\end{bmatrix}
\begin{bmatrix}
x_k^e \\
x_k^d \\
x_k^l
\end{bmatrix}
+ \begin{bmatrix}
\Omega_k D_{1k}^e(\alpha_k)\Gamma_{mk} & \Omega_k D_{2k}^e(\alpha_k)
\end{bmatrix}
\begin{bmatrix}
y_{mk} \\
e_k
\end{bmatrix} \tag{24}\]

The next theorem proves that the linearization of the nonlinear implemented controller, \( K \), at each equilibrium, characterized by \( \alpha_0 \), is the same of the designed \( C(\alpha_0) \), thus establishing Part II of the problem statement.

**Theorem IV.1.** Suppose for each value of the parameter \( \alpha_0 \), the matrix

\[
\begin{bmatrix}
\bar{A}^e(\alpha_0) - I & \bar{B}_2^e(\alpha_0)\Pi_{r'} \\
\Pi_{\Omega}\bar{C}^e(\alpha_0) & \Pi_{\Omega}\bar{D}_2^e(\alpha_0)\Pi_{r'}
\end{bmatrix}
\]  \tag{25}

is full rank, where \((\bar{A}^e(\alpha_0), \bar{B}_2^e(\alpha_0), \bar{C}^e(\alpha_0), \bar{D}_2^e(\alpha_0))\) are the matrices obtained from the lift of \((A_k^e(\alpha_0), B_{2k}(\alpha_0), C_k^e(\alpha_0), D_{2k}(\alpha_0))\), Eq. (18), and \(\Pi_{\Omega}, \Pi_{r'}\) are defined according to Eq. (6). Then the linearization of the gain-scheduled implemented controller \( K \), Eq.(22), at each equilibrium point, is the same as that of the designed controller \( C(\alpha_0) \), Eqs. (20), (21).

Condition Eq. (25) is imposed to guarantee that the trimming values of the controller are unique. Its relation to a similar condition required by the velocity implementation is presented in the next section.

**Proof.** We start by computing \( \Sigma^c \). Eq. (9), the family of equilibrium values of the controller compatible with the equilibrium family of the nonlinear system Eq. (2).

The equations to compute these equilibrium values, denoted with an underbar, \( \underline{x}_k^e, \underline{x}_k^d, \underline{x}_k^l \), are

\[ \begin{bmatrix}
\underline{x}_k^e \\
\underline{x}_k^d \\
\underline{x}_k^l
\end{bmatrix}
= \begin{bmatrix}
A_k^e(\alpha_0) & B_{1k}^e(\alpha_0) & B_{2k}^e(\alpha_0) \\
C_k^e(\alpha_0) & D_{1k}^e(\alpha_0) & D_{2k}^e(\alpha_0)
\end{bmatrix}
\begin{bmatrix}
\underline{x}_k^e \\
\underline{x}_k^d \\
\Gamma_{mk}(y_{r0} - r_0)
\end{bmatrix} \tag{26}\]

\[ \begin{bmatrix}
\underline{x}_k^d \\
\underline{x}_k^l
\end{bmatrix}
= \begin{bmatrix}
I - \Gamma_{mk} & \Gamma_{mk} \\
-\Gamma_{mk} & I
\end{bmatrix}
\begin{bmatrix}
\underline{x}_k^d \\
\Gamma_{mk}y_{m0}
\end{bmatrix} \tag{27}\]

\[ \begin{bmatrix}
\underline{x}_k^e \\
y_{m0}
\end{bmatrix}
= \begin{bmatrix}
I & \Omega_k \\
I & \Omega_k
\end{bmatrix}
\begin{bmatrix}
\underline{x}_k^e \\
y_{m0}
\end{bmatrix} \tag{28}\]

Equation (27) implies that \( \underline{x}_k^d = y_{m0} \) and \( \underline{x}_k^l = 0 \), and equation (28) that \( \Omega_k\underline{x}_k^e = 0 \iff \Omega\underline{x}_k^e = 0 \iff \Pi_{\Omega}\Sigma^c = 0 \). Thus, Eq. (26) reduces to

\[ \begin{bmatrix}
\underline{x}_k^e \\
\underline{x}_k^d \end{bmatrix}
= \begin{bmatrix}
A_k^e(\alpha_0) & B_{2k}^e(\alpha_0) \\
C_k^e(\alpha_0) & D_{2k}^e(\alpha_0)
\end{bmatrix}
\begin{bmatrix}
\underline{x}_k^e \\
\Gamma_{rk}(y_{r0} - r_0)
\end{bmatrix} \]

which in the lifted system corresponds to

\[ \begin{bmatrix}
\bar{x}_k^e \\
\bar{x}_k^d
\end{bmatrix}
= \begin{bmatrix}
\bar{A}^e(\alpha_0) & \bar{B}_2^e(\alpha_0) \\
\bar{C}^e(\alpha_0) & \bar{D}_2^e(\alpha_0)
\end{bmatrix}
\begin{bmatrix}
\bar{x}_k^e \\
\Gamma_r(y_{r0} - \bar{r}_0)
\end{bmatrix} \]

Multiplying the last rows by \( \Pi_{\Omega} \) and taking into account that \( \Pi_{\Omega}\bar{x}_k^e = 0 \) and \( \Gamma_r = \Pi_{r'}^{c}, \Pi_{r'} \) yields

\[ \begin{bmatrix}
\bar{A}^e(\alpha_0) - I & \bar{B}_2^e(\alpha_0)\Pi_{r'} \\
\Pi_{\Omega}\bar{C}^e(\alpha_0) & \Pi_{\Omega}\bar{D}_2^e(\alpha_0)\Pi_{r'}
\end{bmatrix}
\begin{bmatrix}
\bar{x}_k^e \\
\Pi_{r'}(y_{r0} - \bar{r}_0)
\end{bmatrix}
= 0 \]
By assumption Eq. (25) this implies that $\bar{x}^{K} = 0$ and $\Pi_{m}(\bar{y}_{r0} - \bar{r}_0) = 0$ from which we conclude that $y_{r0} - r_0 = 0$. Finally from Eq. (28) we have $x_{k}^{j} = u_0$. Summarizing, the equilibrium variables of the controller are

