Event-driven control with deadline optimization for linear systems with stochastic delays

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Abstract—This work presents a novel control strategy for systems with actuation delays with known stochastic distribution, which improves upon previously proposed deadline-driven and event-driven strategies. In the event-driven strategy the control input is immediately updated after the delay, whereas in the deadline-driven strategy the actuation is updated in a periodic fashion where the sampling period sets a deadline; if the delay is larger than this deadline the actuation is not updated. Our method switches between these two strategies and guarantees a better performance, in an LQG sense, than either method considered separately. An extension of the novel method with a deadline-optimization scheme is shown to improve the performance even further. Simulation results illustrate the effectiveness of the proposed methods.

Index Terms—Stochastic optimal control, Stochastic time-delay, Data loss, Sampled-data control, Event-driven control, Self-triggered control, Dynamic programming

I. INTRODUCTION

DELAYS are present in many control applications, resulting from timing effects in the loop such as control computation, communication between distributed components, or measurement acquisitions [1]. These control delays can lead to significant performance degradation in various control settings, especially in industry that requires embedded hardware [2], [3], (shared) communication networks [3]–[5], and/or data-intensive processing [6], [7].

Many works in the literature addressing control problems with uncertain delays use a worst-case approach, taking into account the largest possible delays, and exploit robust stability analysis techniques (see, e.g., [8]–[10]). In particular, in a traditional control setting, a sufficiently large sampling time is chosen such that the worst-case delays, which may be very large, are accommodated. Naturally, this approach is conservative and can lead to poor closed-loop performance. In the present work, we take an alternative approach, exploiting knowledge of the probability distribution of the delays when selecting the sampling intervals. However, selecting sampling intervals shorter than the delay causes a data dropping effect for which some solutions have been proposed (see, e.g., [11]–[18]). The sampling interval thus plays the role of a deadline and, as such, we denote this approach deadline-driven. This method leads to a trade-off between ‘data-loss’ and control rate. In [13], this trade-off was studied in the context of reliability analysis of a networked control system with energy constraints, while in this work we consider linear quadratic Gaussian (LQG)-type performance. Alternatively, some works address the stochastic nature explicitly [19]–[21], proposing so-called event-driven strategies. Such event-driven strategies (differing from state-dependent event-triggered control, see e.g., [22], [23]) consider that the control input is immediately updated after the delay, and therefore the update intervals are equal to the stochastic delay [24]. Results for systems with stochastic parameters [25] can be used to find optimal control strategies for this case.

The current work extends the results in our preliminary work [24]. The work [24] concluded that event-driven or deadline-driven approaches do not necessarily perform better than one another (in an LQG performance sense). Here, we propose a novel switching strategy that is guaranteed to result in a better performance than that of event-driven and deadline-driven approaches by switching between them. The switching strategy combines strategies that are event-driven, deadline-driven and/or event-driven with a deadline, where the control input is updated after the delay, except when the delay exceeds a deadline, in which case a data drop occurs. Additionally, we show that the novel switching strategy can be combined with a deadline optimization scheme to obtain additional performance benefits.

The new results are obtained in the setting of linear continuous-time systems with Gaussian disturbances.

The stochastic delays in the control-to-actuation link are assumed to be independent and identically distributed (i.i.d.), as is very common in the networked control systems community and justified in several contexts involving computation (see, e.g., [26]) or communication delays (see, e.g., [27], [28]). Digital control with delayed zero-order hold inputs is used, as illustrated in Figure 1. The closed-loop performance is evaluated by an infinite horizon average cost function as in a standard LQG framework. The performance gain of the proposed policies is illustrated by numerical examples. The proofs of the main results resort to Doob’s optional sampling theorem [29].

This work is supported by the Netherlands Organisation for Scientific Research (NWO-TTW) under grant number 12697 “Control based on Data-Intensive Sensing,” and by the Innovational Research Incentives Scheme under the Vici grant “Wireless control systems: A new frontier in automation” (No. 11382) awarded by NWO-TTW.

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Fig. 1. Control loop with actuation delay.
The remainder of the paper is organized as follows. The control setup with actuation delay and the problem formulation are detailed in Section II, where the deadline-driven and event-driven strategies are also discussed as well as the event-driven with deadline strategy. Using a discrete-time model of the system, the performance analysis of the non-switching approaches leads to a preliminary result in Section III. Section IV provides the main results for the proposed control policies. Numerical examples in Section V illustrate the novel results and the benefits of the new method. Concluding remarks are given in Section VI. The proofs of the main results can be found in the appendix.

II. PROBLEM FORMULATION AND BACKGROUND

In this section, we discuss first the control setting and the formal control problem. Subsequently, we discuss several basic control strategies that serve as a benchmark for the methods proposed in this paper.

A. Problem setting

We consider a continuous-time plant modeled by the stochastic differential equation

\[ dx_c = (A_c x_c + B_c u_c)dt + B_w dw(t), \quad x_c(0) = x_0, \quad t \in \mathbb{R}_{\geq 0}, \]

where \( x_c(t) \in \mathbb{R}^{n_c} \) is the state and \( u_c(t) \in \mathbb{R}^{n_u} \) is the applied control input at time \( t \in \mathbb{R}_{\geq 0} \), and \( w \) is an \( n_w \)-dimensional Wiener process with incremental covariance \( I_{n_w} dt \). We assume that \((A_c, B_c)\) is controllable and \( B_c\) has full rank.

As in the standard linear quadratic Gaussian (LQG) framework, the average quadratic cost

\[ J := \limsup_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}[g_c(x_c(t), u_c(t))]dt \]

is chosen as the performance criterion where \( g_c(x, u) := x^T Q_c x + u^T R_c u \) with positive definite matrix \( R_c > 0 \) and positive semi-definite matrix \( Q_c \geq 0 \) for which \((A_c, Q_c^{1/2})\) is observable.

We consider a setup with a simple hold device at the plant input such that the plant actuation signal is held constant between discrete update instances \( t_k, k \in \mathbb{N} \), with \( t_{k+1} > t_k \), for all \( k \in \mathbb{N} \), and \( t_0 = 0 \). In particular, we write

\[ u_c(t) = \hat{u}_k, \quad \text{for all } t \in [t_k, t_{k+1}), \]

where \( \hat{u}_k \in \mathbb{R}^{n_u} \) is the digital input value held at time \( t_k \), \( k \in \mathbb{N} \). We assume that \( \hat{u}_0 := u_c(0) \) is known.

We assume, for now, that the plant may be sampled at any time instance and discuss how to relax this assumption in Remark 2 below. In particular, we choose the sampling instances to coincide with the actuation update instances, i.e., the plant is sampled at times \( t_k, k \in \mathbb{N} \), with \( t_{k+1} > t_k \) for all \( k \in \mathbb{N} \). The time-varying 'sampling' intervals can then be defined as

\[ h_k := t_{k+1} - t_k, \quad k \in \mathbb{N}. \]

At every sampling instance \( t_k \), we assume that the sensor provides a measurement of the full state \( x_c(t_k) \) and denote this by \( x_k := x_c(t_k), k \in \mathbb{N} \).

The sampling intervals \( h_k, k \in \mathbb{N} \), will take different values, detailed later, depending on the chosen control strategy. We assume that there exists a (possibly small) \( h_{\text{min}} \in \mathbb{R}_{> 0} \) such that \( h_k \geq h_{\text{min}} \) for all \( k \in \mathbb{N} \), which imposes a minimal interval.

