Output Regulation for Non-square Linear Multi-rate Systems

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SUMMARY

We address the problem of regulating a subset of outputs of a linear time-invariant plant with multi-rate measurements so as to achieve asymptotic tracking of an exogenous signal generated by the free motion of a linear time-invariant system, denoted by exosystem. A solution to this problem is required to yield closed-loop stability and should be such that output regulation is achieved even in the presence of small plant uncertainties and exogenous disturbances also generated by the exosystem. Contrarily to previous works, we propose a solution to the general case where the plant may have more measured outputs than inputs. We show that this solution allows us to solve simultaneous stabilization and output regulation problems that are not possible to solve through the previous works. Besides incorporating an internal model of the exosystem, the key feature of our proposed controller is that it includes a system that blocks signals generated by the exosystem arriving to the controller from the non-regulated outputs. Copyright © 2011 John Wiley & Sons, Ltd.

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1. INTRODUCTION

One of the celebrated problems in automatic control, commonly known as output regulation, is that of controlling the output of a linear time-invariant system so as to achieve asymptotic tracking of an exogenous signal generated by the free motion of a linear time-invariant system, so-called exosystem, while guaranteeing closed loop stability. Several solutions were proposed in the seventies using different approaches (see, e.g., [1], [2], [3], [4]), and all incorporate an internal model of the exosystem, as well as a controller that stabilizes the closed loop. The necessity of incorporating a model of the exosystem in the controller was proved in [4], and is thereafter known as the internal model principle.

Multi-rate systems appear naturally in multi-input multi-output control, due to the heterogeneity among sensors and possibly among actuators, which are likely to work at different rates. If these rates are sufficiently high, then sensors and actuators may be synchronized to match the rate of the slowest. However in many applications it is desirable, or even necessary, to take advantage of the full capacities of the available sensors and actuators. As an example, taken from [5], consider the control of unmanned vehicles, in which the linear position sensor, consisting of a Global Positioning System (GPS), is typically available at a slower rate than the remaining sensors, such as magnetometers and gyroscopes. It is often the case that delaying all the sensor rates to match the available GPS rate makes closed loop stability difficult to achieve, meaning that a multi-rate scheme must be employed.

The blossom of the research in multi-rate systems took place in the early nineties, where many standard problems for LTI systems have been extended to multi-rate systems. For example,
structural properties for multi-rate systems including stabilizability and detectability notions are provided in [6], a solution to the output pole placement problem can be found in [7], and the linear quadratic gaussian problem for multi-rate systems is solved in [8]. The close relation between multi-rate and periodic systems (see, e.g., [9]), entails that these definitions and problem solutions are often based on the concept of lifting (see [10]), i.e., considering the LTI system obtained by writing the equations of the periodic system along a period.

The output regulation problem for multi-rate systems was addressed in [11]. See also [12] for the special case where the exogenous references are constant signals. However, both in [11] and [12] the analysis is restricted to square systems, i.e., systems with the same number of inputs and measured outputs, which are also the outputs to be regulated. The solution in [11] and [12] consists of designing a stabilizing controller for the system obtained by adding an internal model of the exosystem in series with the input of the plant. The peculiarities of the multi-rate systems that prevented the authors from generalizing the solution to non-square systems, are discussed in [11].

In the present paper, we address the output regulation problem for multi-rate systems for general non-square plants. For example, in the control of unmanned vehicles it is typically the case that some of the variables are required to track constant reference inputs (see [5]). The rotorcraft example given in [5] has only four inputs, while a twelve component vector comprising the linear and angular velocity vectors, linear position, and Euler angles is available for feedback. Naturally, it is desirable to take advantage of the information provided by the twelve measured components, instead of choosing only four in such a way that the system becomes square, and the solution in [11] or [12] can be applied. Moreover, a solution to the output regulation problem for non-square systems that achieves closed-loop stability is required when the non-regulated outputs are needed to guarantee the detectability of the plant. We illustrated this in the present paper through an example.

We propose a controller that achieves stability for the closed loop and output regulation for a number of regulated outputs equal to the number of inputs, while taking advantages of the remaining outputs for feedback. As in [11] and [12], the controller includes an internal model of the exosystem that is placed in series with the input of the plant. The key of our solution is to include a system that blocks signals generated by the exosystem that arrive to the controller from the non-regulated outputs. The concept of a system that blocks signals is made precise, by introducing the notion of blocking zero with respect to a matrix, both for LTI and periodic systems, which generalizes the standard notion of blocking zero for LTI systems (see, e.g., [13]). We show that there exists a stabilizing controller with the proposed structure, i.e., incorporating an internal model of the exosystem and a system that blocks signals generated by the exosystem, that achieves output regulation even in the presence of plant uncertainties and disturbances generated by the exosystem. The present paper may be viewed as a follow up work of [14], and [5], where the output regulation problem was considered for non-square multi-rate systems in the special case where the exogenous reference signals are constant signals. For constant reference signals, the proposed controller structure, is shown in [14], and [5] not only to be suited for output regulation but also that it can be exploited to implement nonlinear gain-scheduled controllers in such a way that a fundamental property known as linearization property is satisfied.

The remainder of the paper is organized as follows. In Section 2, we provide the problem setup and state the output regulation problem for multi-rate systems. In Section 3, we propose a solution to the problem statement summarized in our main result. The main result is proved in Section 4. An illustrative example showing that our solution allows to solve problems not possible to solve through previous works is provided in Section 5. Conclusions and directions for future work are given in Section 6. In the appendix we introduce the notion of blocking zero with respect to a matrix, which is a key concept to prove the main results in Section 4.

Notation: We denote by $I_n$ and $0_{n \times m}$ the $n \times n$ identity and zero matrices, respectively. We drop the dimensions of these matrices when they are clear from the context. The notation $\text{bdiag}(A_1, \ldots, A_n)$
denotes a matrix with blocks $A_i \in \mathbb{R}^{n \times n}$ in the diagonal, i.e.,

$$\text{bdiag}(A_1, \ldots, A_q) := \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix}.$$ 

For a matrix $A$, $A^\top$ denotes its transpose. For dimensionally compatible matrices $A$ and $B$, we define $(A, B) := [A^\top B^\top]^\top$. An eigenvalue of a matrix $A$ is denoted by $\lambda_i(A)$. The nomenclature unstable eigenvalues is used to denote the eigenvalues of a matrix $A$ that have modulus greater than or equal to one. We denote the value at time $t \in \mathbb{R}_{\geq 0}$ of the continuous-time signals $x : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^n$ by $x(t)$, and the value at time $k \in \mathbb{N}$ of the discrete time signals $x : \mathbb{N} \mapsto \mathbb{R}^n$ by $x[k]$.

2. PROBLEM FORMULATION

We describe first the multi-rate setup and the exosystem. Then we state the output regulation problem.

2.1. Multi-Rate Set-up

We consider a continuous-time plant

$$\begin{bmatrix} \dot{x}_P(t) \\ y_P(t) \end{bmatrix} = \begin{bmatrix} A_P & B_P \\ C_P & 0 \end{bmatrix} \begin{bmatrix} x_P(t) \\ u_P(t) \end{bmatrix} + \begin{bmatrix} E_P \\ 0 \end{bmatrix} v_P(t), \ t \geq 0,$$

where $x_P(t) \in \mathbb{R}^n$ is the state, $u_P(t) \in \mathbb{R}^m$ is the input, and $v_P(t) \in \mathbb{R}^{n_v}$ is a disturbance vector generated by the following system

$$\begin{align*}
\dot{v}_P(t) &= S_v v_P(t), \ t \geq 0, \\
v_P(t) &= E_v v_P(t).
\end{align*}$$

The output vector $y_P(t) \in \mathbb{R}^{p}$ can be partitioned into

$$y_P(t) = (y_P^1(t), \ldots, y_P^{n_y}(t))$$

where $y_P^i(t) \in \mathbb{R}^{p_i}$, is associated with sensor $i \in \{1, \ldots, n_y\}$, and $\sum_{i=1}^{n_y} p_i = p$. The sensors are assumed to operate at different sampling rates, with periods that are rationally related. This model for the measurements can be described by

$$y[k] := (y_1^i[k], \ldots, y_{n_y}^i[k]),$$

where

$$y_i^i[k] := \Gamma_k^i y_P(t_k), \ k \geq 0, \ 1 \leq i \leq n_y, \quad (3)$$

$t_k := k t_s, k \geq 0$, for some $t_s > 0$, $\Gamma_k^i = \gamma_k I_{p_i}$, and

$$\gamma_k^i := \begin{cases} 1, & \text{if sensor } i \text{ is sampled at } t_k \\ 0, & \text{otherwise} \end{cases}. \quad (4)$$

Let

$$\Gamma_k := \text{bdiag}(\Gamma_1^k, \ldots, \Gamma_{n_y}^k), \ k \geq 0.$$

Note that, due to our assumption that the periodicities of the sensors sampling are rationally related, $\Gamma_k$ is a periodic function of $k$, i.e., there exists $h$ such that $\Gamma_k = \Gamma_{k+h}, \forall k \geq 0$. We can assume that each diagonal entry of $\Gamma_k$ is non-zero at least once in a period since otherwise a given sensor component would never be sampled and could be disregarded.
The actuator mechanism is assumed to be a standard sample and hold device, and it is assumed to be available for update at every sampling time \( t_k \), i.e.,

\[
 u(t) = u[k], \quad t \in [t_k, t_{k+1}), \quad k \geq 0
\]  

(5)

where \( u[k] \) is the actuation update at time \( t_k \).