$$
\bar{x}^{K} = 0, \quad \bar{x}^{D} = 0,
$$

$$
\bar{x}^{L} = y_{r0}, \quad \bar{x}^{F} = 0,
$$

$$
\bar{x}^{L} = u_0
$$

If we linearize the equations of the gain-scheduled controller $K$, Eq.(22), about this trimming values we obtain

$$
K_{l}(\alpha_k) = \begin{bmatrix}
    x^{K}_{\delta k+1} \\
    y^{K}_{\delta k}
    \end{bmatrix} = \begin{bmatrix}
    A^{K}(\alpha_k) & B^{K}_{1k}(\alpha_k) & B^{K}_{2k}(\alpha_k) \\
    C^{K}(\alpha_k) & D^{K}_{1k}(\alpha_k) & D^{K}_{2k}(\alpha_k)
    \end{bmatrix} \begin{bmatrix}
    x^{K}_{\delta k} \\
    y^{K}_{\delta k}
    \end{bmatrix} + \frac{\partial \alpha(\bar{y}_{r0}, w_{k})}{\partial y} \frac{\partial}{\partial \alpha} \begin{bmatrix}
    A^{K}(\alpha_k) & B^{K}_{1k}(\alpha_k) & B^{K}_{2k}(\alpha_k)
    \end{bmatrix} \begin{bmatrix}
    x^{K}_{\delta k} \\
    y^{K}_{\delta k}
    \end{bmatrix} y^{\delta k}
$$

which evaluated at equilibrium $x^{K} = \bar{x}^{K} = 0$, $y^{D} = \bar{y}^{D} = 0$, $e_k = 0$ and $\alpha_k = \alpha_0$ yields

$$
K_{l}(\alpha_0) = \begin{bmatrix}
    x^{K}_{\delta k+1} \\
    y^{K}_{\delta k}
    \end{bmatrix} = \begin{bmatrix}
    A^{K}(\alpha_0) & B^{K}_{1k}(\alpha_0) & B^{K}_{2k}(\alpha_0) \\
    C^{K}(\alpha_0) & D^{K}_{1k}(\alpha_0) & D^{K}_{2k}(\alpha_0)
    \end{bmatrix} \begin{bmatrix}
    x^{K}_{\delta k} \\
    y^{K}_{\delta k}
    \end{bmatrix}
$$

which is the same as Eqs. (20), (21), concluding the proof.

**B. Relation with the velocity implementation**

The velocity implementation is a method first presented in Ref. 13 to implement a gain-scheduled controller with integral action in order to verify the linearization property. For the discrete single-rate case, see Ref. (14) and Ref. (17). The idea behind the velocity implementation is quite simple to explain. Suppose that, for a family of linear single-rate plants, one has designed a family of linear controllers with the structure $b$

$$
C_{d}(\alpha_0) = \begin{bmatrix}
    x^{K}_{\delta k+1} = A^{K}(\alpha_0)x^{K}_{\delta k} + B^{K}_{1}(\alpha_0)y^{\delta mk} + B^{K}_{2}(\alpha_0)y^{K}_{\delta k} \\
    y^{K}_{\delta k} &= x^{K}_{\delta k} + (y^{\delta mk} - \bar{r}_0) \\
    u^{K}_{\delta k} &= C^{K}(\alpha_0)x^{K}_{\delta k} + D^{K}_{1}(\alpha_0)y^{\delta mk} + D^{K}_{2}(\alpha_0)y^{K}_{\delta k}
    \end{bmatrix}
$$

Suppose also that we implement the non-linear controller moving the integrators to the front of the controller (input of the plant) as shown in Figure 4. Again, $\alpha_k$ considered a constant throughout the design process, $a$ here we consider a non-strictly proper integrator to be according to our formulation.
now becomes a scheduling variable. The equations for this implemented controller are

\[
C_d(\alpha_k) = \begin{cases}
  x_{k+1}^c = A(\alpha_k)x^c_k + B(\alpha_k)(y_{mk} - y_k^c) + B^f(\alpha_k)(y_{rk} - r_k) \\
  x_{k+1}^r = x_k^c + C(\alpha_k)x_k^c + D(\alpha_k)(y_{mk} - y_k^r) + D^f(\alpha_k)(y_{rk} - r_k) \\
  y_{mk}^c = y_k \\
  y_k^c = x_k \\
  \alpha_k = g(y_k, w_k) \\
  u_k = x_k^c + C(\alpha_k)x_k^c + D(\alpha_k)(y_{mk} - y_k^r) + D^f(\alpha_k)(y_{rk} - r_k)
\end{cases}
\]

Figure 4. Velocity implementation

The idea behind moving the integrators is that of building a linear equivalent controller for constant parameter values that when implemented changing \( \alpha_0 \) to \( \alpha_k \) verifies the linearization property. The insight is that for the structure of Figure 4(b), the equilibrium values at the input of \( C_d(\alpha_k) \) are zero and therefore the terms that appear while taking the derivatives for the linearization due to the introduction of the scheduling variable \( \alpha_k \) cancel out (this is the idea of the proof of Theorem (IV.1)). It is easy to see that this does not in general occur for other controller implementations (see Ref. 12, 13, 15, 17 for examples).

There is a close relation between the velocity implementation and the method herein presented when the multi-rate set-up degenerates in the single-rate. In the single rate case, \( \Gamma = I \) and \( \Omega = I \), from which equations (13) and (14) reduce to simple integrators and differentiators and therefore controller Eq. (19) has exactly the same structure as that in Figure 4(b). However the point of view is somehow different. While in the velocity implementation method, the controller could be synthesized by augmenting the state with the integrators at the front of the plant, which includes, for example, P.I. controllers when the state is available, here the synthesized controller takes into account in the design the differentiator at the output and the integrators at the input of the plant. Nevertheless we stress that we do not make assumption on how the controller is synthesized neither for the multi-rate case nor for the particular case of single-rate (Theorem III.2 assures that a stabilizing controller exists) and that, for the single-rate case, the class of stabilizing controllers for either structure Figure 4(a) and Figure 4(b), is the same, for frozen-time values of the parameters.