The computation of a new control action \( u_k \in \mathbb{R}^{n_u} \) by the controller starts immediately after a new sample is obtained, i.e., at time \( t_k \) for all \( k \in \mathbb{N} \) as a function of all the information in the control platform at time \( t_k \). Due to computational delays or communication delays, the new control action can only be applied after a delay \( \tau_k \in \mathbb{R}_{> 0} \) for all \( k \in \mathbb{N} \). The delays \( \tau_k, k \in \mathbb{N} \), are independent and identically distributed (i.i.d.) with known delay distribution defined by the probability measure \( \mu \).

The support of \( \mu \) is allowed to be unbounded, but we assume that \( \mu((0, \infty]) = 1 \) and \( \mu(0) = 0 \). The measure \( \mu \) can be decomposed into continuous and discrete components as \( \mu = \mu_c + \mu_d \) with \( \mu_c((0, s)) = \int_0^s f_c(\tau) d\tau \), where \( f_c \) is a measurable function, and \( \mu_d \) is a discrete measure that captures possible point masses at \( b_i \in \mathbb{R}_{> 0} \cup \{\infty\}, i \in \mathcal{I} \subset \mathbb{N} \), such that \( \mu(\{b_i\}) = w_i, i \in \mathcal{I} \). The Lebesgue-Stieltjes integral of some function \( W \) with respect to the measure \( \mu \) is defined as

\[ \int_0^t W(s) \mu(ds) := \int_0^t W(s) f_c(s) ds + \sum_{i \in \mathcal{I}, b_i \in (0, t]} w_i W(b_i). \]

The cumulative distribution function (cdf) \( F: \mathbb{R}_{> 0} \cup \{\infty\} \to [0,1] \) is given by \( F(\tau) := \mu((0, \tau]) = \int_0^\tau f_c(s) ds + \sum_{i \in \mathcal{I}, b_i \in (0, \tau]} w_i \), for \( \tau \in (0, \infty] \), which is equal to \( \mathbb{P}(\tau_k \leq \tau), k \in \mathbb{N} \), where \( \mathbb{P} \) denotes probability. The probability distribution function (pdf) associated with \( F \) is denoted \( f : \mathbb{R}_{> 0} \cup \{\infty\} \to \mathbb{R}_{\geq 0} \). The expected value of the delay is equal for all \( k \in \mathbb{N} \) and is denoted by \( \bar{\tau} := \mathbb{E}[\tau_k] \).

If the sampling interval \( h_k \) has a maximum value \( D_k \in \mathbb{R}_{> 0}, k \in \mathbb{N} \), imposed, then this works as a deadline. If the delay exceeds the deadline, i.e., if \( \tau_k > D_k \) for some \( k \in \mathbb{N} \), then the newly computed control action \( u_k \) is dropped and the previous actual signal \( \hat{u}_k \) is held constant. We assume that there exists a (possibly large) \( D_{\text{max}} \in \mathbb{R}_{> 0} \) such that \( D_k \leq D_{\text{max}} \) for all \( k \in \mathbb{N} \), which imposes a maximum deadline value. The new plant input \( \hat{u}_{k+1} \), after the interval \( h_k \), becomes, for all \( k \in \mathbb{N}_{\geq 0} \),

\[ \hat{u}_{k+1} = \begin{cases} u_k & \text{if } \tau_k \leq D_k, \\ \hat{u}_k & \text{otherwise.} \end{cases} \]

We assume that only one message, i.e., a control action, is allowed in the actuation channel within each sampling interval, and that a deadline is known at the actuator (if a deadline is applied).

We use a Bernoulli random variable \( \gamma_k, k \in \mathbb{N} \), to capture the occurrence of data drops. In particular, \( \gamma_k = 1 \) denotes that the control input \( u_k \) has been successfully applied to the system while \( \gamma_k = 0 \) denotes that \( u_k \) has been dropped. This is described by the dropping mechanism

\[ \gamma_k = \begin{cases} 1 & \text{if } \tau_k \leq D_k, \\ 0 & \text{if } \tau_k > D_k. \end{cases} \]
As a consequence, equation (3) can be rewritten as
\[
\hat{u}_k = \gamma_{k-1} u_{k-1} + (1 - \gamma_{k-1}) \hat{u}_{k-1}, \quad k \in \mathbb{N}_{>0}.
\] (4)

**Remark 1:** A typical actuation channel cannot be instantaneous, therefore a minimal delay is always present. Hence, it is easy to determine some \( h_{\text{min}} \in \mathbb{R}_{>0} \) such that \( F(h_{\text{min}}) = 0 \). Otherwise, one can consider a new probability measure \( \hat{\mu} \) with the probability \( F(h_{\text{min}}) \neq 0 \) accumulated at \( \mu(\{h_{\text{min}}\}) \) and artificially delay the system if \( \tau_k < h_{\text{min}} \) for some \( k \in \mathbb{N} \).

**Remark 2:** The problem setting can also capture the sampled-data scenario where the sensors can only be sampled at discrete intervals but at a fast rate, in the sense that the sampling period is much smaller than typical delay values. Delaying the actuation updates to the next sampling instance causes the delays to take values in a countable set, which can be captured by a piecewise constant cdf. Since only one message is allowed in the actuation channel, extra samples taken within the actuation update interval are discarded.

**Remark 3:** The setup requires that either the sensor has knowledge of the actuation update instances, which informs the controller by sending a new measurement, or that the controller has knowledge of the channel, such that it can trigger the sensor to provide a new measurement. This can be satisfied by, e.g., a collocated sensor-actuator at the plant or a channel-sensing mechanism at the controller.

### B. Control problem

The control problem is the design of a methodology to obtain suitable control actions \( u_k, k \in \mathbb{N} \), and delay deadlines \( D_k, k \in \mathbb{N} \), such that the performance index (2) is smaller than for known existing methods.

To minimize performance index (2), i.e., to solve the control problem optimally, by, e.g., the use of dynamic programming [31] is not possible due to the curse of dimensionality. As such, we opt to design a suboptimal methodology that is better than current practice. In particular, our goal is to obtain a control policy \( \pi \) that provides \( u_k \) and \( D_k \), i.e.,
\[
(u_k, D_k) = \pi(I_k), \quad k \in \mathbb{N},
\]
as a function of the information available for control at time \( t_k \), being
\[
I_k := \{x_k\} \cup \{x_l, u_l, D_l, h_l, \gamma_l \mid l \in \mathbb{N}_{(0,k)}\} \cup \{\hat{u}_0\}.
\]

In this paper, we provide control policies that are guaranteed to improve over both optimal event-driven and deadline-driven strategies as proposed in [24], in the sense that the performance index (2) is smaller or equal. Simulation results will evidence that significant improvement can be realized.

### C. Basic control strategies (background)

In this work, we consider the following basic strategies or **base policies**, which we will indicate by \( d, e, ed \), respectively.

1) **Periodic deadline-driven control (d):** This typical design approach sets a fixed deadline \( D_d^b \) for each interval and the control update interval coincides with this deadline, i.e., \( D_k = D_d^b \) and \( h_k = D_d^b \) for all \( k \in \mathbb{N} \). This results in dropping \( u_k \) with probability \( 1 - F(D_d^b) \), i.e., \( P(\gamma_k = 0) = 1 - F(D_d^b) \). Note that imposing a deadline is a natural way to deal with large delays. In practice, however, the deadline is imposed without further analysis of the dropping effect while this may significantly impact the stability and/or performance, as we will see.

2) **Event-driven control (e):** This aperiodic strategy updates \( \hat{u}_k \) directly after the delay without considering a deadline, i.e., \( h_k = \tau_k \) and \( D_k = \infty \) for all \( k \in \mathbb{N} \). Note that \( u_k \) is never dropped, i.e., \( P(\gamma_k = 0) = 0 \). When considering this case, we make the additional assumption \( \mu(\{\infty\}) = 0 \), which is necessary for stabilizability, and is equivalent to all \( b_i < \infty \) for all \( i \in \mathcal{I} \).