Denote the sampled state at times \( t_k \) by \( x[k] := x(t_k) \). Then we can write (1), (3) and (5) at times \( t_k \) as

\[
P := \left\{ \begin{bmatrix} x[k+1] \\ y[k] \end{bmatrix} = \begin{bmatrix} A & B \\ \Gamma_k C & 0 \end{bmatrix} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix} + \begin{bmatrix} Bv \\ 0 \end{bmatrix} v[k], \quad k \geq 0, \right. \tag{6}
\]

where \( A = e^{A_{ph}} \), \( B = \int_0^{t_k} e^{A_{ph}} ds B_P \), \( C = C_P \), \( Bv = \int_0^{t_k} e^{A(t_k-s)} E_P E_V e^{S_v s} ds \), and \( v[k] := v_P(t_k) \) is generated by the discretization of (2), which is given by

\[
w_V[k+1] = e^{S_v t_k} w_V[k], \quad k \geq 0,
\]

\[
v[k] = E_V w_V[k]. \tag{7}
\]

2.2. Exosystem

Suppose that we further partition the output vector \( y[k] \) according to

\[
y[k] = \begin{bmatrix} y_m[k] \\ y_r[k] \end{bmatrix} = \begin{bmatrix} \Gamma_{mk} C_{mk} \\ \Gamma_{rk} C_r \end{bmatrix} x[k], \quad k \geq 0, \tag{8}
\]

where \( \Gamma_{mk} \) and \( \Gamma_{rk} \) are \( n_m \times n_m \) and \( m \times m \) matrices, respectively, such that \( \Gamma_k = \text{bdig}(\Gamma_{mk}, \Gamma_{rk}) \). Note that \( n_m = p - m \). The component \( y_r[k] \in \mathbb{R}^m \) is a set of outputs that we wish to asymptotically track a reference signal \( r[k] \), and \( y_m[k] \in \mathbb{R}^{pn} \) is an additional set of measurements available for feedback. Subsumed in this partition is that \( p > m \). A solution to the output regulation problem in the case where \( p \leq m \) can be found in [11].

The reference signal \( r[k] \in \mathbb{R}^m \) and the disturbance signal \( v[k] \in \mathbb{R}^{n_v} \) are assumed to be generated by the following model, which we denote by \textit{exosystem},

\[
w[k+1] = S w[k]
\]

\[
r[k] = C_r w[k]
\]

\[
v[k] = C_v w[k], \quad k \geq 0, \tag{9}
\]

where \( w[k] \in \mathbb{R}^{n_w} \). The matrices \( S \) and \( C_v \) must be compatible with (7), i.e., the same signal \( v[k] \) should be generated by (9) and (7). Consider the Jordan canonical form of \( S \) i.e.,

\[
S = V \text{bdig}(S_1, \ldots, S_{n_w}) V^{-1} \tag{10}
\]

where \( V \) is an invertible matrix and the matrices \( S_j \) take the form

\[
S_j = \begin{bmatrix} \mu_j & 1 & 0 & \ldots \\ 0 & \mu_j & 1 & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_j \end{bmatrix} \in \mathbb{C}^{\kappa_j \times \kappa_j}, \quad 1 \leq j \leq n_s \tag{11}
\]

where \( n_s \leq n_w \), and \( \sum_{j=1}^{n_s} \kappa_j = n_w \). We assume that:

(S1) \( ||\mu_i|| \geq 1, \forall 1 \leq i \leq n_w \).

(S2) \( \mu_i \neq \mu_j \) for \( i \neq j, 1 \leq i, j \leq n_s \).
To see that (S1) and (S2) introduce no loss of generality, note first that the exosystem can generate signals \( r[k] \) and \( v[k] \) taking the form
\[
\xi[k] = \sum_{j=1}^{m} \sum_{l=0}^{n_j-1} b_{jl} \left( \frac{k}{l} \right) \mu_{j}^{k-l}, \quad k \geq 0
\]
(12)
where \( b_{jl} \) can be made arbitrarily by properly choosing \( C_R, C_V \), and \( w[0] \). Since, as we shall see shortly, output regulation is an asymptotic property, if (S1) would not hold then the disturbance and reference terms in (12) corresponding to stable eigenvalues of \( S \) would play no role. If (S2) would not hold, the exosystem would still only be able to generate the same class of reference and disturbance signals (12). Another way of stating (S2) is to say that the characteristic polynomial of \( S \) coincides with the minimal polynomial of \( S \), which is a statement more commonly seen in related works addressing the internal model principle (see, e.g., [15]).

2.3. Problem Statement

Consider a linear controller for the system (6), i.e., a map \( y[k] \mapsto u[k] \) with state \( x_c[k] \in \mathbb{R}^{n_c} \). We say that the closed-loop is stable if \( \langle x[k], x_c[k] \rangle \rightarrow 0 \) as \( k \rightarrow \infty \), when \( r[k] = 0 \), \( \forall k \geq 0 \). Moreover, we say that output regulation is achieved if \( \langle C_r x[k] - r[k] \rangle \rightarrow 0 \) as \( k \rightarrow \infty \).

The problem we are interested in this paper can be stated as follows.

Problem 1

Find a linear controller for the system (6) such that the closed loop is stable and output regulation is achieved.

3. MAIN RESULT

The structure of the controller that we propose to solve the Problem 1 is shown in Figure 1.

The purpose of the systems \( C_I \), \( C_D \) and \( C_K \) and the rationale behind this structure are briefly explained as follows. Since the controller must provide the adequate input value \( u[k] \) such that the output \( y_r[k] \) of the linear plant tracks the desired reference signal \( r[k] \), the system \( C_I \) is such that it is capable of providing such an input to the plant, when the input to \( C_I \) is identically zero. Denote by steady state the state of the plant at which output regulation is achieved. At steady state, due to the linearity of the plant, the non-regulated output \( y_m[k] \) consists of a signal with the same frequency content of the input to the plant. The system \( C_D \) has the purpose of yielding a zero output when the steady state signal \( y_m[k] \) is applied to its input, while assuring that other signals of interest, do not yield a zero output and thus can still be utilized to control the plant.

Closed loop can be guaranteed if the following condition is met

(G1) The system "seen" by the controller, i.e., obtained by computing the series of \( C_I \), \( P \) and \( C_D \), with input \( y_K[k] \) and output \( (y_D[k], y_r[k]) \), is detectable and stabilizable.
In fact, from standard results for linear systems, if (G1) holds one can compute a stabilizing controller \( C_K \). Note that, when output regulation is achieved, both the input to the controller and its output are zero.

Before we state our main result, we provide possible choices for \( C_I \) and \( C_D \). We shall impose in Section 4 some requirements that \( C_I \) and \( C_D \) must meet to prove our main output regulation result. We shall see that the systems proposed below meet these requirements. However other choices may exist that satisfy these requirements.

### 3.1. System \( C_I \)

The system \( C_I \) is an LTI system described by

\[
C_I := \left\{ \begin{bmatrix} x_I[k+1] \\ y_I[k] \end{bmatrix} = \begin{bmatrix} A_I & B_I \\ C_I & 0 \end{bmatrix} \begin{bmatrix} x_I[k] \\ u_I[k] \end{bmatrix}, \quad k \geq 0, \right. \tag{13}
\]

where \( x_I[k] \in \mathbb{R}^{n_I}, u_I[k] \in \mathbb{R}^m \), and \( y_I[k] \in \mathbb{R}^m \). The system \( C_I \) should be capable of providing the adequate input to the plant such that output regulation is achieved. Such an input takes the general form (12). One realization for (13) that achieves this is given by

\[
A_I = \text{bdiag}(S, \ldots, S) \in \mathbb{R}^{(mn_w) \times (mn_w)}, \quad B_I = \text{bdiag}(B_J, \ldots, B_I) \in \mathbb{R}^{(mn_w) \times m}, \quad C_I = \text{bdiag}(C_J, \ldots, C_I) \in \mathbb{R}^{m \times (mn_w)} \tag{14}
\]

where \( B_J \in \mathbb{R}^{n_w \times 1} \) is such that \((S, B_J)\) is stabilizable, and \( C_J \in \mathbb{R}^{1 \times n_w} \) is such that \((C_J, S)\) is detectable. It is straightforward to verify that this implies that \((A_I, B_I)\) is observable, and \((C_I, A_I)\) is detectable, respectively. Note that the system (13) with matrices (14) incorporates an \( m \)-fold reduplication of the exosystem (9), in the sense of [4].