As a final remark we mention that, for the single rate case, condition Eq. (25) of Theorem IV.1 is equivalent to saying that the lift of the controller, which can be assumed time-invariant for the single-rate case, does not have transmission zeros at \( z = 1 \) from the input of the controller corresponding to the regulated output of the system to the output of the controller. It is easy to see that the fact that the lift system does not have transmission zeros at \( z = 1 \) implies that the original time-invariant controller does not have either, which is a condition of the similar Theorem in Ref. 14.

C. Outline of the proposed method

The following procedure outlines the proposed method for the design and implementation of gain-scheduled controllers for multi-rate systems:

1. Given a nonlinear system, Eq. (1), with multi-rate input and output interface SH, Eq. (5), obtain a family of parameter-dependent linear models, Eq. (4).

2. Design a family of linear controllers with the structure of Eqs. (20), (21). Theorems III.3 and III.2, guarantee that for each fixed value of the schedule parameter a stabilizing controller exists, under the conditions therein stated.

3. Implement the non-linear gain-scheduled controller according to Eq. (22). Theorem IV.1 assures that the linearization property is verified.
V. Application to the control of an autonomous rotorcraft

In this section the method proposed is applied to the control of an autonomous rotorcraft. The dynamic model of a small-scale helicopter, parameterized for the Vario X-treme R/C Helicopter presented in Ref. 18 is used, and briefly presented first. The problem addressed is the path-following problem which consists of steering the autonomous vehicle along a predefined path while tracking a given velocity profile. The solution relies on the definition of a path-dependent error space to express the model of the vehicle.

A. Vehicle Dynamic Model

The helicopter dynamics are described using the conventional six degree of freedom rigid body equations

\[
\begin{align*}
\dot{\mathbf{r}}_B &= f(\mathbf{v}_B, \mathbf{\omega}_B, u) + \mathbf{\omega}^T \times \mathbf{g}, \\
\dot{\mathbf{\omega}}_B &= \mathbf{\Phi}(\mathbf{v}_B, \mathbf{\omega}_B, u), \\
\dot{\mathbf{p}}_B &= \mathbf{\Phi}^T \mathbf{v}_B, \\
\dot{\mathbf{\lambda}}_B &= \mathbf{Q}(\phi_B, \theta_B) \mathbf{\omega}_B,
\end{align*}
\]

where \((^{T}p_B, ^{T}R) \in SE(3) \cong \mathbb{R}^3 \times SO(3)\) denotes the configuration of the body frame attached to the vehicle’s center of mass, \(\{T\}\), with respect to the inertial frame, \(\{I\}\) and the rotation matrix \(^{T}_{B} \mathbf{R} = ^{T}_{I} \mathbf{R}^B\) can be parameterized by the Z-Y-X Euler angles \(\lambda_B = [\phi_B \, \theta_B \, \psi_B]^T\), \(\theta_B \in [-\pi/2, \pi/2]\), \(\phi_B, \psi_B \in \mathbb{R}\), \(^{T}_{B} \mathbf{R} = \mathbf{R}_z(\psi_B) \mathbf{R}_y(\theta_B) \mathbf{R}_x(\phi_B)\). The linear and angular body velocities, \(\mathbf{v}_B = [u_B \, v_B \, w_B]^T \in \mathbb{R}^3\) and \(\mathbf{\omega}_B = [\mathbf{\omega}_B]^T \in \mathbb{R}^3\), are given respectively by \(\mathbf{v}_B = ^{I}_{B} \mathbf{R} \dot{\mathbf{p}}_B\) and \(\mathbf{\omega}_B = ^{I}_{B} \mathbf{R} \mathbf{\omega}_B\), where \(^{I}_{B} \mathbf{\omega}_B \in \mathbb{R}^3\) is the angular velocity of \(\{B\}\) with respect to \(\{I\}\).

The actuation \(u = [\delta_0, \delta_{1x}, \delta_{1y}, \delta_{1z}]^T\) comprises the main rotor collective input \(\delta_0\), the main rotor cyclic inputs, \(\delta_{1x}, \delta_{1y}\), and the tail rotor collective input \(\delta_{1z}\). The dynamic equations for the helicopter are highly non-linear and its derivation is only accomplishable assuming several simplifications. For a detailed explanation of the modeling of the small-scale helicopter used in this paper the reader is referred to Ref. 18.

B. Path-dependent Error Dynamic Model

An integrated guidance and control strategy proposed in Ref. 4 in order to solve the path-following problem, consists of defining a convenient path-dependent non-linear transformation applied to the vehicle dynamic and kinematic model. In the new error dynamic variables the problem of steering the autonomous vehicle along a predefined path with a given velocity profile, is reduced to that of regulating the error variables to zero. To present this transformation we first introduce frames \(\{T\}\) and \(\{C\}\), shown in Figure 5, where the references signals are defined. These frames can be briefly described as (see Ref. 4 for further details)

- **Frame** \(\{T\}\) There is an almost exact correspondence between \(\{T\}\) and the standard Serret-Frenet frame.\(^4\) The x axis, \(x_T\), is aligned with the tangent vector to the path, which allows to define the linear velocity reference in this frame as

\[v_T = V_T \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\]

where \(V_T\) is the desired velocity modulus. The desired angular velocity is also defined in this frame and is given by

\[\omega_T = V_T \begin{bmatrix} \tau & 0 & \kappa \end{bmatrix}\]

where \(\tau\) and \(\kappa\) are respectively the torsion and the curvature of the curve defining the path.\(^4\) This coordinate frame moves along the path attached to the point on the path closest to the vehicle, which allows to define the two components position error \(d_t = [d_x \, d_y]^T \in \mathbb{R}^2\) as

\[
\begin{bmatrix} 0 \\ d_t \end{bmatrix} = \tau T \mathbf{R} (^{T}_{B}p_T - ^{T}_{B}p_T)
\]

where \(^{T}_{B}p_T\) is the position of the origin of frame \(\{T\}\) with respect to \(\{I\}\). It will also be useful to consider, for frame \(\{T\}\), the Z-Y-X Euler angles \(\lambda_T = [\phi_T \, \theta_T \, \psi_T]^T\) and the linear speed \(V_T\), which is related to the vehicle’s velocity by\(^4\)

\[V_T = \frac{1}{1 - \kappa d_y} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tau B R v_B\]
Frame \{C\} The need to define \{C\} arises from the fact that while following a path, the vehicle may take different orientations or even rotate with respect to the path. The desired orientation can be represented by the Z-Y-X Euler angles $\lambda_C = [\phi_C \ \theta_C \ \psi_C]^T$, $\theta_C \in [-\pi/2, \pi/2]$, $\phi_C, \psi_C \in \mathbb{R}$ and the angular velocity of \{C\} with respect to \{T\} expressed in \{T\} is represented by $^T\omega_C$. Its origin coincides with the origin of frame \{T\}.