3) **Periodic event-driven control with deadline (ed):** This aperiodic strategy updates \( \hat{u}_k \) directly after the delay if the delay is less than the set deadline \( D_{ed}^b \), and at the deadline when the delay is larger, i.e., \( h_k = \min(\tau_k, D_k) \) and \( D_k = D_d^b \) for all \( k \in \mathbb{N} \). This results in dropping \( u_k \) with probability \( 1 - F(D_{ed}^b) \), i.e., \( P(\gamma_k = 0) = 1 - F(D_{ed}^b) \), but updating \( \hat{u}_{k-1} \) earlier at time \( t_k + \tau_k \) with probability \( f(\tau_k) \) for each value of \( \tau_k \leq D_k \). Note that assuming \( \mu(\{\infty\}) = 0 \) is not necessary for stabilizability in this case.

The methods \( d \) and \( e \) were previously discussed in our preliminary work [24], where it was suggested to use event-driven approaches to improve performance over conservative but easy-to-implement deadline-driven approaches that are typically adopted in practice.

Through an example, it was found that event-driven approaches can indeed give significant improvement, but this is not necessarily always the case, as was illustrated by another example that showed better performance for periodic control. This motivated the design and investigation of the \( ed \) strategy, proposed here.

Building upon the results in [25], for each of the above methods, an analytical expression for the value of performance index (2) can be obtained. A method to calculate this cost value is explained in Section III-B. The cost is given by
\[
J_b := \frac{1}{b} c_b, \quad c_b := \text{tr}(P_b W_b) + \alpha_b, \quad b \in \{d, e, ed\}, \tag{5}
\]
where \( \bar{h}_b \) denotes the average interval \( h_k \), i.e., \( \bar{h}_b := \mathbb{E}[h_k] \), and where, as in standard LQG, \( P_t \) corresponds to the solution of a Riccati equation, as described in Appendix A. \( W_b \) is a noise term from the Wiener process \( w \), also given in Appendix A, and \( \alpha_b \) is a term resulting from the intersampling behaviour of the system, given in Appendix B. \(^1\)

The values of \( J_b, b \in \{d, e, ed\} \) are called the **base costs** and will serve as a reference to compare our newly proposed methods. The main results in this work derive control policies \( \pi \) that guarantee that \( J_\pi \leq J_b \) whilst typically performing

\(^1\)The additional factor \( \alpha_b \) in (5) was not considered in our preliminary work [24], where the cost due to intersampling behaviour was neglected. Typically, the value of \( \alpha_b \) is small compared to \( \text{tr}(P_b W_b) \), as was the case in [24], and a good approximation of the actual cost can be obtained by neglecting \( \alpha_b \).
(significantly) better in the sense that \( J_\pi < J_b \) for all \( b \in \{d, e, ed\} \). For \( d \) and \( ed \), the cost (5) depends on the chosen deadline \( D \), and the corresponding costs can be denoted by \( J_d(D) \) and \( J_{ed}(D) \), respectively. Optimal deadline values that minimize the cost (5), are denoted \( D_d^* \) and \( D_{ed}^* \), respectively. The costs (5) corresponding to the basic strategies (with optimal deadline) are denoted by \( J_* : = J_d(D_0^*), J_e : = J_e(D_0^*), J_{ed} := J_{ed}(D_0^*) \) respectively. These costs correspond to parameters with the same subscripts \( h_d^*, P_d^*, W_d^*, \alpha_d \), for periodic deadline-driven, and analogously \( e \) and \( ed^* \) for event-driven and periodic event-driven with deadline, respectively. Note that \( h_d^* = D_d^* \), \( h_e = \tau \), and \( h_{ed}^* = \mathbb{E}[\min(\tau, D_0^*)] \).

Remark 4: Although of interest, it is beyond the scope of the present paper to establish a guarantee of strict performance improvement of the proposed strategies. However, we do prove \( J_\pi \leq J_b \) and show the strict improvements via various numerical examples. In addition, note that we believe that strict performance improvement guarantees could be derived by following a similar approach to the one in [32] where a switched system derived in a different context was studied. Such an approach entails rather long arguments, requiring concepts such as ergodicity, and it is therefore not pursued here.

### III. Preliminary results

In this section, for reasons of completeness and self-containment, we discuss shortly the analysis needed to obtain the results in our preliminary work [24], which will be used as a benchmark, and how this leads to an initial result for the ed policy in Lemma 1 below. In order to analyze the proposed strategies, it is convenient to obtain a discrete-time description of the system, which we provide next.

#### A. Discretization

By discretization of system (1) at times \( t_k, k \in \mathbb{N} \), we obtain

\[
x_{k+1} = A(h_k)x_k + B(h_k)\hat{u}_k + w_k, \tag{6}
\]

where \( A(h) := e^{A_c h} \) and \( B(h) := \int_0^h e^{A_c s}B_e ds \). The disturbance is a sequence of zero-mean independent random vectors \( w_k \in \mathcal{R}^{n_w}, k \in \mathcal{N} \), with covariance \( \mathbb{E}[w_k(w_k)^T] = W(h) \).

We augment the state with the current input and define \( \xi_k := \begin{bmatrix} x_k^T & \hat{u}_k^T \end{bmatrix}^T \). The state evolution of the augmented system can then be written as

\[
\xi_{k+1} = A_{\gamma_k}(h_k)\xi_k + B_{\gamma_k}u_k + \tilde{w}_k, \tag{7}
\]

where \( A_{\gamma}(h) := \begin{bmatrix} A(h) & B(h) \\ 0 & (1 - \gamma)I_{n_u} \end{bmatrix}, B_{\gamma} := \begin{bmatrix} 0 \\ I_{n_u} \end{bmatrix} \), and

\[
\tilde{w}_k := \begin{bmatrix} w_k^T \\ 0 \end{bmatrix}^T \text{ with covariance } \tilde{W}(h) := \begin{bmatrix} W(h) & 0 \\ 0 & 0 \end{bmatrix}.
\]

The average cost can be written as

\[
J = \lim_{T \to \infty} \sup_{\tau} \frac{1}{T} \mathbb{E} \left[ \sum_{k=0}^{N(T)-1} g(\xi_k, h_k) \right], \tag{8}
\]

where \( N(T) := \min\{L \in \mathbb{N}_{[1, \infty]} \mid \sum_{k=0}^L h_k > T \} \) and \( g(\xi, h) := \xi^T Q(h) \xi + \alpha(h) \), with

\[
Q(h) := \int_0^h e^{A_c s}B_w^sB_e^s e^{A_c^T s} ds, \tag{9}
\]

and

\[
\alpha(h) := \text{tr}(Q_c \int_0^h e^{A_c s}B_wB_e^T e^{A_c^T s} ds dt),
\]

which is the cost associated with the intersampling behaviour of (1).

The model (7) can be used to describe the behaviour for all proposed strategies. Later, we sometimes use the notation \( \gamma[D, h] \) to indicate that the probability distributions of those variables depend on the deadline \( D \).

#### B. Performance of the basic control strategies

Each cost \( J_b \) in (5) is associated with an optimal control policy (note that \( \tau_k \) is not known at time \( t_k \))

\[
u_k = -K_b \xi_k, \quad b \in \{d^*, e, ed^*\}, \tag{10}
\]

where the expressions for the gains are given in Appendix A. We assume that mild conditions for mean-square stabilizability hold, see [33, Proposition 3.42] and [25, Theorem 6.1], such that solutions (10) are well-defined. Note that for the event-driven case \( e, \gamma_k = 1 \) for all \( k \in \mathbb{N} \). While for the \( d \) and \( e \) cases the use of the results in [25] is straightforward, for the \( ed \) case it is possible to observe that a new probability distribution for \( h_k \) can be defined as a function of the probability distribution of \( \tau_k \), determined by the cumulative distribution function

\[
F_h(\tau, D) := \begin{cases} F(\tau) & \text{if } \tau < D, \\ 1 & \text{if } \tau \geq D,
\end{cases}
\]

and the results in [25] also apply.