### 3.2. System \( C_D \)

The system \( C_D \) is a linear periodically time-varying system described by

\[
C_D := \left\{ \begin{bmatrix} x_D[k+1] \\ y_D[k] \end{bmatrix} = \begin{bmatrix} A_{Dk} & B_{Dk} \\ C_{Dk} & D_{Dk} \end{bmatrix} \begin{bmatrix} x_D[k] \\ u_D[k] \end{bmatrix}, \quad k \geq 0, \right. \tag{15}
\]

where \( x_D[k] \in \mathbb{R}^{n_D}, u_D[k] \in \mathbb{R}^{n_m}, \) and \( y_D[k] \in \mathbb{R}^{m_m}, \) and \( A_{Dk}, B_{Dk}, C_{Dk} \) and \( D_{Dk} \) are \( h \)-periodic, i.e., e.g., \( A_{Dk} = A_{Dk+h}, \forall k \geq 0. \)

The system \( C_D \) has the purpose of blocking the signals that can be generated by the exosystem. By this we mean that the output of (15) is zero for every input generated by the exosystem, which takes the general form (12). The notion is made precise in the appendix (cf. Definition 18). One realization for (15) that achieves this is given by

\[
A_{Dk} = \text{bdiag}(A^i_k, \ldots, A^{n_m}_k) \in \mathbb{R}^{n_m n_w \times n_m n_w}, \quad B_{Dk} = \text{bdiag}(B^i_k, \ldots, B^{n_m}_k) \in \mathbb{R}^{(n_m n_w) \times n_m}, \quad C_{Dk} = \text{bdiag}(C^i_k, \ldots, C^{n_m}_k) \in \mathbb{R}^{n_m \times (n_m n_w)} \tag{16}
\]

where, corresponding to each output \( i \in \{1, \ldots, n_w\}, \) these matrices are given by

\[
A^i_k = \begin{bmatrix} 0 \\ I_{n_w-1} \\ 0 \end{bmatrix}, \quad B^i_k = \begin{bmatrix} 1 \\ 0_{n_w-1} \end{bmatrix}, \quad C^i_k = (e^{ik})^T, \quad D^i_k = 1, \tag{17}
\]

if the output \( i \) is sampled at \( t_k \), and

\[
A^i_k = I_{n_w}, \quad B^i_k = 0_{n_w \times 1}, \quad C^i_k = 0_{1 \times n_w}, \quad D^i_k = 0, \tag{18}
\]
otherwise, where \( c_{ik} \) is a \( h \)-periodic vector, i.e., \( c_{ik} = c_{i(k+h)} \), which is described shortly. The matrices \( A_k^i \) and \( B_k^i \), which correspond to the \( i \)th component of the output vector \( y_m[k] \), are such that the system \( C_D \) holds the last \( n_w \) sampled values of the output \( i \) when \( k \) is sufficiently large. A condition for \( k \) that assures this is \( k \geq n_w h \). In fact, from (17) we see that if the output is sampled at \( t_k \) then the new measurement is introduced in the state while the least recent is dropped. From (18) we see that if the output is not sampled at \( t_k \) the system \( C_D \) holds the previous state. The matrices \( C_k^i \) and \( D_k^i \) are such that the output is zero when steady state is achieved, in which case \( y_m[k] \) is a signal taking the form (12). This will be shown in Proposition 7. The \( n_w \) dimensional periodic vector \( c_{ik} \) can be determined as follows. Since the \( c_{ik} \) are \( h \)-periodic we need only to specify \( c_{ik} \) along a period, i.e., e.g., for \( k \in \{1, \ldots, h\} \), and for values for which \( \gamma_k^i = 1 \) (otherwise \( C_k^i = 0 \)), where \( \gamma_k^i \) is given by (4). Let \( [k] \) be the remainder of the division of \( k \) by \( h \) if \( k \geq 1 \), i.e., e.g., \( [k+1] = k+1 \) if \( 1 \leq k \leq h-1 \), and \( [k+1] = 1 \) if \( k = h \). Moreover if \( k \leq 0 \) we use the same notation to denote \( [k] := [k+rh] \) for some \( r \in \mathbb{N} \) such that \( k+rh \geq 1 \). For each \( k \in \{1, \ldots, h\} \) such that \( \gamma_k^i = 1 \), define a set of \( n_w + 1 \) indexes \( \{ \tau_{ik}^l, 0 \leq l \leq n_w \} \) by

\[
\tau_{ik}^l = \begin{cases} 
0, & \text{if } l = 0, \\
\tau_{ik}^l + \min\{ k_1 > 0 : \gamma_k^i = \gamma_{k-\tau_{ik}^l}^i = 1 \}, & \text{if } 1 \leq l \leq n_w.
\end{cases}
\]

Define also the set of vectors

\[
b_k = \begin{bmatrix} d_1 \\
\vdots \\
d_{n_s} \end{bmatrix} \in \mathbb{R}^{n_w \times 1}
\]

where \( k \in \{1, \ldots, h\} \)

\[
d_j := \begin{bmatrix} 1 & 0_{1 \times (s_j-1)} \end{bmatrix}^T, 1 \leq j \leq n_s.
\]

We make the following assumption:

\[
M_k^i \text{ is invertible for every } k \in \{1, \ldots, h\}, 1 \leq i \leq n_y.
\]

(19)

Take \( e_{ik} = [c_{ik}^1 \ldots c_{ik}^{n_w}] \) as the solution to

\[
M_k^i e_{ik} = -b_k,
\]

(20)

which is unique due to (19) and it is real. To see that it is real note that if \( \mu_i \) is a complex eigenvalue of \( S \) then so is its conjugate since \( S \) is real. Note also that to the eigenvalue \( \mu_i \), and to its conjugate, correspond complex conjugate rows of \( M_k^i \) and therefore both \( e_{ik} \) and its conjugate satisfy (20).

Assumption (19) holds in the special case where the sensors are sampled at a single-rate. In fact, in this case we have that \( \tau_{ik}^l = l, \forall 1 \leq k \leq h, 1 \leq i \leq n_y, 0 \leq l \leq n_w \), and the rows of \( M_k^i \) correspond to linear independent vector \( l' \mu_j^{-l} \) where \( 0 \leq r \leq n_j - 1 \). This is also in general true in the multi-rate case where the \( \tau_{ik}^l \) are in general not equal to \( l \). However, Assumption (19) may fail in some pathological cases as we illustrate in the next example.
Example 2
Suppose that \( y_m[k] \) is one dimensional and corresponds to a sensor which is sampled once every five times in a period, i.e., \( h = 5 \), \( \gamma_1^k = 1 \), if \( k = 1 \) and \( \gamma_1^k = 0 \) if \( k \in \{2, 3, 4, 5\} \). If \( n_w = n_s = 3 \), then we can obtain that \( \tau_1^k = 5 \), \( \tau_2^k = 10 \), and \( \tau_3^k = 15 \) if \( k = 1 \) and \( \tau_1^k \) do not need to be specified if \( k \in \{2, 3, 4, 5\} \). Suppose that \( \mu_1 = e^{i2\pi/5}, \mu_2 = e^{-i2\pi/5}, \mu_3 = 1 \). Then the matrices \( M_k^1 \), which needs only be specified for \( k = 1 \), is given by

\[
M_k^1 = \begin{bmatrix}
(e^{i2\pi/5})^{-5} & (e^{-i2\pi/5})^{-10} & (e^{i2\pi/5})^{-15} \\
(e^{-i2\pi/5})^{-5} & (e^{-i2\pi/5})^{-10} & (e^{-i2\pi/5})^{-15} \\
1 & 1 & 1
\end{bmatrix},
\]

which is singular and therefore Assumption 19 does not hold.

3.3. Main Result
We make the following assumptions on the multi-rate discrete-time plant (6):

(P1) \((A, B)\) is stabilizable and \((C, A)\) is detectable.

(P2) There are no invariant zeros from the input of the plant to the regulated output that coincide with the eigenvalues of \( S \), i.e.,

\[
\begin{bmatrix}
A - \mu_j I_n \\
C_r
\end{bmatrix}
\begin{bmatrix}
B \\
0
\end{bmatrix}
\]

is invertible \( \forall 1 \leq j \leq n_s \).

(P3) There does not exist an initial condition \( x[0] \) for the unforced plant (6) (i.e., \( u[k] = v[k] = 0, \forall k \geq 0 \)) such that \( \hat{y}_r[k] = 0, \forall k \geq 0 \) and such that \( y_m[k] = \Gamma_{mk} \xi[k] \), for some non-zero signal \( \xi[k] \) generated by the exosystem taking the general form (12) for some constants \( b_{jl} \in \mathbb{R}^{n_m} \), \( 1 \leq j \leq n_s, 0 \leq l \leq \kappa_j - 1 \).

We assume (P1)-(P3) to obtain (G1). The assumptions (P1) and (P2) are typical in related problems (cf. [4]). The assumption (P3) is closely related to the following assumption, which is easier to test:

(P3') \( A \) and \( S \) do not share an eigenvalue, or if \( A \) and \( S \) share an eigenvalue \( \mu_j, 1 \leq j \leq n_s \), then the corresponding eigenvector of \( A \) is not in the kernel of \( \Gamma_{rk} C_r \) for every \( k \in \{1, \ldots, h\} \).

While (P3) and (P3') are not equivalent, it is straightforward to show that (P3) implies (P3'). However if (P3') holds then (P3) may not hold in pathological cases.

Example 3
Suppose that \( \Gamma_{rk} = 1, \forall k \geq 0 \) and \( \Gamma_{mk} = 1 \) if \( k = 0, \Gamma_{mk} = 0 \) if \( k \in \{1, 2, 3, 4\} \) and \( \Gamma_{mk} = \Gamma_{m(k+5)} \), \( \forall k \geq 0 \). Let \( S = 1, C_R = 1, C_m = [1 0 0], C_r = [0 0 1] \), and

\[
A = \begin{bmatrix}
\cos\left(\frac{2\pi}{5}\right) & -\sin\left(\frac{2\pi}{5}\right) & 0 \\
\sin\left(\frac{2\pi}{5}\right) & \cos\left(\frac{2\pi}{5}\right) & 0 \\
0 & 0 & \frac{1}{2}
\end{bmatrix}.
\]

Although \( A \) does not have eigenvalues that coincide with the eigenvalues of \( S \), and therefore (P3') holds, if we make \( x[0] = [0 1 0]^T \) and \( w[0] = 1 \) then we have that \( \hat{y}_r[k] = 0, \forall k \geq 0, \) and \( \hat{y}_m[k] = \Gamma_{mk} \xi[k], \forall k \geq 0 \), where \( \xi[k] = 1, \forall k \geq 0 \), which means that (P3) does not hold.