Given the definitions of \{T\} and \{C\}, the error state vector is given by

$$x_e = \begin{bmatrix} v_e \\ \omega_e \\ d_t \\ \lambda_e \end{bmatrix} = \begin{bmatrix} v_B - \frac{\eta}{\tau} R v_r \\ \omega_B - \frac{\eta}{\tau} (\omega_r + ^T\omega_C) \\ \Pi_{yz} R (\lambda_B - \lambda_C) \end{bmatrix} \in \mathbb{R}^{11} \quad \Pi_{yz} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{30}$$

As mentioned in Ref. 4 the vehicle follows the path with the desired velocity profile and orientation if and only if $x_e = 0$ (regulation problem). Furthermore, if we restrict the set of possible paths to the set of trimming paths, the error dynamics can be parameterized by a set of parameters that only depend on the trimming path. A trimming path corresponds to a curve that the vehicle can follow while satisfying the trimming condition, which is equivalent to having $\dot{v}_B = 0$, $\dot{\omega}_B = 0$, and $\dot{u} = 0$ in (29). It is well known that, for a vehicle with dynamics described by (29), the set of trimming trajectories comprises all $z$-aligned helices ($\kappa = 0$, $\tau = 0$, $\lambda_r = [0 \ \theta_r \ \psi_r]^T$, and $\lambda_t = \text{sign}(\kappa)V_r \sqrt{\kappa^2 + \tau^2} [0 \ 0 \ 1]^T$), followed at constant speed ($V_r = 0$) and constant orientation relative to the path ($^T\omega_C = 0$). For helices, the flight path angle $\theta_r$ is given by $\theta_r = \arctan(-\tau/\kappa)$, while in the case of straight lines ($\kappa = 0$, $\tau = 0$), $\theta_r$ is a predefined constant. For these trimming paths the error dynamics can be written as

$$\dot{x}_e = f_e(x_e, \eta, u). \tag{31}$$

where the constant path dependent set of parameters $\eta$ is given by

$$\eta = (V_r, \psi_r, \theta_r, \psi_{ct}, \phi_C, \theta_C)$$

with $\psi_{ct} = \psi_c - \psi_t$ and $\dot{\psi}_r = V_r \sqrt{\kappa^2 + \tau^2}$. If the vehicle under consideration is fully actuated, then the set of operating points can be parameterized by $\eta$. However for the case of model-scale helicopters which typically have four actuators forming the input vector, we need to define a suitable parameterization of lower dimension. As explained in Ref.4, the first four elements of $\eta$, which we now denote by $\xi = (V_r, \dot{\psi}_r, \theta_r, \psi_{ct})$, adequately parameterize the operating points of a helicopter over its flight envelope, automatically constraining the roll and pitch angles $\phi_C, \theta_C$. We are also interested in defining an output to be driven to zero at steady-state by means of integral action. The following output is proposed

$$y_e = \begin{bmatrix} v_e + \frac{\eta}{\tau} R \\ \psi_e \end{bmatrix} \in \mathbb{R}^4. \tag{32}$$
It can be shown that regulating this output to zero $y_e = 0$ implies $x_e = 0.4$.

To summarize, the error system for a helicopter can be described as

$$\mathcal{P}(\xi) := \begin{cases} 
\dot{x}_e &= f_e(x_e, \xi, u) \\
y_e &= g(x_e, \xi)
\end{cases},$$  \hspace{1cm} (33)

Recalling that $\xi$ is a constant parameter vector, the linearization of $\mathcal{P}(\xi)$ about $(x_e = 0, u = u_\xi)$ results in a time-invariant system of the form

$$\mathcal{P}_t(\xi) = \begin{cases} 
\dot{x}_{e\delta} &= A_e(\xi)x_{e\delta} + B_e(\xi)u_{\delta} \\
y_{e\delta} &= C_e(\xi)x_{e\delta}
\end{cases},$$  \hspace{1cm} (34)

where $A_e(\xi) = \frac{\partial f_e}{\partial x_e}(0, u_\xi, \xi)$, $B_e(\xi) = \frac{\partial f_e}{\partial u}(0, u_\xi, \xi)$, and $C_e(\xi) = \frac{\partial g}{\partial x_e}(0, \xi)$.

For simplicity, we restrict our analysis to paths consisting of straight lines ($\psi_r = 0$) followed by the vehicle with constant velocity $(V_r = \text{const})$ and constant orientation, with $\psi_{ct} = 0$. Therefore the only parameter not restricted is the flight path angle $\theta_r$, which allows to consider climbing and descending straight lines that the vehicle follows with constant orientation. As already mentioned, with the choice of $\xi$, $\phi_r$ and $\theta_r$ are automatically constraint. If the inertial frame $\{I\}$ is chosen with its $z$-axis aligned with the gravity vector and $y$-axis orthogonal to the straight line path which the vehicle is required to follow, these angles are approximately zero, $\phi_r \approx 0$, $\theta_r \approx 0$. This means that while following a straight line with a given flight-path angle, the orientation of the vehicle at equilibrium is such that its $z$-axis, $z_n$, is approximately aligned with the gravity vector. This set of paths is considered in Ref.19 in the context of terrain-following. The errors dynamics and its linearization is therefore assumed to only depend on $\alpha_0 = \theta_r$ and is given by

$$\mathcal{P}_t(\alpha_0) = \begin{cases} 
\dot{x}_{\alpha\delta} &= A_e(\alpha_0)x_{\alpha\delta} + B_e(\alpha_0)u_{\alpha\delta} \\
y_{\alpha\delta} &= C_e(\alpha_0)x_{\alpha\delta}
\end{cases},$$  \hspace{1cm} (35)

where we have added the measured output variables available for control, $y_{em\delta}$, and renamed $y_{e\delta}$ to $y_{er\delta}$ to indicate that these are the outputs to be regulated to zero. With respect to Section II, $\mathcal{P}_t(\alpha_0)$ is the parameterized local system model Eq. (4) for which the controller is to be designed.