For compactness, we introduce the following notation. For a Bernoulli variable \( \gamma \) and a random variable \( h, \) random matrices \( X \) and \( Y \) as in Section III-A that depend on \( \gamma \) and \( h, \) and some matrix \( P, \) the expected value \( \mathbb{E}_{\gamma, h}[X_\gamma(h)Y_\gamma(h)] \) is denoted \( X_\gamma(h)^T Y_\gamma(h) \), and analogously \( \mathbb{E}_{\gamma, h}[X_\gamma(h)] = X_\gamma(h) \).

Intuitively, the \( ed^* \) strategy seems to be better than both the \( d^* \) and \( e \) strategies. From the derivation in this section, we obtain directly the following result.

**Lemma 1** (ed*-better than e): The cost (2) of the event-driven policy with optimal deadline is not larger than that of the event-driven policy, i.e.,

\[
J_{ed^*} \leq J_e.
\]
The proof follows directly from the fact that \( J_{ed}(D) \rightarrow J_e \) for \( D \rightarrow \infty \) and the policy \( e \) is contained in the class of policies \( ed \) parameterized by \( D \).

There may exist (pathological) cases for which updating the control before the deadline has a negative effect on performance. Thus, a guarantee analogous to Lemma 1 for \( d \) does not directly exist. However, the main results of this paper, for a new strategy, do give such a guarantee.

### IV. CONTROL POLICY AND MAIN RESULTS

In this section, the main results are presented. First, we propose the novel switching strategy that leads to a performance guarantee, which is formalized in the main theorem. Subsequently, we present a switching strategy that extends the main result with a deadline optimization scheme.

#### A. Two-policy control (\&)

We propose to use a switched approach to the problem formulated in Section II-B. In particular, we allow the system to choose online which type of strategy, i.e., \( d/e/ed \), to use for the next update instance. Actually, the result is derived for the combination of only \( d^* \) and \( ed^* \) because Lemma 1 shows that the performance of \( ed^* \) is not larger than that of \( e \). The proposed control policy for this case is denoted \( d^* \& ed^* \). The result in Theorem 1 shows that this policy leads to a better performance than using either \( d^* \) or \( ed^* \) all the time.

The idea behind our policy is to choose, at each sampling instance, the control strategy to use during the next interval, denoted by \( \sigma_k \in \{d^*, ed^*\} \), while assuming that either of the base policies, denoted by \( b_k \in \{d^*, ed^*\} \), can be used all the time afterwards, such that the expected future cost is the smallest. After the next interval, the impact of disturbances is neglected in the switching condition to ensure that the cost of the lookahead predictions can be computed. Now, at each \( t_k \), \( k \in \mathbb{N} \), four switching options are available and we establish switching conditions to determine the best option.

Let \( p^* := \arg\min_{p \in \{d^*, ed^*\}} J_p \) select the best periodic base policy with an optimal deadline from the possible base policies, whose cost were defined as \( J_b \) in (5). Now, we define two functions that are to be used in the switching conditions. First, we define a value function \( V_p(\xi) := \xi^\top P_p \xi \), where \( p \in \{d^*, ed^*\} \) and \( P_p \) is defined as in (5), i.e., as solutions to the Riccati equations given in Appendix A. Second, we define a difference function

\[
V^\Delta(\xi_k, m, b) := E[V_{p^*}(\xi_{k+1}) - V_b(\xi_{k+1}) \mid \xi_k, \begin{bmatrix} \sigma_k \\ b_k \end{bmatrix} = \begin{bmatrix} m \\ b \end{bmatrix}],
\]

where \( \xi_{k+1} \) follows (by a prediction step) from (7). In particular, the value of \( \xi_{k+1} \) follows from (7) that \( \sigma_k = m \) and \( b_k = b \), meaning that in (7) \( u_k \) is the optimal input for the next interval, which is defined below in (12), and that \( \gamma_k \) and \( h_k \) are random variables that depend on the deadline \( D_{zk} \) corresponding to the choice \( \sigma_k = m \), i.e., \( \gamma_k|D_{\sigma_k} \) and \( h_k|D_{\sigma_k} \) (see also (13) below). Furthermore, we define, a set-valued function

\[
S(\xi) := \{m, b \in \{d^*, ed^*\} \mid V^\Delta(\xi, m, b) \leq 0\},
\]

mapping \( \xi \) into the choices \( m \) and \( b \) that guarantee that \( V^\Delta \) is non-positive. Note that, by definition, for any \( \xi, m \), \( V^\Delta(\xi, m, p^*) = 0 \) and therefore the set \( S(\xi) \) is non-empty.

The proposed control policy \( d^* \& ed^* \) is the following function of the state \( \xi_k \) to be evaluated for all \( t_k, k \in \mathbb{N} \).

\[
\begin{aligned}
\sigma_k &= \arg\min_{(m,b) \in S(\xi_k)} \xi_k^\top Z^*_{m,b} \xi_k + \beta^*_{m,b}, \\
u_k &= -K^*_{\sigma_k,b_k} b_k,
\end{aligned}
\]

(11)

where the arguments are given by

\[
\begin{aligned}
Z^*_{m,b} &= A^*_m(h_k) P_b A^*_m(h_k) + Q(h_k) - K^*_s B^*_m P_b A^*_m(h_k), \\
K^*_m,b &= B^*_m P_b A^*_m(h_k), \\
\beta^*_{m,b} &= tr(P_b W_m) + \alpha_m - \frac{\eta_m}{h_{p^*}} c_{p^*},
\end{aligned}
\]

with \( c_{p^*} \) given in (5), and \( \alpha_m = E[\alpha(h_k) \mid \sigma_k = m] \) as defined in Appendix B, and where the distribution for \( h_k \) and \( \gamma_k \) depends on the value of the deadline \( D^* \) in

\[
h_k = \begin{cases} 
D_{d^*} & \text{if } \sigma_k = d^* \\
\min\{\tau_k, D_{ed^*}\} & \text{if } \sigma_k = ed^* 
\end{cases}
\]

(13)

corresponding to the value of \( \sigma_k = m \). The symbol \( \dagger \) denotes the pseudo-inverse. The expectations can be numerically computed (using footnote 2). In the first two terms, the scalars \( \beta^*_{m,b} \) contain the cost due to noise over interval \( h_k \), and, in the third term, they contain a correction for the time difference between (the expectation of) the interval \( h_k \) and that of the base policy. The policy selects \( (\sigma_k, b_k) \) as the values that minimize the right-hand side of (11) subject to the condition that \( V^\Delta(\xi_k, \sigma_k, b_k) \leq 0 \). The condition \( V^\Delta(\xi_k, \sigma_k, b_k) \leq 0 \) guarantees that switching to a different base policy whilst neglecting the disturbances after the next interval does not cause a performance loss.

Let the value of performance index (2) obtained for the policy (11)-(13) be denoted \( J_{d^* \& ed^*} \). The following result is the main result of the paper.

**Theorem 1:** The cost (2) of the two-policy approach given by (11)-(13) is not larger than that of both base policies \( d \) and \( e \) in the sense that

\[
J_{d^* \& ed^*} \leq J_{d^*}, \quad \text{and} \quad J_{d^* \& ed^*} \leq J_{d^*}.
\]

The proof is given in Appendix C and resorts to Doob's optional sampling theorem [29].