The following is the main result of the paper. We denote by plant uncertainties the fact that the matrices \( A, B, C \) in (6) might not be known exactly, i.e., although the controller of Fig. 1 is designed for the model (6), the actual plant is described by the matrices \( \hat{A}, \hat{B}, \hat{C} \) and is given by

\[
\hat{P} := \begin{bmatrix}
x[k+1] \\
y[k]
\end{bmatrix} = \begin{bmatrix}
\hat{A} & \hat{B} \\
\Gamma_k \hat{C}
\end{bmatrix} \begin{bmatrix}
x[k] \\
u[k]
\end{bmatrix} + \begin{bmatrix}
B_\gamma \\
0
\end{bmatrix} v[k] \tag{21}
\]
Since asymptotic stability is a robust property, if \( \hat{A}, \hat{B}, \) and \( \hat{C} \) are sufficiently close to \( A, B, \) and \( C, \) respectively, and if the controller of Fig. 1 designed for (6) asymptotically stabilizes the closed-loop, then asymptotic stability is preserved when \( P \) is replaced by the actual plant \( \tilde{P} \).

\textbf{Theorem 4} \\
Suppose that (P1)-(P3) hold for the plant \( P \), and that \( C_I \) is given by (13), (14) and \( C_D \) is given by (16), (17), and (18). Then there exists matrices \( B_I, C_I \) for \( C_I \), and \( C_K \) such that the closed-loop in Figure 1 is stable and output regulation is achieved even in the presence of plant uncertainties that do not destroy closed-loop stability.

\[ \square \]

4. PROOF OF THE MAIN RESULT

We start by reviewing some general definitions for periodically time-varying systems. Then we state the assumptions that we make on \( C_I, C_D \) that lead to establishing that there exists a system \( C_K \) that yields the closed-loop of Figure 1 stable. After establishing the existence of such a system \( C_K \) we prove the main result. In the appendix we introduce the notion of blocking zero with respect to a matrix, which is key to understand the assumptions on the block \( C_D \), and to prove some of the results in the present section.

4.1. Periodic Systems

Consider a discrete-time linear periodic system

\[ R = \begin{cases} \dot{x}[k + 1] = A_k x[k] + B_k u[k] \\ y[k] = C_k x[k] + D_k u[k], \quad k \geq 0, \end{cases} \tag{22} \]

where \( x[k] \in \mathbb{R}^{n_A}, u[k] \in \mathbb{R}^{n_B}, y[k] \in \mathbb{R}^{n_C}, \) and \( A_k, B_k, C_k, D_k \) are \( h \)-periodic matrices, e.g., \( A_k = A_{k+h} \). Many system analytical notions for (22) are defined by considering the lifted time-invariant system \( \bar{R} \) associated with \( R \), which is defined as

\[ \bar{R} = \begin{cases} \ddot{x}[l + 1] = \ddot{A} \ddot{x}[l] + \ddot{B} \ddot{u}[l] \\ \ddot{y}[l] = \ddot{C} \ddot{x}[l] + \ddot{D} \ddot{u}[l], \quad l \geq 0, \end{cases} \tag{23} \]

where \( \ddot{x}[l] = x[lT], \)

\[ \ddot{u}[l] := (u[lh], u[lh + 1], \ldots, u[lh + h - 1]), \]

\[ \ddot{y}[l] := (y[lh], y[lh + 1], \ldots, y[lh + h - 1]), \]

and the system matrices in (23) are given by \( \ddot{A} := \Phi(h, 0), \)

\[ \ddot{B} := [\Phi(h, 1)B_0 \ \Phi(h, 2)B_1 \ldots B_{h-1}], \]

\[ \ddot{C} := (C_0, C_1 \Phi(1, 0), \ldots, C_{h-1} \Phi(h - 1, 0)), \]

\[ \ddot{D} := \begin{bmatrix} E_{11} & \ldots & E_{1h} \\ \vdots & \ddots & \vdots \\ E_{h1} & \ldots & E_{hh} \end{bmatrix}, \]

\[ E_{ij} := \begin{cases} C_{i-1} \Phi(i - 1, j)B_{j-1} & i > j \\ D_i & i = j, \\ 0 & i < j \end{cases} \]

where \( \Phi(i, j) := \prod_{k=i}^{j} A_k, \) for \( i > j \) and \( \Phi(i, i) := I. \)

The system \( R \) is stable, stabilizable and detectable if and only if \( \bar{R} \) is stable, stabilizable or detectable, respectively. Equivalently, stability of (22) is characterized by all the eigenvalues of
the matrix $\bar{A}$ having norm less that one, i.e., $\|\lambda_i(\bar{A})\| < 1$, $\forall i$, stabilizability of (22), denoted by $(A_k, B_k)$ is stabilizable, is characterized by there exists a set of periodic matrices $F_k, F_k = F_{k+h}$, such that $x[k + 1] = (A_k + B_k F_k)x[k]$ is stable, and detectability of (22), denoted by $(C_k, A_k)$ is detectable, is characterized by there exists a set of periodic matrices $G_k, G_k = G_{k+h}$, such that $x[k + 1] = (A_k + G_k C_k)x[k]$ is stable (cf. [16]).

4.2. Assumptions on $C_I$:

The system $C_I$, described by (13), must be such that:

(I1) $(A_I, B_I)$ is stabilizable and $(C_I, A_I)$ is detectable,

(I2) $C_I$ does not have invariant zeros at the unstable eigenvalues of the plant $P$, i.e.,

$$\begin{bmatrix} A_I - \rho I_{n_I} & B_I \\ C_I \\ 0 \end{bmatrix}$$ is invertible \hspace{1cm} (24)

for $\rho \in \{\lambda_i(A) : \|\lambda_i(A)\| \geq 1\}$.

(I3) For every $Z \in \mathbb{R}^{m \times n_w}$, the following equation

$$\begin{align*}
A_I \Pi_I &= \Pi_I S \\
C_I \Pi_I &= Z
\end{align*}$$ \hspace{1cm} (25)

has a solution $\Pi_I \in \mathbb{R}^{n_I \times n_w}$. Moreover such solution is unique.

(I4) The eigenvalues of $A_I$ belong to the set of eigenvalues of $S$.

We assume (I1)-(I2) to obtain (G1). Assumption (I4) is required to limit the set of possible systems $C_I$. As stated in the next proposition Assumption (I3) is closely related to the purpose of system $C_I$, i.e., to provide the adequate input to the plant so that it tracks the reference signal. Note that due to the linearity of the plant, an adequate input takes the same form of the reference signal one wishes to follow, i.e., it is generated by

$$\begin{align*}
w[k + 1] &= Sw[k], \hspace{0.5cm} k \geq 0, \\
u[k] &= C_U w[k].
\end{align*}$$ \hspace{1cm} (26)

**Proposition 5**

If (I3) holds then for any signal $u[k]$ generated by (26) there exists an initial condition $x_0$ such that the free motion of

$$\begin{align*}
x_I[k + 1] &= A_I x_I[k], \hspace{0.5cm} x_I[0] = x_0 \\
y_I[k] &= C_J x_I[k], \hspace{0.5cm} k \geq 0
\end{align*}$$

is such that $y_I[k] = u[k]$.

**Proof**

See [17, Ch. 6].

Due to the following proposition it is always possible to find $B_J$ and $C_J$ such that (I1)-(I3) hold for the system (13) with matrices (14).

**Proposition 6**

The set of matrices $(B_J, C_J) \in \mathbb{R}^{n_w \times 1} \times \mathbb{R}^{1 \times n_w}$ for which (I1)-(I3) do not hold for the system (13) with matrices (14) is a set of measure zero in $\mathbb{R}^{n_w \times 1} \times \mathbb{R}^{1 \times n_w}$.

**Proof**

See [17, Ch. 6].
4.3. Assumptions on $C_D$:

The system $C_D$ must be such that:

(D1) $(A_{Dk}, B_{Dk})$ is stabilizable and $(C_{Dk}, A_{Dk})$ is detectable.

(D2) The following equations

$$
A_{Dr} \Pi_{D[r+1]} + B_{Dr} \Gamma_{mr} Y = \Pi_{Dr}, S \\
C_{Dr} \Pi_{Dr} + D_{Dr} \Gamma_{mr} Y = 0, \quad 1 \leq r \leq h
$$

have a solution $\Pi_{Dr} \in \mathbb{R}^{n_D \times n_w}$, $1 \leq r \leq h$ such that (27) holds for every $Y \in \mathbb{R}^{n_m \times n_w}$.

(D3) Consider the system

$$
\dot{w}[k] = \hat{S} \dot{w}[k], \quad \dot{w}[0] = \hat{w}_0 \\
\dot{r}[k] = C_R \dot{w}[k], \quad k \geq 0,
$$

and suppose that $\hat{S}$ is such that there exists $\hat{w}_0$ such that $\dot{r}[k] \neq \dot{r}[k]$ for any arbitrary $w_0$, $C_R$ in (9). Then (D2) does not hold when $S$ is replaced by $\hat{S}$.

We assume (D1)-(D3) to obtain (G1). From Definition 18 of a time-invariant blocking zero given in the appendix, we see that (D2) is equivalent to the system obtained by computing the series of $C_D$ and $\Gamma_m$ having a time-invariant blocking zero with respect to $S$. From the interpretation of blocking zeros given in Proposition 19, we see that the assumption (D2) is closely related to the purpose of the system $C_D$, i.e., to block the signals that are generated from the exosystem (9). Moreover, using again the same interpretation of Proposition 19, Assumption (D3) states that $C_D$ does not block any signal other than the ones generated by the exosystem.

Proposition 7

The system (15) with matrices (16), (17) satisfies (D1)-(D3).