### C. Multi-rate output nature of the sensors

As already mentioned, for an autonomous vehicle the dynamic and kinematic state-variables comprise linear and angular velocities, position and orientation. Usually the kinematic variables are available at a lower rate (for example the position is typically measured by a GPS receiver which imposes a slow rate).

In this work we assume that the state variables of the vehicle corresponding to linear and angular velocities and orientation are measured at a sampling rate of 50 Hz, which corresponds to a sampling period $t_s = 0.02$ s, whereas the position variables are measured at a sampling rate of 2.5 Hz. The actuators are assumed synchronous and updated at a rate of 50 Hz.

Taking into account the definition of the error space, Eq. (30), and the set of paths considered, this implies that all the component of $x_e$ are available at a rate of 50 Hz, except for $d_t$, which is available at a rate of 2.5 Hz. From Eq. (32) we also have that the first three components of the regulated output $y_e$ are available at a rate of 2.5 Hz, whereas $\psi_e$ is available at a rate of 50 Hz.

With respect to Section II, the regulated output is $y_r = y_{er}$, and since all states are available, the measured output is given by $y_m = y_{em} = x_e$. The $h$-periodic matrices $\Gamma_{mk}$, $\Gamma_{rk}$ and $\Omega_k$ that characterized the multi-rate setup are given by

$$\Omega_k = I_{4\times 4}, \quad \Gamma_{mk} = \begin{cases} 
I_{1\times 1} & k = 1 \\
\text{diag}([1_3, 1_3, 0_2, 1_3]) & \text{otherwise}
\end{cases}, \quad \Gamma_{rk} = \begin{cases} 
I_{4\times 4} & k = 1 \\
\text{diag}([0_3, 1]) & \text{otherwise}
\end{cases},$$

where $h = 20$. 

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D. Controller Synthesis and Implementation

There are two methods commonly used in the design of a family of controllers for the parameterized family of models, described by Eq. (35), to overcome the impracticality of synthesizing an infinite number of controllers for the allowed values of the continuous schedule parameter.

The first method can be introduced by the following steps: i) Obtain a finite set of parameter values from discretization of the continuous parameter space, ii) synthesize a linear controller for each linear plant, Eq. (35), obtained from the linearization of the nonlinear plant for each value of the schedule parameter, iii) interpolate the linear controllers coefficients to obtain a continuously parameter dependent family of controllers.

The second method consist of imposing a structure for the set of parameterized family of systems, by approximating the describing matrices, typically by affine parameter dependent matrices, and use available results in the literature for the design of controllers for this type of systems.\textsuperscript{20}

In the work herein presented we have followed the first method. The finite set of values for the parameter, \{\alpha_0\} was chosen to be

\[
\alpha_0 = [-50 -40 -30 -20 -10 0 10 20 30 40 50] \frac{\pi}{180} \text{ rad.}
\]

which means that the conditions under which the vehicle is expected to operate include straight lines, with a flight path angle between -50 and 50 degrees that the vehicle is required to follow at equilibrium.

For each finite value of the parameter the standard output feedback \(H_2\) formulation for periodic system was used to synthesize a controller \(C_k\) for the augmented system \(G_{\alpha}\), Eq. (16), where \(A_d, B_d, C_d\) are now the discretizations of \(A, B, C, \) Eq. (35). Notice that \(G_{\alpha}\) has a number of states equal to \(n + m + n_y = 11 + 4 + 11 = 26\) and a number of outputs equal to \(n_u + n_y = 11 + 4 = 15\). The solution used for the \(H_2\) problem is the convex LMI formulation presented in Ref. 21 which consists on separately designing the controller and the observer, based on the solution of a LMI problem and then constructing the resulting dynamic compensator since the separation principle is also valid for periodic systems. For a system \(G\)

\[
G := \begin{cases}
  x_{k+1} = A_kx_k + B_{1k}w_k + B_{2k}u_k \\
  z_k = C_{1k}x_k + D_{11k}w_k + D_{12k}u_k \\
  y_k = C_{2k}x_k + D_{21k}w_k + D_{22k}u_k
\end{cases}
\]

the \(H_2\) problem for periodic system has the interpretation of minimizing the norm of the impulse response, defined as \(\|G\|_2 = (\frac{1}{n} \sum_{j=0}^{n-1} \sum_{m=1}^{m} ||G\delta(k-j)e_i||_2^2)^{\frac{1}{2}}\).

The matrices associated with the performance channel \(w_k \mapsto z_k\) can be found from the matrices associated with the performance channel \(w(t) \mapsto z(t)\) specified in Eq. (4), to guarantee \(H_2\) performance for the continuous time linear plant (see Ref. 22 for the LTI case). However, both continuous and discrete performance matrices, are usually used as tuning knobs to improve the performance during extensive simulations. After this simulation phase the matrices found to yield acceptable local performance were

\[
\begin{align*}
C_{1k}(\alpha_0) &= [Q^T \ 0]^T \\
D_{12k}(\alpha_0) &= [0 \ \bar{R}^T]^T \\
B_{1k}(\alpha_0) &= [G \ 0] \\
D_{21k}(\alpha_0) &= [0 \ \sqrt{L}] \\
D_{11k}(\alpha_0) &= 0
\end{align*}
\]

which means that the chosen matrices are independent of \(k\) and \(\alpha_0\).