The following remark explains a relaxation of the switching condition, which will be used in the numerical example in Section V.

**Remark 5:** From the proof of Theorem 1, one can see that Theorem 1 also holds if the condition \( V^\Delta(\xi_k, m, b) \leq 0 \) on \( S(\xi) \) is relaxed to \( V^\Delta(\xi_k, m, b) \leq \Delta(\xi_k, p^*, p^*, m, b) \), where \( \Delta \) is defined in (18).

**Remark 6:** Note that \( V^\Delta(\xi_k, m, b) \leq 0 \) is directly satisfied for all \( \xi_k \) and \( m \) if \( P_{\mu_p} \leq P_b \) for all \( b \).

**Remark 7:** The result of Theorem 1 would directly extend to a policy \( d^* \& ed \) if \( D_{ed^*} = \infty \) would be selected. The derivation
of the policy is omitted for brevity. The result for this case is summarized in the following corollary.

**Corollary 1:** The cost (2) of the two-policy approach given by (11)-(13) when \( D_{eq} \to \infty \), is not larger than that of both base policies \( e \) and \( d \) in the sense that

\[
J_{d^*kec} \leq J_{e}, \quad \text{and} \quad J_{d^*kec} \leq J_{d^*}. \quad \Box
\]

The proof follows the same arguments as the ones used to prove Theorem 1.

### B. Online deadline optimization (s)

In our preliminary work [24], we proposed the idea of deadline-optimization, in the form of a self-triggered policy for the periodic deadline-driven controller. Here, we show that our idea of online deadline optimization can be extended to all policies that consider a deadline, including the two-policy case and the \( ed \) case.

Next, we describe extended switching conditions that include an optimization procedure for the deadline for the two-policy strategy. Analogous to \( S(\xi) \), we define the extended set that includes a deadline variable

\[
S^s(\xi) := \{ m \in \{d, ed\}, D \in D, b \in \{d^*, ed^* \} \mid V^{D}(\xi, m, D, b) \leq 0 \},
\]

where, for mathematical and practical convenience \( D^s \in D \) for all \( b \in \{d^*, ed^*\} \), \( D \subset \mathbb{R}_{>0} \), is a finite but possibly arbitrarily large set of allowable deadlines, and where \( m \) corresponds to a method with deadline and

\[
V^{D}(\xi_k, m, D, b) := \mathbb{E}[V^s(\xi_{k+1}) - V_k(\xi_{k+1}) \mid \xi_k, \begin{bmatrix} \sigma_k \\ b_k \\ D_k \end{bmatrix} = \begin{bmatrix} m \\ b \\ D \end{bmatrix}]
\]

now has an additional argument \( D \in D \) for the choice of deadline compared to \( V^{D}(\xi_k, m, b) \).

We propose to use the following control policy

\[
\begin{bmatrix} \sigma_k \\ D_k \\ b_k \end{bmatrix} = \arg \min_{(m,D,b) \in S^s(\xi_k)} \xi_k^T Z_{m,b}(D) \xi_k + \beta_{m,b}(D), \quad (14)
\]

\[
u_k = -K_{s,b_k}^*(D_k) \xi_k, \quad (15)
\]

where

\[
Z_{m,b}(D) := A_{\gamma_k}(h_k)^T P_b A_{\gamma_k}(h_k) + Q(h_k) - K_{m,b}(D)^T (B_1^T P_b B_1 F(D)) K_{m,b}(D),
\]

\[
K_{m,b}(D) = (B_1^T P_b B_1 F(D))^T (B_{\gamma_k}^T P_b A_{\gamma_k}(h_k)),
\]

\[
\beta_{m,b}(D) = \text{tr}(P_b W_m(D)) + \alpha_m(D) - \frac{h_{m,D}}{h_{p}p},
\]

and where the distribution for \( h_k \) and \( \gamma_k \) depend on the value of the deadline \( D \) in

\[
h_k = \begin{cases} D_k & \text{if } \sigma_k = d, \\ \min\{\tau_k, D_k\} & \text{if } \sigma_k = ed, \end{cases} \quad (16)
\]

corresponding to the value of \( \sigma_k = m \). In particular, \( h_{m,D} = \mathbb{E}[h_k \mid D_k = D, \sigma_k = m] \) and \( W_{m}(D) := \mathbb{E}[W(h_k) \mid D_k = D, \sigma_k = m] \) and \( \alpha_m(D) := \mathbb{E}[\alpha(h_k) \mid D_k = D, \sigma_k = m] \).

Let the cost of the above policy (14)-(16) be denoted \( J^s_{dke} \). We obtain the following result, which can be seen as an extension of Theorem 1.

**Theorem 2:** The cost (2) of the online deadline optimization policy (14)-(16) is not larger than that of the two-policy method \( d&cd \) with fixed optimal deadlines in the sense that

\[
J^s_{dke} \leq J_{d^*kec*}. \quad \Box
\]

The proof is given in Appendix D. Again, the condition \( V^{D}(\xi_k, m, D, b) \leq 0 \) can be relaxed as explained in Remark 5.

Let the cost of the policy (14)-(16), when \( b_k = ed^* \) and \( \sigma_k = ed \) for all \( k \in \mathbb{N} \), be denoted \( J^*_{ed} \). By restricting the choice of the base policy, we obtain the following result.

**Corollary 2:** The performance (2) of the online deadline optimization policy (14)-(16), when \( b_k = ed^* \) for all \( k \in \mathbb{N} \), is not larger than that of the base policy \( ed^* \) in the sense that

\[
J^*_{ed} \leq J_{ed^*}. \quad \Box
\]

In the next section, we show numerical results for the proposed policies and the performance gain that can be achieved by addressing the delay probability directly.

**Remark 8:** Note that the above approach requires the computation of the argument in (14) for many different options. By reducing the search space for the deadline, the computational load can easily be reduced to match the available computational capacity.

### C. Computational complexity

Here, we consider the computational complexity of the online optimization methods considered. For each argument for the minimization (11) and (14), the matrices \( Z \) and the scalars \( \beta \) can be computed offline a priori and, e.g., stored in memory. The same holds for the gains in (12) and (15).

Therefore, to compute the optimal arguments, it is required to compute and compare the terms \( \xi^T Z \xi + \beta \) for \#deadlines \x \#switchingpolicies \x \#basepolicies switching options in the sets \( S \) and \( S^s \). Note that the size of the sets \( S \) and \( S^s \) depends on the current state and is upper bounded by the total number of switching options. Knowledge of sets \( S \) and \( S^s \) may be used to reduce computations by limiting online the number of options in (11) and (14), but in some implementations it may be more convenient to compute all possible options.

The computation of \( V^{D} \) for the sets \( S \) and \( S^s \) requires two computations of the form \( \xi^T X \xi + Y \) where \( X \) and \( Y \) are a matrix and a scalar, which can be computed offline for each switching option, taking the forms \( (A - BK)^T P(A - BK) \) and \( tr(PW) \), respectively. The number of computations in the terms \( \xi^T Z \xi + \beta \) scales quadratically with the state dimension and linearly with the number of switching parameters, but they can be computed in parallel for all elements in \( S \) or \( S^s \) (or all switching options). For the control inputs, the multiplication \( Kx \) scales linearly with the state.

From this analysis we conclude that, typically, the time required to compute (11) and (14) is small when compared to the communication or data processing computation delay.
modeled by $F$. However, in cases in which these computation times are non-negligible (because the initial computation/communication delay modeled by $F$ can be small), it can be incorporated in a new probability distribution modeling the sum of delays, say $\bar{F}$. Therefore, the methods in this paper can be used to analyze this case as well. Note, however, that in the latter case one should compare the simpler $d^*$ and $e$ methods considering the initial distribution $F$ with the $ed^*$ method considering $\bar{F}$ and therefore the method $ed^*$ does not guarantee a better performance than $e$ a priori.