Proof

Note that (15) with matrices (16), (17) is a stable system. In fact, one can check that all the eigenvalues of $\hat{A}_D = A_{D[k-1]} \ldots A_{D2} A_{D1} A_{D0}$ are equal to zero, and therefore not only it is stable, but also the corresponding lifted system is a deadbeat system. Thus, its lifted system is stabilizable and detectable, and therefore so is (15) with matrices (16), (17).

To prove (D2) it suffices, from Proposition 19 in the appendix, to prove that there exists an initial condition for $C_D$ such that $C_D$ has zero output for every signal generated by the following system

$$
w[k+1] = Sw[k] \\
r[k] = \Gamma_{mk} C_R w[k], \quad k \geq 0.
$$

(29)

Denote the $n_w$ basis functions that generate (12) by

$$
\begin{align*}
  f_l[k] &= \binom{k\ i}{l} \mu_j^{k-l}, \quad \text{if } l \in \left(\sum_{q=1}^{j-1} \kappa_q, \sum_{q=1}^{j} \kappa_q\right], \quad k \geq 0,
\end{align*}
$$

(30)

where $l = k - \sum_{q=1}^{j-1} \kappa_q - 1, \quad 1 \leq l \leq n_w$. Recall that at a given time $k \geq h n_w$ the state of $C_D$ holds the last $n_w$ sampled values corresponding to each output component of $y_j[k]$. Then, also by construction of the $C_{Dr}$ and $D_{Dr}$ of the system $C_D$, proving that for any input signal taking the form (12), there exists an initial condition $x_D[0]$, such that the output of $C_D$ is zero, is equivalent to proving that the following holds

$$
\begin{bmatrix}
  f_l[k] \\
  \vdots \\
  f_{n_w}[k]
\end{bmatrix}
+ 
\begin{bmatrix}
  f_l[k - \tau_{ik}^1] & \ldots & f_l[k - \tau_{ik}^{n_w}]
  \\
  \vdots & \vdots & \vdots \\
  f_{n_w}[k - \tau_{ik}^1] & \ldots & f_{n_w}[k - \tau_{ik}^{n_w}]
\end{bmatrix}
\begin{bmatrix}
  c_{ik}^1 \\
  \vdots \\
  c_{ik}^{n_w}
\end{bmatrix}
= 0,
$$

(31)
for every index \( i \) corresponding to sensor \( i, 1 \leq i \leq n_s \), and for every \( k \geq 0 \), where (31) only needs to be verified for \( k \in \{1, \ldots, h\} \) such that sensor \( i \) is sampled at \( t_k \). The \( c^{ik} \) are obtained from (20). We establish this by proving that each row of (31) imposes the same restriction as each row of (20). It is easy to see that to do so, it suffices to consider, without loss of generality, a single sensor \( (i = n_s = 1) \) and a single set of \( \kappa_1 \) rows corresponding to the eigenvalue \( \mu_1 \), i.e., prove that (31) with \( n_w = \kappa_1 \), imposes the same restrictions on \( c^{1k} \) as the following set of equations

\[
\begin{bmatrix}
\mu_1^{1-k} & \mu_1^{2-k} & \cdots & \mu_1^{n_w-k} \\
\tau_1^{1-k} & \tau_1^{2-k} & \cdots & \tau_1^{n_w-k} \\
\vdots & \vdots & \ddots & \vdots \\
(\tau_1^{1-k})^{n_s-1} & (\tau_1^{2-k})^{n_s-1} & \cdots & (\tau_1^{n_s-k})^{n_s-1}
\end{bmatrix}
\begin{bmatrix}
c_1^{1k} \\
c_1^{2k} \\
\vdots \\
c_1^{n_w-k}
\end{bmatrix}
= \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]  

(32)

We argue by induction. The first row of (32) imposes the restriction

\[
\begin{bmatrix}
\mu_1^{1-k} & \mu_1^{2-k} & \cdots & \mu_1^{n_w-k}
\end{bmatrix}
\begin{bmatrix}
c_1^{1k}
\end{bmatrix}
= 1
\]  

(33)

while the first row of (31) imposes the restriction

\[
\begin{bmatrix}
m_1^{k-1-k} & m_1^{k-2-k} & \cdots & m_1^{k-n_w-k}
\end{bmatrix}
\begin{bmatrix}
c_1^{1k}
\end{bmatrix}
= m_1^k
\]  

(34)

which are obviously equivalent since \( m_1 \neq 0 \). It is also insightful to see that the second row of (32), given by

\[
\begin{bmatrix}
\tau_1^{1-k} & \tau_1^{2-k} & \cdots & \tau_1^{n_w-k}
\end{bmatrix}
\begin{bmatrix}
c_1^{1k}
\end{bmatrix}
= 0,
\]  

(35)

imposes the same restriction as the second row of (31), given by

\[
\begin{bmatrix}
(k-\tau_1^{1-k})m_1^{k-1-k} & \cdots & (k-\tau_1^{n_w-k})m_1^{k-n_w-k}
\end{bmatrix}
\begin{bmatrix}
c_1^{1k}
\end{bmatrix}
= km_1^k
\]  

(36)

where we used (33).

Now assuming that the first \( 1 \leq r-1 < n_w - 1 \) rows of (31) impose the same restriction as the first \( 1 \leq r-1 < n_w - 1 \) rows (32), we prove that the same is true for the row \( r \), i.e.,

\[
\begin{bmatrix}
\tau_r^{1-k} & \tau_r^{2-k} & \cdots & \tau_r^{n_w-k}
\end{bmatrix}
\begin{bmatrix}
c_1^{1k}
\end{bmatrix}
= \tau_r^k
\]  

(37)

To this effect, note that we can write (30) as

\[
f_r[k-\tau] = \binom{k-\tau}{r} m_1^{k-r-\tau}
\]

\[
= \sum_{m=0}^{r-1} a_m k^{(r-m)} m_1^{k-r-\tau}
\]

(38)

for some coefficients \( a_m \) implicitly defined from

\[
\binom{k-\tau}{r} = \frac{1}{r!} (k-\tau)(k-\tau-1) \cdots (k-\tau-(r-1))
\]

\[
= \sum_{m=0}^{r} a_m k^{(r-m)} m.
\]
Note that \( a_r = \frac{(-1)^r r!} \). If we replace (38) in (37) and use the fact that the first \( r - 1 \) restrictions of (32) hold, we obtain
\[
a_r \left( (\tau_1^{1k})^r \mu_j^{k-r-\tau_1^{1k}} \cdots (\tau_n^{1w})^r \mu_j^{k-r-\tau_n^{1w}} \right) c_k^1 = 0
\]
which is equivalent to the restriction associated with row \( r \) of (32) since \( a_r \neq 0 \) and \( a_r \neq 0 \).

To prove that (15) with matrices (16), (17) satisfies (D3), note that using the interpretation of a blocking zero given in Proposition 19, the assumption (D3) states that \( C_D \) does not block any signal other than the ones generated by the exosystem. To see that this is true, suppose that there exists a signal \( g[k] \) such that there exists an initial condition to the system \( C_D \) such that the output of the system \( C_D \) is identically zero when \( g[k] \) is applied to its input. Then, by construction of \( C_D \), i.e., by the fact that for \( k \geq n_w h, x_D[k] \) will hold the value of the last \( n_w \) samples of every sensor \( 1 \leq i \leq n_s, g[k] \) must satisfy
\[
g[k] + \sum_{i=1}^{n_s} \left[ g[k - \tau_1^i] \right] \left[ g[k - \tau_2^i] \right] \cdots \left[ g[k - \tau_n^i] \right] c_i^k = 0
\]
for every, \( 1 \leq i \leq n_s \) and \( k \in \{1, \ldots, h\} \) such that sensor \( i \) is sampled at \( t_k \). From (31) and uniqueness of the \( c_i^k \) (cf. (19), (20)) we conclude that \( g[k] \) must be a linear combination of the \( f_i[k], 1 \leq i \leq n_w \) and therefore \( g[k] \) can be generated by the exosystem characterized by \( S \).

4.4. System \( C_K \):
The system \( C_K \) takes the form
\[
\begin{align*}
C_K := \{ & \begin{cases} x_K[k+1] = A_{Kk} x_K[k] + B_{Kk} u_K[k], & k \geq 0, \\
y_K[k] = C_{Kk} x_K[k] \end{cases} \\
\end{align*}
\tag{39}
\]
where the matrices \( A_{Kk}, B_{Kk}, C_{Kk}, \) and \( D_{Kk} \) are \( h \)-periodic, i.e., e.g., \( A_{Kk} = A_{K(k+h)} \) and is such that the closed loop of Figure 1 is stable. The next lemma shows that such controller always exists.

**Lemma 8**
Suppose that \( P, C_I, C_D \) are such that (P1)-(P3), (I1)-(I4), (D1)-(D3) hold. Then there exists a stabilizing controller \( C_K \) for the closed loop system of Figure 1 taking the form (39).