The resulting finite set of synthesized controller coefficients, for example \(\{A_k^c(\alpha_0)\}\), were interpolated using least squares yielding a family of continuously parameter dependent controllers, Eq. (20), where the describing matrices are quadratically parameter dependent, for example

\[
\begin{align*}
A_k^c(\alpha_0) &= A_k^{c1} + \alpha_0 A_k^{c2} + \alpha_0^2 A_k^{c3} \\
B_{1k}^c(\alpha_0) &= B_{1k}^{c1} + \alpha_0 B_{1k}^{c2} + \alpha_0^2 B_{1k}^{c3}
\end{align*}
\]

The disadvantage of this technique is that by the interpolation process there is no guarantees that, even for fixed parameter values, the controller obtained by interpolation stabilizes the closed loop. This analysis was made a posteriori, verifying that for a dense grid of fixed values of \(\alpha_0\) the closed loop is stabilized.
Having designed the family of controllers $C_k(\alpha_0)$, the non-linear controller $K$, is implemented according to Eq. (22), where $\alpha_k$ becomes a time-varying parameter. An important question is whether to schedule the controller based on the values of the reference $\theta_T$ which is the flight path angle and only depends on the reference path, or on current values of the state variable that at equilibrium have the same values of the reference (see Ref. 12 for a discussion). The answer is in general problem dependent. Since in the work herein presented the reference can change abruptly, we have opted to scheduled on the current values of the state variable. The following scheduling variable is defined

$$\alpha = \arctan(-\frac{w_B}{u_B})$$

that with assumptions $\phi_c \approx 0$, $\theta_c \approx 0$, is equal to the reference value $\theta_T$ at equilibrium. With this choice of function $g(y, w)$ we have completed the description of the controller synthesis and implementation. The final implementation scheme is shown in Figure 6.

![Block diagram with the final implementation scheme](image)

Some important features of this implementation are worthwhile emphasizing. The placement of the integrators at the front of the plant, has the following advantages: i) the implementation of anti-windup schemes is straightforward ii) auto-trimming property- the controller automatically generates adequate trimming values for the actuation signals. In this case the integrators are simple integrators since the actuators are synchronously updated but the same advantages hold for system $\Omega_I$, Eq. (13), when the inputs are updated at different rates.

### E. Simulation Results

To illustrate the effectiveness of the proposed design technique we start with a simple example that compares the performance of a multi-rate guidance and control law with that obtained using a standard single-rate $H_2$ compensator designed using equivalent weighting matrices.

During the maneuver the vehicle is required to follow a horizontal straight line, $\theta_T = 0$, with constant velocity $V_r = 1.5 m/s$.

A single controller of the form Eq. (20), periodically time-varying, was synthesized for $\alpha_0 = 0$. In the single-rate case every output is assumed to be available at the sampling rate of $50Hz$, $t_s = 0.02$, whereas for the multi-rate case the sampling rate of the outputs has already been discussed in Subsection (V.C). The results are shown in Fig.(7). The outputs available for each case are shown in sub-figures 7(e), 7(f) and the errors that have different rates are shown in 7(g), 7(h). Comparing the errors, 7(a) and 7(b), and the actuation, 7(c) and 7(d), for the single and multi-rate cases, we can see that there is a performance degradation, as expected, when the multi-rate case is considered. However the performance of both cases is fairly similar indicating that, although the position sensor is available at a much lower rate, the multi-rate control law is able to still achieve good performance. Using the fact that equivalent weighting matrices were used to derive both compensators the previous observations can easily be confirmed by a simple comparison of the $H_2$ obtained for both designs, 8.231 for the single-rate case and 10.628 for the multi-rate case.
In the second example we allow the controller to be gain-scheduled so that it can perform well in a wide variety of working conditions. Here the helicopter is required to follow the path depicted in Figure (8), which may arise from a terrain following maneuver, with constant linear speed $V_r = 1.5$ m/s. The path is divided in the four following segments: i) a level flight segment along the $x$ axis, ii) a climbing ramp with a flight path angle of $\theta_r = 0.5236$ rad, iii) a level flight segment along the $x$ axis, iv) finally, a descendant ramp with a flight path angle of $\theta_r = -0.2618$ rad. Figure (9) depicts the time evolution of the actuation and errors signals. From the figure we can conclude that the helicopter with the gain-scheduling multi-rate controller efficiently performs the desired task and that after each transition, quickly converges to the reference path, while keeping the actuation within the limits of operation.

VI. Conclusion

A new method was proposed for the design and implementation of gain-scheduled controllers for multi-rate systems with application to the integrated guidance and control problem for autonomous vehicles. Departing from previous approaches, where the multi-rate characteristics of the sensors are handled by the navigation system, this approach provides a new framework that directly takes into account these characteristics in the control system.

A theoretical formulation was offered to tackle the problem of designing and implementing gain-scheduled controllers for non square multi-rate systems. The formulation is valid for a wide class of non-linear plants and its application to other control problems is an interesting topic for future work.

The proposed technique was applied to the solution of a path-following problem for small-scale helicopters. Simulation results show the effectiveness of the method along a simple terrain-following maneuver where the vehicle linear position is measured at a lower rate than that of the remaining state variables.

Directions for future work include the application of the method to the solution of trajectory tracking control problems as well as the precise characterization of the impact of sensor noise on the performance of the overall closed loop system.

Appendix

Proof. (of Theorem (III.3)) System $G_a$, Eq. (16), can be seen as the series connection of two systems. The first system is described by

$$
x_{k+1} = F x_k + G \Omega_k u_k \\
y_k = \begin{bmatrix} y_{mk} \\ y_{rk} \end{bmatrix} = \Gamma_k \begin{bmatrix} H_m \\ H_r \end{bmatrix} x_k
$$

(37)

where

$$
F = \begin{bmatrix} A_d & B_d \\ 0 & I \end{bmatrix} \\
G = \begin{bmatrix} B_d \\ I \end{bmatrix} \\
H_m = \begin{bmatrix} C_{dm} & 0 \end{bmatrix} \\
H_r = \begin{bmatrix} C_{dr} & 0 \end{bmatrix}
$$

(38)

which is stabilizable and detectable under conditions 1, 2 and 3 stated in the theorem, as proved in Ref. 9 where the necessity of these conditions can be found. The second system is described by

$$
x_{k+1}^p = (I - \Gamma_{mk}) x_k^p + \begin{bmatrix} \Gamma_{mk} \\ 0 \end{bmatrix} u_k^p \\
y_k^p = - \begin{bmatrix} \Gamma_{mk} \\ 0 \end{bmatrix} x_k^p + \begin{bmatrix} \Gamma_{mk} \\ 0 \end{bmatrix} u_k^p
$$

(39)