### V. Numerical Results

In this section, we compare the performance of the proposed strategies on a second-order system taking the form (1) with

$$A_c = \begin{bmatrix} 0 & \frac{1}{t} \\ -\frac{d}{mt} & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix},$$

which represents a linearized inverted pendulum system with force input, gravitational acceleration $g = 10 ms^{-1}$, mass of pendulum $m = 0.25 kg$, length $l = 0.5 m$ and damping coefficient $d = 1 \text{Nm/srad}^{-1}$.

The cost function matrices in (2) are taken as

$$Q_c = \begin{bmatrix} 20 & 1 \\ 1 & 20 \end{bmatrix}, \quad R_c = [3].$$

We consider for the delay both a Gamma distribution $f_1$ with shape and scale parameters $k = 3$ and $\theta = 4/100$, respectively, and the piecewise constant two-block distribution

$$f_2(\tau) = \begin{cases} 4.5 & \text{if } \tau \in [0.05, 0.25), \\ 0.1 & \text{if } \tau \in [0.50, 0.60), \\ 0 & \text{otherwise}. \end{cases}$$

<table>
<thead>
<tr>
<th>Method</th>
<th>Base cost $J_{ba}$ (analytical)</th>
<th>Base cost $J_{ba}$ (simulation)</th>
<th>Cost with deadline optimization $J^*_{ba}$ (simul.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$</td>
<td>7.9507</td>
<td>7.9591 (≈)</td>
<td>7.9591 (≈)</td>
</tr>
<tr>
<td>$e$</td>
<td>7.9507</td>
<td>7.9591 (≈)</td>
<td>7.9591 (≈)</td>
</tr>
<tr>
<td>$ed$</td>
<td>7.9507</td>
<td>7.9591 (≈)</td>
<td>7.9591 (≈)</td>
</tr>
<tr>
<td>$dke$</td>
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<td>7.9793 (≈)</td>
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<tr>
<td>$dkce$</td>
<td>n.a.</td>
<td>7.8075 (≈)</td>
<td>7.8075 (≈)</td>
</tr>
</tbody>
</table>

**TABLE I**

VALUES OF PERFORMANCE INDEX (2) FOR $f = f_1$.

Note that although $f_1$ does not satisfy the condition $F_1(\epsilon) = 0$ it can easily be adapted (see Remark 1) to meet such an assumption, with a small $\epsilon$, without impacting on the results.

First, for $f_1$, the optimal solutions for the base policies are computed. For $D$ in the interval $[10^{-3}, 1]$, the stochastic Riccati equations for $Pu(D)$, $P_\epsilon$, and $P_{ed}(D)$ (see Appendix A) are solved iteratively, with initial value $P = 10^{-4} I_{nx}$, up to accuracy $10^{-4}$ of the mean square error. All cost values are depicted as a function of the deadline in Figure 2. Subsequently, the optimal values $J_{ba}$, $J_{ed}$, and $J_{ed}(D)$, respectively, are found for their respective optimal deadlines $D_0^*$ and $D_{ed}^*$. The optimal deadline values, are found to be $D_0^* = 0.1508$ and $D_{ed}^* = 0.3307$. Furthermore, $\bar{D} = 0.1200$ and $h_{ed^*} = 0.1194$. The optimal costs $J_{ba^*}$, $J_{ed^*}$, and $J_{ed}(D)$, for $f_1$ are given in Table I. Although the contribution of the intersampling behaviour is very small, since $\alpha_0 \approx 0.0014$ for all the base policies, it has been taken into account in the calculation of the costs and our simulations.

For this example, observe that the green line in Figure 2 is below the blue line everywhere, indicating that even for a suboptimal deadline, the $ed$ approach outperforms the deadline-driven approach. As expected from Lemma 1, $J_{ed}(D)$ performance approximates $J_e$ for large $D$.

For each delay value and each switching option, all variables are computed a priori to speed-up computation for Monte-Carlo (MC) simulations. For $\xi_0 = [0, 0, 0]^T$, we run 40 “long” Monte-Carlo simulations for $t \in [0, 24000]$ such that the average costs have approximately converged for each simulation. Then, the costs are averaged over the MC simulations and the values are given in Table I. Due to the limited simulation time and limited number of MC simulations, the cost is not completely averaged over the probability space, leaving a small error.

The cost differences that support our theorems are indeed visible, thereby underlying the results. Note that for the two-policy approaches, the relaxed switching conditions (see Remark 5) are used, giving a small additional performance gain of $1 - 2\%$. It is notable that, while adding an optimal deadline only gives small improvement over the event-driven case ($J_{ed^*} \approx J_e$), the fact that deadline-optimization can be enabled brings significant advantage of $6 - 9\%$ compared to the non-switching case (e.g., $J_{ed} < J_{ed}$ and $J_{dke} < J_{dke^*}$). Furthermore, the strategy with deadline optimization on $ed$, which builds upon our previously proposed policies, performs better than the two-policy approach in the sense that $J_{ed}^* < J_{dke^*}$. However, such a performance improvement is not guaranteed formally and the converse, i.e., $J_{ed}^* > J_{dke^*}$ may also occur for different examples. Furthermore, a quantification of the performance difference for the deadline-optimization approaches may only be obtained through simulation or experiments.
with the proposed policies. Performance relations between switched policies with deadline optimization are still subject of study. Moreover, future work also includes the output-feedback counterpart, which adds an estimation problem influenced by stochastic delays, and studies of robustness with respect to model uncertainty.

APPENDIX A
Riccati equations for the base policies

To compute the cost for event-driven control with deadline, it is required to solve, for \( P_{ed}(D) > 0 \), the generalized Riccati equation

\[
\begin{align*}
P_{ed}(D) &= \mathcal{A}_0(D) P_{ed}(D) \mathcal{A}_0(D) + Q(D) - K_{ed}(D) \mathcal{G}_{ed}(D) K_{ed}(D), \\
G_{ed}(D) &= \mathcal{B}_{ed}^T P_{ed}(D) \mathcal{B}_{ed} = F(D) \mathcal{B}_{ed}^T P_{ed}(D) \mathcal{B}_{ed}, \\
K_{ed}(D) &= \mathcal{G}_{ed}(D) \left( \mathcal{B}_{ed}^T P_{ed}(D) \mathcal{A}_0(D) \right) = (\mathcal{B}_{ed}^T P_{ed}(D) \mathcal{A}_0(D),
\end{align*}
\]

and to compute \( W_{ed}(D) := \hat{W}(|D|) = \int_0^D \hat{W}(s) dF(s) + (1 - F(D)) \hat{W}(D) \), see also footnote 2 on page 4. The solution to the Riccati equation can be found by, e.g., the iteration \( P_{ed}^{k+1}(D) = Ric(P_{ed}^k(D)) \) for \( k \geq 0 \) where \( Ric(\cdot) \) is the function of the right-hand side of (17). One can recover the Riccati equations for \( d \) and \( e \), respectively, since considering any new probability measure with \( \mu(\{0, D\}) = 0 \) and \( \mu(\{D\}) = F(D) \) gives

\[
\begin{align*}
P_d(D) &= \mathcal{A}_d(D) P_d(D) \mathcal{A}_d(D) + Q(D) - K_d(D) \mathcal{G}_{d}(D) K_d(D), \\
G_d(D) &= \mathcal{B}_{d}^T P_{d}(D) \mathcal{B}_{d} = F(D) \mathcal{B}_{d}^T P_{d}(D) \mathcal{B}_{d}, \\
K_d(D) &= \mathcal{G}_{d}(D) \left( \mathcal{B}_{d}^T P_{d}(D) \mathcal{A}_d(D) \right) = (\mathcal{B}_{d}^T P_{d}(D) \mathcal{A}_d(D),
\end{align*}
\]

and, alternatively, by letting \( D \to \infty \) we have

\[
\begin{align*}
P_e &= \mathcal{A}_e(D) P_e(D) \mathcal{A}_e(D) + \bar{Q}(\tau) - K_e(D) \mathcal{G}_{e}(D) K_e(D), \\
G_e &= \mathcal{B}_{e}^T P_{e}(D) \mathcal{B}_{e} = F(D) \mathcal{B}_{e}^T P_{e}(D) \mathcal{B}_{e}, \\
K_e &= \mathcal{G}_{e}(D) \left( \mathcal{B}_{e}^T P_{e}(D) \mathcal{A}_e(D) \right),
\end{align*}
\]