To prove Lemma 8 we need the following two results. For two dimensionally compatible LTI systems described by
\[
C_1 := \{ \begin{bmatrix} x_1[k+1] \\ y_1[k] \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1[k] \\ u_1[k] \end{bmatrix}, & k \geq 0, \quad \tag{40} \end{align*}
\]
and
\[
C_2 := \{ \begin{bmatrix} x_2[k+1] \\ y_2[k] \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} x_2[k] \\ u_2[k] \end{bmatrix}, & k \geq 0. \quad \tag{41} \end{align*}
\]
The series of the system \( C_2 \) and \( C_1 \) obtained by making \( u_1[k] = y_2[k] \), is defined by:
\[
C_3 := \{ \begin{bmatrix} x_3[k+1] \\ y_1[k] \end{bmatrix} = \begin{bmatrix} A_3 & B_3 \\ C_3 & 0 \end{bmatrix} \begin{bmatrix} x_3[k] \\ u_2[k] \end{bmatrix}, & k \geq 0, \quad \tag{42} \end{align*}
\]
where
\[
A_3 = \begin{bmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad C_3 = \begin{bmatrix} C_1 & D_1 C_2 \end{bmatrix}. \]
Recall that for an LTI system, say \( (40) \), an eigenvalue \( \lambda \) of \( A_1 \), i.e., there exists \( v_1 \) and \( w_1 \) such that
\[
A_1 v_1 = \lambda v_1 \quad \text{and} \quad w_1^T A_1 = \lambda w_1^T, \quad \text{is observable if} \ C_1 v_1 \neq 0, \quad \text{and controllable if} \ w_1^T B_1 \neq 0. \quad \text{The pair} \ (A_1, C_1) \text{is detectable if all the unstable eigenvalues of} \ A_1 \text{are observable, and the pair} \ (A_1, B_1) \text{is stabilizable if all the eigenvalues of} \ A_1 \text{are contained in} \ \mathbb{C} \}. \]
is stabilizable if all the unstable eigenvalues of $A_1$ are controllable. One can find these definitions in [13] for continuous-time systems, which have an obvious counterpart for discrete-time systems. Denote the set of eigenvalues of $A$ by

$$
\Lambda_A := \{ \lambda : Av = \lambda v, \text{ for some } v \},
$$

where $A$ can be replaced by $A_1$, $A_2$, $A_3$. Also denote the set of eigenvalues of $A$ which are not eigenvalues of $B$ by

$$
\Lambda_{A/B} := \{ \lambda : \lambda \in \Lambda_A \text{ and } \lambda \notin \Lambda_B \},
$$

where $A$, $B$ can be replaced by $A_1$ and $A_2$.

**Proposition 9**

Consider $C_1$, $C_2$ and the series $C_3$. Then

(i) $\Lambda_{A_3} = \Lambda_{A_1} \cup \Lambda_{A_2}$

(ii) If $\lambda \in \Lambda_{A_1} \cap \Lambda_{A_2}$ then $\lambda$ is an observable eigenvalue of $C_3$ if $\lambda$ is an observable eigenvalue of $C_1$.

(iii) If $\lambda \in \Lambda_{A_1}$ then $\lambda$ is an observable eigenvalue of $C_3$ if $\lambda$ is an observable eigenvalue of $C_2$ and $C_1$ has no invariant zeros at $\lambda$.

(iv) If $\lambda \in \Lambda_{A_2}$ then $\lambda$ is a controllable eigenvalue of $C_3$ if $\lambda$ is a controllable eigenvalue of $C_2$.

(v) If $\lambda \in \Lambda_{A_1}$ then $\lambda$ is a controllable eigenvalue of $C_3$ if $\lambda$ is a controllable eigenvalue of $C_1$ and $C_2$ has no invariant zeros at $\lambda$.

**Proof**

See [17, Ch. 6].

The series of two periodically time-varying systems is defined similarly to the series of two LTI systems.

**Proposition 10**

The lift of the series of two periodic systems is the series of the LTI lift systems of each periodic system.

**Proof**

Obtained by direct replacement.

We shall need the following proposition.

**Proposition 11**

Consider the system

$$
x[k + 1] = Ax[k]
$$

$$
y[k] = \text{bdig}(\gamma^1_k, \ldots, \gamma^n_k)Cx[k], \quad k \geq 0,
$$

where $y[k] \in \mathbb{R}^{n_y}$. The $\gamma^i_k \in \{0, 1\}$, $1 \leq i \leq n_y$ are $h-$periodic, i.e., $\gamma^i_k = \gamma^i_{k+h}$, and for each $i$ are equal to 1 at least once in a period. The system (43) is detectable if and only if the pair $(A, C)$ is detectable.

**Proof**

See [7]

We shall also need to take into account that if we apply the lift for an LTI system we obtain the following eigenvalues of the lifted system.

$$
\Lambda_{Ah} := \{ \lambda^h, \lambda \in \Lambda_A \}
$$

(44)

Lemma 8 is proved next.
Proof

In [18], it is shown that a periodic system $R$ taking the form (22) is detectable and stabilizable if and only if there exists a periodic linear controller, taking the form (39) such that the closed loop system is asymptotically stable. Thus it suffices to prove the stabilizability and detectability of the periodic system obtained by computing the series of $C_I$, $P$, and $C_D$, which is a periodic system that we shall term $C_A$. This can be proved by establishing stabilizability and detectability of the lifted LTI system, denoted by $\tilde{C}_A$. From Proposition 10 the lift of $C_A$ is the series of the lift of each individual system $C_I$, $P$, and $C_D$, which are denoted by $\tilde{C}_I$, $\tilde{P}$, and $\tilde{C}_D$, respectively. Thus, we only have to prove the observability and controllability of the unstable eigenvalues of $\tilde{C}_A$, corresponding to the unstable eigenvalues of $\tilde{C}_I$ and $\tilde{P}$, since $\tilde{C}_D$ is a deadbeat system and thus stable.

We start by establishing stabilizability. From Proposition 9(iv) the controllability of the unstable eigenvalues of $C_A$ associated with $P$ or with $C_I$, does not depend on $C_D$, and since $C_I$, $P$ are LTI systems, this amounts to an LTI test. In fact, for an LTI system, considered to be a special case of a periodic system with period $h$, stabilizability of the lift is equivalent to stabilizability of the original LTI system, which can be concluded from the definition of stabilizability for periodic systems. Thus, it suffices to prove that the LTI series of $C_I$, $P$, which we denote by $C_{IP}$, is stabilizable. The eigenvalues of $C_{IP}$ belonging to $\Lambda_{C_I/C_P}$ are controllable due to (I1) and Proposition 9(iv). The eigenvalues of $C_{IP}$ belonging to $\Lambda_{C_P}$ are controllable due to (P1), (I2), and Proposition 9(v).

We prove next detectability. To prove that observability of the unstable eigenvalues of $\tilde{C}_A$ belonging to $\Lambda_{C_I}$, which in turn correspond to unstable eigenvalues of $C_I$, due to (44), it suffices to prove that these eigenvalues are detectable from the plant output $y_r[k]$, i.e., if we consider the system obtained by computing the series of $C_I$ and $P_r$ where $P_r$ is the system obtained by considering the plant with $y_r[k]$ as the only output. Again this amounts to an LTI test, since the series of $C_I$ and $P_r$ is an LTI system with multi-rate measurements as in Proposition 11, where we state that for such systems detectability can be proved by an LTI test. The desired conclusion follows then from Assumptions (I1), (I4), (P2), and Proposition 9(iii).

To see the observability of the unstable eigenvalues of $\tilde{C}_A$ that correspond to those of $P$ it suffices to consider those not observable through $y_r[k]$. By this we mean, the eigenvalues of the plant $P$ that are not an observable eigenvalue of the pair $(A, C_P)$. The existence of such eigenvalue, is equivalent to existing an initial condition for the plant $x[0]$ such that

$$\begin{align*}
x[k + 1] &= Ax[k] \\
y_r[k] &= \Gamma_C C_r x[k], \quad k \geq 0,
\end{align*}$$

has zero output, i.e., $y_r[k] = 0, \forall k \geq 0$. From assumption (P3) there cannot exist an initial condition for the unforced plant such that (45) has a zero output $y_r[k] = 0$ and $y_m[k]$ can be generated by the exosystem. Since by assumption (D3), the system $C_D$ does not cancel signals other than those generated by the exosystem, we obtain that this yields a non-zero output for $C_D$ and thus these eigenvalues of $A$ are also observable for the system $\tilde{C}_A$.

\[ \square \]

4.5. Output regulation

The following result and Lemma 8 allows to conclude the main result of the paper, i.e., Theorem 4.

**Theorem 12**

Suppose that (P1)-(P3) hold for the plant $P$. Then one can find $C_I$ such that (I1)-(I4) holds, $C_D$ such that (D1)-(D3) hold, and $C_K$ such that the closed-loop in Figure 1 is stable. Moreover, output regulation is achieved even in the presence of plant uncertainties that do not destroy closed-loop stability.

\[ \square \]

**Proof**

The first part of the Theorem has been established in Proposition 6 and 7 and Lemma 8. Thus, it suffices to prove that output regulation is achieved. We start by writing the equations for the series
connection of \(C_I, P,\) and \(C_D,\) which are given by
\[
x_A[k+1] = A_{Ak}x_A[k] + B_{Ak}u_A[k] + B_{Awk}w[k], \quad k \geq 0,
\]
\[
y_A[k] = C_{Ak}x_A[k] + D_{Awk}w[k]
\]
where
\[
A_{Ak} = \begin{bmatrix}
A & BC_I & 0 \\
0 & A_I & 0 \\
B_{Dk} \Gamma_{mk} C_m & 0 & A_{Dk}
\end{bmatrix}, \quad B_{Ak} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]
\[
C_{Ak} = \begin{bmatrix}
D_{Dk} \Gamma_{mk} C_m & 0 & C_{Dk} \\
\Gamma_{rk} C_r & 0 & 0
\end{bmatrix}, \quad B_{Awk} = \begin{bmatrix} B_V C_V \\ 0 \\ 0
\end{bmatrix}, \quad D_{Awk} = \begin{bmatrix} 0 \\ -\Gamma_{rk} C_R
\end{bmatrix}.
\]