For simplicity, we will restrict the prove to the case where the input and outputs are one dimensional signals $u_k, y_{mk}, y_{rk} \in l(Z^+, R)$. The general case can be proved using the same ideas. Using Theorem (III.2), we will establish stabilizability and detectability for this series connection by proving stabilizability and detectability for its time-invariant lift. We start by noticing that to compute the lift of the series of two periodic systems, we can first compute the lift of each one of them and then the series. Defining a permutation
matrix $P$, as

$$P^{-1} \begin{bmatrix} y_{ml} \\ y_{rl} \end{bmatrix} = \begin{bmatrix} \bar{y}_{ml} \\ \bar{y}_{rl} \end{bmatrix}$$

where $\begin{bmatrix} y_{ml} \\ y_{rl} \end{bmatrix} = \begin{bmatrix} y_{m1} & y_{r1} & y_{m1} & y_{r1} & \cdots & y_{m1} & y_{r1} \end{bmatrix}^T$, $\bar{y}_{ml} = \begin{bmatrix} y_{m1} & y_{m1} + 1 & \cdots & y_{m1} + h \end{bmatrix}^T$, $\bar{y}_{rl} = \begin{bmatrix} y_{r1} & y_{r1} + 1 & \cdots & y_{r1} + h \end{bmatrix}^T$ and defining also the following matrices

$$H_m = \begin{bmatrix} H_m \\ H_m F \\ \vdots \\ H_m F^{N-1} \end{bmatrix}, \quad H_r = \begin{bmatrix} H_r \\ H_r F \\ \vdots \\ H_r F^{N-1} \end{bmatrix}$$

$$D_m = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ H_m G & 0 & \cdots & \cdots & 0 \\ H_m F G & H_m G & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ H_m F^{N-2} G & H_m F^{N-3} G & \cdots & H_m G & 0 \end{bmatrix}, \quad D_r = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ H_r G & 0 & \cdots & \cdots & 0 \\ H_r F G & H_r G & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ H_r F^{N-2} G & H_r F^{N-3} G & \cdots & H_r G & 0 \end{bmatrix}$$

the lift of system described by Eq. (37) is given by

$$\bar{x}_{l+1} = \bar{F} \bar{x}_l + \bar{G} \bar{u}_l$$

$$\bar{y}_l = P \begin{bmatrix} \Gamma_m & 0 \\ 0 & \Gamma_r \end{bmatrix} \begin{bmatrix} H_m \\ H_r \end{bmatrix} \bar{x}_l + \begin{bmatrix} \Gamma_m & 0 \\ 0 & \Gamma_r \end{bmatrix} \begin{bmatrix} D_m \\ D_r \end{bmatrix} \Omega \bar{u}_l$$

(41)

Defining matrices

$$\bar{B}_l = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \Pi \Gamma_m, \quad \bar{C}_l = \Pi \Gamma_m$$

$$\bar{D}_l = \Pi \Gamma_m$$

(42)

where $\Pi \Gamma_m$, $\Pi \Gamma_r$ are projection matrices defined at the end of Subsection (II.B), Eq. (6), the lift of system Eq. (39) can be verified to be given by

$$\bar{x}_{l+1}^D = \begin{bmatrix} \bar{B}_l & 0 \end{bmatrix} \Gamma P^{-1} \bar{u}_l^D$$

$$\bar{y}_{l+1}^D = P \begin{bmatrix} \Pi \Gamma_m \bar{C}_l & 0 \\ 0 & \Pi \Gamma_r \end{bmatrix} \bar{x}_l^D + P \begin{bmatrix} \Gamma_m \bar{D}_l \Gamma_m & 0 \\ 0 & \Gamma_r \end{bmatrix} P^{-1} \bar{u}_l^D$$

(43)

Computing the series of systems described by Eq. (41) and Eq. (43) yields

$$\begin{bmatrix} \bar{x}_{l+1} \\ \bar{x}_{l+1}^D \end{bmatrix} = \begin{bmatrix} \bar{F} & 0 \\ \bar{B}_l \Gamma_m \bar{H}_m & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_l \\ \bar{x}_l^D \end{bmatrix} + \begin{bmatrix} \bar{G} \\ \bar{B}_l \Gamma_m \bar{D}_m \end{bmatrix} \Omega \bar{u}_l$$

$$\begin{bmatrix} \bar{y}_l \\ \bar{y}_l^D \end{bmatrix} = P \begin{bmatrix} \Pi \Gamma_m & 0 \\ 0 & \Pi \Gamma_r \end{bmatrix} \begin{bmatrix} \Pi \Gamma_m \bar{D}_l \Gamma_m \Pi \Gamma_m \bar{C}_l & 0 \\ 0 & \Pi \Gamma_m \Gamma_r \end{bmatrix} \begin{bmatrix} \bar{x}_l \\ \bar{x}_l^D \end{bmatrix} + P \begin{bmatrix} \Pi \Gamma_m \bar{D}_l \Gamma_m & 0 \\ 0 & \Pi \Gamma_r \end{bmatrix} \begin{bmatrix} \bar{D}_m \\ \bar{D}_r \end{bmatrix} \Omega \bar{u}_l$$

(44)

Since system described by Eq. (37) is stabilizable and detectable, we have that the following conditions (PHB test) must hold for its lift, Eq. (41),

For $\lambda_F : \|\lambda_F\| \geq 1$

$$w^T F = \lambda_F w^T \Rightarrow w^T \bar{G} \Omega \neq 0 \Rightarrow w^T \bar{G} \neq 0$$

(45)
\[
\tilde{F}v = \lambda_F v \Rightarrow \begin{bmatrix} \Gamma_m & 0 \\ 0 & \Gamma_r \end{bmatrix} \begin{bmatrix} \tilde{H}_m \\ \tilde{H}_r \end{bmatrix} v \neq 0 \Rightarrow \begin{bmatrix} \tilde{H}_m \\ \tilde{H}_r \end{bmatrix} v \neq 0
\] (46)
and we want to prove that the system described by Eq.(44) is stabilizable and detectable, i.e,

For \( \lambda_F : \|\lambda_F\| \geq 1 \)

\[
\begin{bmatrix} w_T^1 & w_T^2 \\ \tilde{F} & 0 \\ \tilde{B} & \tilde{H}_m \end{bmatrix} = \lambda_F \begin{bmatrix} w_T^1 & w_T^2 \\ \tilde{B} & \tilde{H}_m \end{bmatrix} \Omega \neq 0 \Leftrightarrow \begin{bmatrix} w_T^1 & w_T^2 \\ \tilde{B} & \tilde{H}_m \end{bmatrix} \neq 0
\] (47)

From the left side of Eq. (47) we can see that \(w_T^1 = w^T, \quad w_T^2 = 0\) and therefore the condition of Eq. (47) reduces to the condition of Eq. (45) establishing stabilizability.