Furthermore, \( W_{d}(D) = \hat{W}(D) \) for a given value of \( D \), and \( W_{e} = \hat{W}(\tau) \).

APPENDIX B
Cost due to intersampling behaviour

For the base policies \( b \in \{d, e, ed\} \), where \( h_k, k \in \mathbb{N} \), are i.i.d., in (5), the contribution \( \alpha_{b}b \in \{d, e, ed\} \), of the intersampling behaviour of the Wiener process is given by the average value \( \mathbb{E}[\alpha(h_k)] \), where \( \alpha(h) \) is given in (9). This follows from \( \lim_{N \to \infty} \frac{1}{N} \mathbb{E}\left[ \sum_{k=0}^{N} \alpha(h_k) \right] = \frac{1}{\mathbb{E}[h_k]} \mathbb{E}[\alpha(h_k)] = \frac{1}{\mathbb{E}[h_k]} \beta_B \) which can be concluded from [34, Prop. 3.4.1]. Specifically, for a given deadline \( D \in \mathbb{R} \) and policy \( b \in \{d, e, ed\} \), \( \alpha_{b}(D) := \mathbb{E}[\alpha(h_k) | D_k = D, \sigma_k = b] \). Then, in (5), we have \( \alpha_{d}(D) := \alpha(D), \alpha_{e} := \alpha(\tau), \alpha_{ed}(D) := \alpha(h_kD) \), see also footnote 2 on page 4.
APPENDIX C

PROOF OF THEOREM 1

We drop the superscript ⋆ for Z and β for brevity and let d and ed be represented by m or b or p. Note that for each option, $\xi^*_{m,b}\xi_k + \text{tr}(P_bW(h_k|D,m)) + \alpha(h_k|D,m)$ is the minimal value of the optimization

$$
\min_{u_k} \mathbb{E}\left[ \int_{t_k}^{t_k+h_k|D} x(t)^\top Q_c x(t) + u(t)^\top R_c u(t) dt \right. \\
+ \xi(t_k + h_k|D)^\top P_b \xi(t_k + h_k|D) \left. \right] 
$$

This follows from standard LQG arguments (see [30]). Define the difference in arguments in (11) by

$$
\Delta(\xi, m_1, b_1, m_2, b_2) := \left[ \xi^\top Z_{m_1,b_1} \xi + \text{tr}(P_{b_1} W_{m_1}) + \alpha_{m_1} - \frac{\bar{h}_{m_1}}{h_p} c_p \right] \\
- \left[ \xi^\top Z_{m_2,b_2} \xi + \text{tr}(P_{b_2} W_{m_2}) + \alpha_{m_2} - \frac{\bar{h}_{m_2}}{h_p} c_p \right] 
$$

where $c_p$ is given in (5), such that, for $p \in \{d^\ast, ed^\ast\}$,

$$
\Delta(\xi, p, p, p, p) := 0,
$$

$$
\Delta(\xi, p, m, p, b_2) := \left[ \xi^\top P_{p} \xi + c_p \right] \\
- \left[ \xi^\top Z_{m_2,b_2} \xi + \text{tr}(P_{b_2} W_{m_2}) + \alpha_{m_2} - \frac{\bar{h}_{m_2}}{h_p} c_p \right],
$$

where $\bar{h}_m = \mathbb{E}[h_k | \sigma_k = m]$, such that, e.g., $\bar{h}_m = D$ if $m = d$, and we use the fact that $Z_{p,p} = P_p$. Observe that the switching condition (11) aims to maximize the value of $\Delta(\xi_k, p, p, \sigma_k, b_k)$ for $p = p^\ast$. Note that it is always allowed to choose the optimal base policy $\sigma_k = p, b_k = p$ since it directly satisfies the condition on $V^\Delta$. It corresponds to a value $\Delta(\xi_k, p, p, p, p) = 0$. Hence, $\Delta(\xi_k, p, p, \sigma_k, b_k)$ is non-negative since any choice of $\sigma_k, b_k$ following from (11) satisfies $\Delta(\xi_k, p, p, \sigma_k, b_k) \geq \Delta(\xi_k, p, p, p, p) = 0$.

Moreover, we consider

$$
\mathbb{E}[g(\xi_k, h_k) + V_b(\xi_{k+1}) | \xi_k, \left[ \begin{array}{c} \sigma_k \\ b_k \end{array} \right] = \left[ \begin{array}{c} m \\ b \end{array} \right]] \\
= \xi_k^\top Z_{m,b}\xi_k + \text{tr}(P_{b} W_{m}) + \alpha_{m} \\
= \xi_k^\top Z_{m,b}\xi_k + \text{tr}(P_{b} W_{m}) - \xi_k^\top P_{p}\xi_k + \xi_k^\top P_{p}\xi_k \\
+ \frac{\bar{h}_m - \bar{h}_p}{h_p} c_p + \alpha_{m} \\
= \frac{\bar{h}_m}{h_p} c_p - \xi_k^\top P_{p}\xi_k + \xi_k^\top P_{p}\xi_k \\
+ \frac{\bar{h}_m}{h_p} c_p - \Delta(\xi_k, p, p, p, b) + V_p(\xi_k).
$$

As defined in Section IV-A, $p^\ast = \arg\min_{p \in \{d^\ast, ed^\ast\}} J_{P^\ast}$. The one-step cost of the proposed control policy is then given by the difference

$$
\mathbb{E}[g(\xi_k, h_k) | \xi_k, \left[ \begin{array}{c} \sigma_k \\ b_k \end{array} \right] = \left[ \begin{array}{c} m \\ b \end{array} \right]] = \frac{\bar{h}_m}{h_p} c_{p^\ast} - \Delta(\xi_k, p^\ast, p^\ast, m, b) + V_{p^\ast}(\xi_k) - \mathbb{E}[V_b(\xi_{k+1}) | \xi_k, \left[ \begin{array}{c} \sigma_k \\ b_k \end{array} \right] = \left[ \begin{array}{c} m \\ b \end{array} \right]].
$$

Note that the condition $V(\xi_k, m, b) \leq 0$, guarantees that

$$
\mathbb{E}[V_p^\ast(\xi_{k+1}) - V_b(\xi_{k+1}) | \xi_k, \left[ \begin{array}{c} \sigma_k \\ b_k \end{array} \right] = \left[ \begin{array}{c} m \\ b \end{array} \right]] \leq 0.
$$

Define

$$
G_N := \sum_{k=0}^{N} g(\xi_k, h_k), \quad E_N := \sum_{k=0}^{N} \mathbb{E}[g(\xi_k, h_k) | I_k].
$$