Let \(C_K\) be a stabilizing controller described by (39), whose existence is established in Lemma 8. Then the closed-loop is described by
\[
\begin{bmatrix}
x_A[k+1] \\
x_K[k+1]
\end{bmatrix} = \begin{bmatrix} A_{Ak} & B_{Ak} C_{Dk} \\
B_{Kk} C_A & A_{Kk}
\end{bmatrix} \begin{bmatrix} x_A[k] \\
x_K[k]
\end{bmatrix} + \begin{bmatrix} B_{Awk} \\
B_{Kk} D_{Awk}
\end{bmatrix} w[k]. \tag{46}
\]
where \(w[k]\) is described by (9). Using the fact that the periodic system (46) is stable, since \(C_K\) is a stabilizing controller, from Proposition 19 in the appendix, we conclude that there exists unique \(\Pi_k, k \in \{1, \ldots, h\},\) such that
\[
\begin{bmatrix} A_{Ak} & B_{Ak} C_{Kk} \\
B_{Kk} C_A & A_{Kk}
\end{bmatrix} \Pi_k + \begin{bmatrix} B_{Awk} \\
B_{Kk} D_{Awk}
\end{bmatrix} = \Pi_{|k+1]} S \tag{47}
\]
and for any initial condition \((x_A[0], x_K[0])\) the state of the system tends asymptotically to
\[
\begin{bmatrix} x_A[k] \\
x_K[k]
\end{bmatrix} = \Pi_k w[k]. \tag{48}
\]

We provide a solution to (47), which is unique as mentioned above, and see that corresponding to such solution, we have that (48) is such that output regulation is achieved, i.e., \(y_{rk} = r[k],\) or equivalently, using (48),
\[
[C_r \ 0 \ 0] \Pi_k w[k] = C_R w[k]. \tag{49}
\]
Such solution \(\Pi_k\) is obtained as follows. Make
\[
\Pi_k = [\Pi_{Ak}^T \ 0]^T \tag{50}
\]
where
\[
\Pi_{Ak} = [\Pi_P^T \ \Pi_I^T \ \Pi_{DK}^T]^T. \tag{51}
\]
Then, from (47) we obtain that (51) must be such that
\[
\begin{bmatrix} \Pi_P \\
\Pi_I \\
\Pi_{D[k+1]} \\
0 \\
0
\end{bmatrix} S = \begin{bmatrix} A & BC_I & 0 \\
0 & A_I & 0 \\
B_{Dk} \Gamma_{mk} C_m & 0 & A_{Dk} \\
B_{K1k} D_{Dk} \Gamma_{mk} C_m & 0 & B_{K1k} C_{Dk} \\
B_{K2k} \Gamma_{rk} C_r & 0 & 0
\end{bmatrix} \begin{bmatrix} \Pi_P \\
\Pi_I \\
\Pi_{D[k+1]} \\
0 \\
0
\end{bmatrix} + \begin{bmatrix} B_V C_V \\
0 \\
0 \\
0 \\
-B_{K2k} \Gamma_{rk} C_R
\end{bmatrix} \tag{52}
\]
where \(B_{K1k}\) and \(B_{K2k}\) are appropriate partitions of \(B_{Kk} = [B_{K1k} B_{K2k}].\)
The matrices \(\Pi_P, \Pi_I, \Pi_{DK}\) are obtained as follows.
(i) Take $\Pi_P$ to be the solution, along with $E \in \mathbb{R}^{m \times n_w}$, to

$$
\Pi_P S = A\Pi_P + BE + B_V C_V \\
C_r \Pi_P = C_R
$$

which as explained in [15] exists if and only if (P2) holds.

(ii) Take $\Pi_I$ to be the unique solution to

$$
A_I \Pi_I = \Pi_I S \\
C_I \Pi_I = E
$$

where $E$ is, along with $\Pi_P$, the solution to (53). Note that such solution $\Pi_I$ exists due to Assumption (I3).

(iii) Take $\Pi_{Dk}$ to be the unique solution to

$$
A_{Dk} \Pi_{Dk} + B_{Dk} \Gamma_{mk} C_m \Pi_P = \Pi_{D[k+1]} S \\
C_{Dk} \Pi_{Dk} + D_{Dk} \Gamma_{mk} C_m \Pi_P = 0
$$

which exists due to Assumption (D3) and is unique due to Proposition 19 and the fact that $C_D$ is a stable system.

By construction, we conclude that $\Pi_k$ given by (50), where $\Pi_{Ak}$ is given by (51), and $\Pi_P$, $\Pi_I$ and $\Pi_{Dk}$ are described by (i), (ii) and (iii), respectively, satisfies (47) and (49) and therefore output regulation is achieved.

Note that in the proof we only required the controller $C_K$ to stabilize the closed-loop. If the plant describing matrices are not known but $C_K$ still stabilizes the closed-loop the proof remains unchanged.

Proof

(of Theorem 4) Follows as a Corollary of Lemma 8 and Theorem 12, provided that we notice that the matrices $B_J$ and $C_J$ can be chosen such that $C_I$, given by (13), satisfies (I1)-(I4) (cf.Propriosition 6) and that we notice that $C_D$, given by (16), (17), (18) satisfies (D1)-(D3) (cf.Propriosition (7)).

5. ACADEMIC EXAMPLE

Example 13

The following continuous-time linear system is considered in [14].

$$
P_C = \begin{cases}
\dot{x}_1(t) = -x_1(t) - x_2(t) \\
\dot{x}_2(t) = -x_2(t) + u(t) \\
\dot{x}_3(t) = -x_2(t) + 0.5x_3(t) + u(t), & t \geq 0.
\end{cases}
$$

A sensor measuring $x_1(t)$ works at a fixed sampling period of $t_s = 0.25$, while the actuator update mechanism can be done at a sampling period of $t_u = 0.05$. We wish that $x_1(t)$ tracks a prescribed reference signal. However, one can verify that $P_C$ is not detectable from $x_1(t)$. Therefore we cannot use the solution provided in [11] for output regulation of square multi-rate systems, since this solution would not guarantee closed-loop stability. We consider that $x_3(t)$ is also available for feedback at a sampling period of $t_2 = 0.1$. According to the framework of Section 2 we have that $t_s = 0.05$, $h = 10$. The system is now detectable from $(x_1(t), x_3(t))$, and since the sampling period $t_s = 0.05$ is not pathological (see, e.g., [19], the discretization of (55) is also detectable with respect to the multi-rate outputs $y_r[k] := \Gamma_{rk} x_1(kt_s)$, and $y_m[k] := \Gamma_{mk} x_3(kt_s)$ where the $h$-periodic matrices $\Gamma_{mk}$, $\Gamma_{rk}$, $\Omega_k$ are determined by

$$
\Gamma_{mk} = \begin{cases}
1 & k \text{ odd} \\
0 & \text{otherwise}
\end{cases}, \quad \Gamma_{rk} = \begin{cases}
1 & k = 1, 6 \\
0 & \text{otherwise}, \quad k \geq 0.
\end{cases}
$$
and we also used Proposition 11 to conclude the detectability of the multi-rate discretization. Consider the problem of designing a linear controller for $P_C$ that achieves closed loop stability and such that the output $y_r[k]$ tracks the reference $r[k]$ with zero steady-state error, where $r[k]$ is described by

$$w[k+1] = Sw[k], \quad w[0] = w_0,$$
$$r[k] = C_Rw[k], \quad k \geq 0,$$

where

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C_R = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$  

Thus, the reference $r[k]$ takes the form

$$r[k] = c_1 + c_2 k, \quad k \geq 0,$$

where $c_1, c_2$ can be made arbitrary. Such non-constant references prevent the use of the results from [14]. Contrarily to [11], this problem can be solved with the solution we provide in the present paper. In fact, the discretization of $P$ can be verified to satisfy (P1)-(P3) and therefore a linear controller with the structure depicted in Figure 1 can be synthesized for this system. The stabilizing controller $C_K$ in Figure 1 can be obtained, e.g., from the solution provided in [7] and the gains $c_k$ determining the system $C_D$ are in this special case time-invariant and given by

$$c^{1k} = [-2 \ 1]^T, \forall k \text{ odd},$$

and do not need to be specified for $k$ even since $y_m[k]$ is only sampled for odd $k$. In Figure 2, we show the response of the output $x_1[k]$ when a reference signal $r[k]$ consisting of a concatenation of signals taking the general form (56) is applied to the closed-loop system. We see that zero steady-state error is achieved after a transitory period, as desired. In Figure 3 we show several signals of the closed-loop for a short period of time where a transition of the reference signal occurs. Note that before the transition, in steady state, the output $y_D[k]$ of the blocking system $C_D$ is zero as desired, and that at steady state $u[k]$ has the desired value to be applied to the plant so that output regulation can be achieved.

6. CONCLUSIONS AND FUTURE WORK

We propose a solution to the output regulation problem for non-square multi-rate systems. This solution makes possible to control a plant with several sensors sampled at different frequencies.
taking advantage of all the available measurements for feedback, while driving a subset of the outputs to a prescribed reference.

A topic for future work is to consider the case where some or all the components of the actuation update mechanism may not be available at the sampling period at which the controller operates, i.e., the actuators may also be multi-rate. Note that due to the linearity of the plant, if output regulation of some variables of the output is to be achieved then in general the actuation signal should be composed of the same frequency content signals as the desired output values. Thus, if one considers standard sample and hold device, as in the present paper, but working at different rates, it does not appear to be simple in general to guarantee that such signal is generated. Note that we were able to consider a multi-rate actuation updating mechanism in [5], where we restrict the reference to constant signal, since the standard sample and hold mechanism is capable of generating constant references signals. One solution that appears to be promising is to consider a generalized sample-and-hold device, as in [20], i.e., a hold device that provides signals to the plant between sampling instants generated by a linear time invariant system, instead of simply holding its input as a standard zero order hold. It is expectable that if one combines the solution we provide here with the work of [20], that as in [20] asymptotic tracking can be achieved with zero steady-state error also in continuous-time.