To establish detectability we start by determining the eigenvalues and eigenvectors of \(\tilde{F}\). As noted in Ref. 9, condition 3 implies that \(v\) is an eigenvalue corresponding to the eigenvalue \(\lambda_F : \|\lambda_F\| \geq 1\) if it is a eigenvector of \(\tilde{F}\) corresponding to the eigenvalue \(\lambda_F = (\lambda_F)^h\). Moreover from (Eq. (38)), the eigenvalues of matrix \(F\) are the union of the eigenvalues of matrix \(A_d\) and of unitary eigenvalues. The eigenvectors, \(v^T = [v_a^T \quad v_b^T]\) corresponding to \(\lambda_F = \lambda_{A_d} : \|\lambda_{A_d}\| \geq 1\) are given by

\[
\begin{bmatrix} A_d & B_d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix} = \lambda_{A_d} \begin{bmatrix} v_a \\ v_b \end{bmatrix} \implies A_d v_a = \lambda_{A_d} v_a, v_b = 0
\] (49)

and the eigenvectors \(v^T = [v_a^T \quad v_b^T]\) corresponding to \(\lambda_F = 1\) verify

\[
\begin{bmatrix} A_d & B_d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix} = \lambda_{A_d} \begin{bmatrix} v_a \\ v_b \end{bmatrix} \implies A_d v_a = \lambda_{A_d} v_a, v_b = 0
\] (50)

From the left side of Eq. (48) we see that \(v_1 = v\) and \(v_2 = \tilde{B}_1 \Gamma_m \tilde{H}_m v = \tilde{B}_1 \Gamma_m \tilde{H}_m v\). From the definitions of \(\tilde{H}_m\) and \(\tilde{B}_1\), Eq. (40) and Eq. (42), it is easy to see that \(\tilde{B}_1 \Gamma_m \tilde{H}_m v = \lambda_F^j [C_{dm} \quad 0] v\) for some \(j \in \{0, ..., h-1\}\), and thus \(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{v}{\lambda_{\lambda_F}^j} \cdot \begin{bmatrix} 1 & 0 & \ldots & 0 \\ -1 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 \end{bmatrix} \Gamma_m \tilde{H}_m \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \) (51)

One can also verify that

\[
\begin{bmatrix} 1 & 0 & \ldots & 0 \\ -1 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 \end{bmatrix} \Gamma_m \tilde{H}_m = [C_{dm} \quad 0] \lambda_F^j v \quad \text{for some} \quad j \in \{0, ..., h-1\}
\]

and

\[
\begin{bmatrix} 1 & 0 & \ldots & 0 \\ -1 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 \end{bmatrix} \Gamma_r \tilde{H}_r = [C_{dr} \quad 0] \lambda_F^k v \quad \text{for some} \quad k \in \{0, ..., h-1\}
\]

from which two of the lines of Eq. (51) are given by

\[
\begin{bmatrix} (C_{dm} \lambda_F^j - C_{dm} \lambda_{\lambda_F}^j) & 0 \\ C_{dr} \lambda_F^k & 0 \end{bmatrix} v \Leftrightarrow \begin{bmatrix} (C_{dm} \lambda_F^j - C_{dm} \lambda_{\lambda_F}^j) & 0 \\ C_{dr} \lambda_F^k & 0 \end{bmatrix} v_a \Leftrightarrow \begin{bmatrix} (\lambda_F^j - \lambda_{\lambda_F}^j) & 0 \\ 0 & \lambda_F^k \end{bmatrix} C_{dr} v_a
\]

where \(v_a\) is an eigenvector of \(A_d\) corresponding to an unstable mode \(\lambda_F = (\lambda_F)^h = (\lambda_{A_d})^h : \|\lambda_{A_d}\| > 1\) (Eq.(49)). From detectability of (\(A_d, C_d\)) we have \(C_d v_a \neq 0\) which allows to establish detectability for these unstable modes.

For \(\|\lambda_F\| = 1\) it can be checked that the first rows of Eq. (48), Eq. (51) are zero \(\Gamma_m \tilde{B}_1 \Gamma_m \tilde{H}_m v_1 + \Gamma_m \tilde{C}_j v_2 = 0\). However assumption 17 implies that the last rows are different from zero \(\Gamma_r \tilde{H}_r v_1 \neq 0\), since
otherwise the condition $\Pi_r, \Pi_r v_1 = 0$ would imply $[C_{dr} \ 0] v = 0 \Leftrightarrow [C_{dr} \ 0] \begin{bmatrix} \bar{v}_c \\ v_d \end{bmatrix} = 0$ and along with Eq.(50) we would have
\[
\begin{bmatrix} A_d - I B_d \\ C_{dr} \ 0 \end{bmatrix} \begin{bmatrix} \bar{v}_c \\ v_d \end{bmatrix} = 0
\]
which contradicts assumption 17 of the theorem.

\[\square\]

**Proof.** (of Theorem (III.3)) Suppose the equation for the controller $C_K$ that asymptotically stabilizes $G_a$, Eq. (16), are
\[
C_K = \left\{ \begin{bmatrix} x_k^{a+1} \\ u_k^a \end{bmatrix} = \begin{bmatrix} A_k^c & B_k^c \\ C_k^c & D_k^c \end{bmatrix} \begin{bmatrix} x_k^c \\ y_k^c \end{bmatrix} \right. 
\]

It is straightforward to check that the equilibrium values (denoted with an underbar) for the feedback connection of $C_K$ and $G_a$ are

\[
x_k^c = 0, \quad u_k^a = 0, \quad y_k^a = 0,
\]

\[
x_k = 0, \quad x_{k+1}^c = u_0, \quad x_{k+1}^c = 0,
\]

\[
y_{mk} = y_{m0}, \quad y_{rk} = r_0
\]

Due to linearity and asymptotic stability the system trajectories will tend to this unique equilibrium point even in the presence of disturbances that do not affect asymptotic stability.

\[\square\]

**Acknowledgments**

**References**


Figure 7. Regulation to forward trajectory
Figure 8. Helicopter in the terrain-following task

Figure 9. Terrain-following task