Note that $g(\xi_k, h_k)$ given $I_k$ is a random variable since $I_k$ includes $\xi_k$ but $h_l$ only for $l < k$. We have that the process $X := (X_k)_{k \in \mathbb{N}}$, with $X_k := G_k - E_k$, is a martingale with respect to the filtration associated with $I_{k+1}$, since

$$
\mathbb{E}[X_{k+1} | I_{k+1}] = \mathbb{E}[X_k + g(\xi_{k+1}, h_{k+1}) - \mathbb{E}[g(\xi_{k+1}, h_{k+1}) | I_{k+1}] | I_{k+1}] = \mathbb{E}[X_k | I_{k+1}] = X_k.
$$

Note that $N(T)$ is a stopping time w.r.t. $(X_k, I_{k+1})$, which has finite expectation for given $T$, i.e., $\mathbb{E}[N(T)] < \infty$ since $h_k \geq h_{\min} > 0$ for all $k \in \mathbb{N}$, and that $N(T) \rightarrow \infty$ as $T \rightarrow \infty$. Provided that we prove that there exists some constant $c \in \mathbb{R}$ such that $\mathbb{E}[|X_{k+1} - X_k| | I_{k+1}] \leq c$ for all $k < N(T)$ for $k \in \mathbb{N}$, which we will do in the sequel, we can apply Doob's optional sampling theorem (see, e.g., [35, Th. 9, Sec. 12.5] or [29, Th. 2.2, Ch. VII]) and have

$$
\mathbb{E}[X_{N(T)}] = \mathbb{E}[X_0] = 0.
$$

where we use the fact that

$$
\mathbb{E}[X_0] = \mathbb{E}[g(\xi_0, h_0) - \mathbb{E}[g(\xi_0, h_0) | I_0]] = 0.
$$

This implies that $\mathbb{E}[G_{N(T)}] = \mathbb{E}[E_N]$. Furthermore, since by the law of total expectation (or tower rule) $\mathbb{E}[g(\xi_{N(T)}, h_{N(T)}) | I_{N(T)}] = 0$, we have that

$$
\mathbb{E}\left[ \sum_{k=0}^{N(T)-1} g(\xi_k, h_k) \right] = \mathbb{E}\left[ \sum_{k=0}^{N(T)-1} \mathbb{E}[g(\xi_k, h_k) | I_k] \right].
$$

We have that

$$
\mathbb{E}[|X_{k+1} - X_k| | I_{k+1}] = \mathbb{E}[g(\xi_{k+1}, h_{k+1}) - \mathbb{E}[g(\xi_{k+1}, h_{k+1}) | I_{k+1}] | I_{k+1}] \leq \mathbb{E}[g(\xi_{k+1}, h_{k+1}) | I_{k+1}] + \mathbb{E}[g(\xi_{k+1}, h_{k+1}) | I_{k+1}] | I_{k+1}] = 2 \mathbb{E}[g(\xi_{k+1}, h_{k+1}) | I_{k+1}] \leq 2 \mathbb{E}[g(\xi_{k+1}, h_{k+1}) | I_{k+1}] + V_{p^\ast}(\xi_{k+1}),
$$

where the last inequality follows from (19). The first term is bounded since $\bar{h}_{\sigma_{k+1}} \leq \bar{\tau}$ for all $k \in \mathbb{N}$ for strategies $c$ and $ed$, and $\bar{h}_{\sigma_{k+1}} \leq D_{\max}$ for all $k \in \mathbb{N}$ for strategy $d$. The fact that $\mathbb{E}[|X_{k+1} - X_k| | I_{k+1}] \leq c$ follows then from mean-square stability of $\xi_k$, which is proven by boundedness of $\mathbb{E}[V_{p^\ast}(\xi_k)]$ for all $k \in \mathbb{N}$ as $k \rightarrow \infty$, which follows similar arguments as a similar proof in [32, Theorem 4].
Summing (19) for $k \in \{0, \ldots, N(T) - 1\}$, we have
\[
N(T) - 1 \sum_{k=0}^{N(T) - 1} \mathbb{E}[g(\xi_k, h_k) \mid I_k] = \sum_{k=0}^{N(T) - 1} h_{\sigma_k} \frac{1}{h_{p^*}} c_{p^*} - \delta_k + \nu_k
+ V_{p^*}(\xi_0) - V_{p^*}(\xi_{N(T)}),
\]
where
\[
\delta_k := \Delta(\xi_k, p^*, p^*, \sigma_k, b_k) - V^\Delta(\xi_k, \sigma_k, b_k),
\]
and
\[
\nu_k := V_{p^*}(\xi_{k+1}) - \mathbb{E}[V_{p^*}(\xi_{k+1}) \mid I_k].
\]
Next, we substitute (20) and (21) in (8). Taking the expectation, we have that $\mathbb{E}[\nu_k] = 0$ for all $k \in \mathbb{N}$. Then, when taking the limit $T \to \infty$ in (8), the last two terms in (21) vanish, since $\mathbb{E}[V_{p^*}(\xi_{N(T)})]$ is bounded as $T \to \infty$, as explained before, and $\mathbb{E}[V_{p^*}(\xi_0)]$ is bounded by the initial condition. Furthermore, the first term becomes equal to $J_{p^*}$ since
\[
\limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{k=0}^{N(T) - 1} \tilde{h}_{\sigma_k} \right] = 1. \tag{22}
\]
This holds by the fact that $(\tilde{H}_k)_{k \in \mathbb{N}}$, with $\tilde{H}_N := \sum_{k=0}^N \tilde{h}_k - \tilde{h}_{\sigma_k}$, is a martingale with respect to the filtration associated with $I_{k+1}$ and again the fact that $\tilde{h}_{\sigma_k}$ is bounded by $\bar{\sigma}$ or $D_{\text{max}}$. These conditions, again by Doob’s optional sampling theorem [29], [35], imply that
\[
\mathbb{E} \left[ \sum_{k=0}^{N(T) - 1} \tilde{h}_{\sigma_k} \right] = \mathbb{E} \left[ \sum_{k=0}^{N(T) - 1} h_k \right].
\]
Then (22) holds by the fact that $\lim_{T \to \infty} \frac{1}{T} \mathbb{E} \sum_{k=0}^{N(T) - 1} h_k = 1$, since the discretization error vanishes in the limit.

As a result, we get, for $\pi = d^* \land \epsilon d^*$ that
\[
J_{d^* \land \epsilon d^*} = \frac{1}{h_{p^*}} c_{p^*} - \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{k=0}^{N(T) - 1} \delta_k \right]
\leq \frac{1}{h_{p^*}} c_{p^*} = J_{p^*} = \min\{J_{d^*}, J_{\epsilon d^*}\}. \tag{23}
\]
with $\delta_k \geq 0$ by the fact that $\Delta$ is non-negative and $V^\Delta$ is non-positive by definition of (11). Note that $\delta_k = 0$ for the choice $(\sigma_k, b_k) = (p^*, p^*)$. This proves the theorem.

APPENDIX D

PROOF OF THEOREM 2 AND COROLLARY 2

Consider all allowable choices of combinations $(m, D)$, where $D$ is in the finite set $\mathcal{D}$, as new methods $\hat{m}$ such that $(\hat{m}, b) \in S^* (\epsilon)$. Each method $\hat{m}$ has the particular value of $D$ as its optimal choice of deadline $d^\ast$. By reformulation, the switching condition (14) then takes the same form as (11) and the proof of Theorem 1 applies. For Corollary 2, the switching options are limited to $b_k = ed^\ast$ and $\sigma_k = ed^\ast$ for all $k \in \mathbb{N}$, hence always $V^\Delta D \leq 0$, and the deadline $D$ is the only switching parameter.

REFERENCES


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