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A. BLOCKING ZEROS WITH RESPECT TO A MATRIX

Consider an LTI system

\[
\begin{align*}
x[k+1] &= Ax[k] + Bu[k] \\
y[k] &= Cx[k] + Du[k], \quad k \geq 0,
\end{align*}
\]
and a periodic linear system
\[ x[k + 1] = A_k x[k] + B_k u[k] \]
\[ y[k] = C_k x[k] + D_k u[k], \quad k \geq 0, \tag{58} \]

where both (57) and (58) have the same number of inputs and outputs, and \( x[k] \in \mathbb{R}^n, u[k] \in \mathbb{R}^m, y[k] \in \mathbb{R}^n \).

We generalize the definition of blocking zero for LTI systems (see, e.g. [13]) and subsequently introduce the notion for periodic system. This generalization lies at the heart of the solution we propose in Figure 1 since it is the property that the key system \( C_D \) is required to satisfy (cf. Subsection 4.3).

**LTI systems**

We start by introducing the following definition.

**Definition 14**

We say that (57) has a *blocking zero with respect to a matrix* \( R \in \mathbb{R}^n \), if there exists \( \Pi \in \mathbb{R}^{n \times n} \) such that for every \( E \in \mathbb{R}^{n \times n} \) we have that

\[
P R = A \Pi + B E \tag{59}
\]
\[
0 = C \Pi + D E \tag{60}
\]

To interpret the nomenclature blocking zero used in the previous definition, we need the following proposition. Consider the system

\[
w_R[k + 1] = R w_R[k], \quad k \geq 0,
\]
\[
u[k] = C_R w_R[k]. \tag{61}
\]

**Proposition 15**

Suppose that the input of the system (57), is generated by (61). Then there exists a solution \( \Pi \in \mathbb{R}^{n \times n} \) to

\[
P R = A \Pi + B C_R \tag{62}
\]

if and only if the solution to (58) satisfies \( x[k] = \Pi w[k] \) when \( x[0] = \Pi w[0] \) for an arbitrary \( w[0] \in \mathbb{R}^n \). Moreover, if \( A \) has all its eigenvalues inside the open unit disc and \( R \) has all its eigenvalues outside the open unit disk then the solution to (62) is unique, and \( x[k] \to \Pi w[k] \) as \( k \to \infty \) for every initial condition \( x[0], w[0] \).

**Proof**

See [17, Ch. 6].

From Proposition 15, we can conclude that according to Definition 14, the system (57) has a blocking zero with respect to \( R \) if the following holds. If the input of (57) is generated by (61) for an arbitrary matrix \( C_R = E \) and the initial condition of (57) satisfies \( x[0] = \Pi w[0] \), then the output of (57) is identically zero. This suggests the nomenclature blocking zero with respect to the matrix \( R \).

According to [13, Def. 3.14] the system (57) has a blocking zero at a complex number \( z_0 \in \mathbb{C} \) that does not belong to the spectrum of \( A \), if

\[
C(z_0 I - A)^{-1} B + D = 0. \tag{63}
\]

As stated in the next proposition, in the case where \( R \) is scalar our definition coincides with the one from [13, Def. 3.14].

**Proposition 16**

Suppose that \( R = z_0 \in \mathbb{C} \) is a complex number that does not belong to the spectrum of \( A \). Then, the system (57) has a blocking zero according to Definition 14 if and only if (63) holds.

**Proof**

To prove sufficiency it suffices to multiply (59) by \( C(z_0 I - A)^{-1} \) and sum the result to (60). One obtains \( (C(z_0 I - A)^{-1} B + D)E = 0 \) for every \( E \), which implies (63). To prove necessity take \( \Pi = (z_0 I - A)^{-1} BE \), and see that (59)-(60) holds if (63) holds.
Note that if the matrix \( R \) has only simple eigenvalues, then having a blocking zero with respect to \( R \), is equivalent to having \( n_R \) zeros with respect to all the eigenvalues of \( R \). Our definition of a blocking zero is broader since it allows to consider a matrix \( R \in \mathbb{R}^{n_R} \) with a Jordan block structure

\[
R = \begin{bmatrix}
\lambda & 1 & 0 & \ldots & 0 \\
0 & \lambda & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & \lambda
\end{bmatrix}
\]  

and conclude if (57) has a blocking zero with respect to \( R \), then every signal taking the form

\[
u[k] = \sum_{l=0}^{n_R-1} c_l \left( \frac{k}{l} \right) \lambda^{k-l},
\]

is blocked (yields a zero output) by (57).

It is also important to note that Definition 14 is distinct from the definition of blocking zero structure (for multi-variable systems) given in [21, p. 241, eqs. (11),(12)]. In fact, in the notation of the present paper, the definition of blocking zero structure would say that there exist matrices \( \Pi \) and \( E \) such that the equations (59), (60) hold, while the definition given here says that there exists a matrix \( \Pi \) such that the equations hold for every \( E \). The following example illustrates the difference.

**Example 17**
Suppose that \( R = 1, C_R = [1 \ 0]^\top \). Consider two linear systems taking the form (57) characterized by

\[P_1 : A = 0_{2\times 2}, B = I_{2\times 2}, C = -I_{2\times 2}, D = I_{2\times 2}\]

and

\[P_2 : A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, B = I_{2\times 2}, C = -I_{2\times 2}, D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \].

While \( P_1 \) has a blocking zero with respect to \( R = 1 \), characterized by \( \Pi = E \) in (59), (60) for any \( E \), \( P_2 \) does not have a time-invariant blocking zero with respect to \( R = 1 \), but has a blocking zero structure, i.e., the equations (59), (60) have a solution when \( E = C_R = [1 \ 0]^\top \) (given by \( \Pi = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)), but do not necessarily have a solution for an arbitrary \( E \).

**Periodic Systems**

We extend the definition of blocking zero to periodic systems as follows. Recall that \([r]\) denotes the remainder of the division by \( h \).

**Definition 18**
We say that (58) has a **blocking zero with respect to \( R \in \mathbb{R}^{n_R} \)**, if there exists \( \Pi_r \in \mathbb{R}^{n_A \times n_R} \) such that for every \( E_r \in \mathbb{R}^{n_B \times n_R} \), \( 1 \leq r \leq h \), we have that

\[
\Pi_{r+1}R = A_r\Pi_r + B_rE_r \quad 0 = C_r\Pi_r + D_rE_r, \quad 1 \leq r \leq h.
\]

Moreover, if (66) holds when \( E_r = E \) for an arbitrary matrix \( E \in \mathbb{R}^{n_B \times n_R} \) we say that (58) has a **time-invariant blocking zero with respect to \( R \)**.

To interpret the nomenclature used in the previous proposition, we need the following proposition. Consider the system

\[
w_R[k+1] = R w_R[k], \quad k \geq 0, \quad u[k] = C_{Uk} w_R[k].
\]

where \( C_{Uk} = C_{U(k+h)}, \forall k \geq 0 \) and \( R \) is now assumed to be invertible.

**Proposition 19**
Suppose that the input of the system (58), is generated by (67). Then, there exists a solution \( \Pi_r \in \)
\[ \mathbb{R}^{n_A \times n_B}, 1 \leq r \leq h \] 
\[ \Pi_{[r+1]} R = A_r \Pi_r + B_r C_U R, 1 \leq r \leq h \]  
(68)

if and only if the solution to (58) satisfies \( x_k = \Pi_{[k]} w_k, k \geq 1 \) when \( x[0] = \Pi_k w[0] \) for an arbitrary \( w[0] \in \mathbb{R}^{n_B} \). Moreover, if (58) is stable, and \( R \) has all its eigenvalues outside the open unit disk, then the solution to (68) is unique, and \( x[k] \to \Pi_{[k]} w[k] \) as \( k \to \infty \) for every initial condition \( x[0], w[0] \).

\[ \square \]

**Proof**

See [17, Ch. 6].

From Proposition 19, we can conclude that according to Definition 18, the system (58) has a blocking zero with respect to \( R \) if the following hold. If the input of (58) is generated by (67) for an arbitrary matrices \( C_{Uk} = E_k \), then the output of (57) is identically zero. If this hold when (67) is time-invariant, then (58) with \( n_A = n_B \) has a time-invariant blocking zero with respect to \( R \).

The relation between the blocking zeros of a periodic system and the blocking zeros of its lift is provided in the next result.

**Proposition 20**

The periodic system (58) has a blocking zero with respect to \( R \) if and only if its LTI lifted system has a blocking zero with respect to \( R^h \).

\[ \square \]

**Proof**

From Proposition 19 we conclude that (58) has a blocking zero with respect to \( R \), if and only if for every signal generated by (67), the output is zero. This holds if and only if (23) has zero output when the input is generated by

\[ \hat{u}[l + 1] = R^h \hat{w}[l] \]
\[ \hat{u}[l] = F w[k], \]  
(69)

where

\[ F = \begin{bmatrix} C_{U0} & C_{U1} R & \cdots & C_{U(h-1)} R^{h-1} \end{bmatrix} \]

Since the \( C_{Uk}, k \in \{1, \ldots, h\} \) are arbitrary and \( R \) is invertible, we see that \( F \) can be made arbitrary, and using Proposition 15 we conclude that (23) has a blocking zero with respect to the matrix \( R^h \).

\[ \square \]

**REFERENCES**