Rollout Event-Triggered Control: Beyond Periodic Control Performance

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Abstract—Cyber-Physical Systems (CPSs) resulting from the interconnection of computational, communication, and control (cyber) devices with physical processes are wide spreading in our society. In several CPS applications it is crucial to minimize the communication burden, while still providing desirable closed-loop control properties. To this effect, a promising approach is to embrace the recently proposed event-triggered control paradigm, in which the transmission times are chosen based on well-defined events, using state information. However, few general event-triggered control methods guarantee closed-loop improvements over traditional periodic transmission strategies. Here, we provide a new class of event-triggered controllers for linear systems which guarantee better quadratic performance than traditional periodic time-triggered control using the same average transmission rate. In particular, our main results explicitly quantify the obtained performance improvements for quadratic average cost problems. The proposed controllers are inspired by rollout ideas in the context of dynamic programming.

I. INTRODUCTION

Cyber devices capable of sensing, processing, and communicating information of interest are wide spreading in our society, creating new opportunities to make our physical processes operate exceedingly better. In fact, the number of applications in which communication, computation and control elements (the cyber part) go hand in hand with motion, energy, climate, and human processes (the physical part) is steadily growing in intelligent transportation, smart buildings, energy networks, healthcare, and robotics (see, e.g., [2], [3], [4], [5], [6], respectively). To meet the challenges arising in many of these applications the traditional separation-of-concerns principle in designing control, communication, and computational algorithms, must be abandoned in favor of an integrated approach. This can lead to dramatic communication and computation savings in control applications, which is crucial to prevent overloading existing and future communication networks, to extend the battery life of cyber devices, and to enable cost-efficient control solutions (see, e.g., [7], [8]).

A research area providing integrated communication and control algorithms that deal with the need to reduce the communication load in (networked) control systems, while at the same time guaranteeing desirable stability and performance properties, is that of event-triggered control (ETC). The key idea of ETC is that transmission times in a networked control loop are triggered based on events (using, e.g., state or output information), as opposed to being time-triggered as in traditional periodic control.

Extensive research has been conducted on ETC over the past few years leading to various types of ETC strategies; see [9] for a recent overview. For instance, [10] proposes that transmissions should only be triggered when needed to guarantee a certain decrease condition for a Lyapunov function; [11], [12] analyze, in different contexts, the case in which transmissions are triggered only when the loop tracking error exceeds a given threshold; in [13] transmissions are triggered when the error between the measured state and the state of a model-based estimator used by a control input generator is large. Several related problems have been studied in the literature, including self-triggered implementations [14]–[17], co-design [18], [19], discrete-time variants [20]–[23], and periodic event-triggered control [24]. Another line of research formulates ETC in the scope of optimal control by considering cost functions that penalize transmissions [25]–[30]. Some recent works, e.g., [31]–[35], propose model predictive control methods to address related optimal event-triggered control problems. See also [36] for an early work using model predictive control to minimize bandwidth utilization.

Although the large majority of the works on ETC show very promising results, there are few ETC methods which guarantee better closed-loop performance/average transmission rate trade-offs than traditional periodic control. The works [12], [37], [38] proposed event-triggered control laws which have this property, considering a quadratic performance index, but the analysis is restricted to first-order systems. Recently, [39] extended the ideas of [12] to a class of second-order systems, formally establishing the desired ETC performance improvement property over periodic control. However, as acknowledged in [39], it is difficult to extend the results for the considered class of event-triggered controllers to higher order systems. Also in the context of first-order systems, [40], [41] optimally solve estimation and control problems, respectively, in which a quadratic cost is to be minimized, subject to constraints on the number of samples. Yet, in general, it is extremely difficult to obtain optimal event-triggered controllers for higher order systems, although several structural properties of optimal event-triggered controllers can still be inferred (see [25]–[30]).

In the present paper we present a novel class of event-triggered controllers for linear systems of arbitrary (finite)
order which achieve better performance than periodic strategies using the same average transmission rate. Performance is measured by a quadratic cost as in the well-known Linear Quadratic Regulator (LQR) and Linear Quadratic Gaussian (LQG) problems (see, e.g., [42]–[44]). Our method, inspired by rollout ideas in the context of dynamic programming [42], consists in choosing, in a receding horizon fashion, optimal control inputs and transmission decisions over a horizon assuming that a base policy, conveniently picked as the optimal periodic control strategy, is used after the horizon.

For this new ETC scheme, we show that, under mild conditions, a \textit{strict} performance improvement with respect to periodic control can be guaranteed both for average and discounted quadratic costs using the same average transmission rate. For the average cost problem we explicitly quantify these performance improvements. As quantifying the performance improvements of rollout algorithms is a hard problem\footnote{As stated in [42, p. 338]: ‘Empirically, it has been observed that the rollout policy typically produces considerable (and often dramatic) cost improvement over the base policy. However, there is no solid theoretical support for this observation.’}, this latter result is the main technical contribution of the paper.

We illustrate the applicability of our event-triggered control method in the problem of controlling a mass-spring linear system. The results show that our method can achieve a closed-loop performance significantly beyond the performance of periodic control using the same average transmission rate.

The remainder of the paper is organized as follows. Section II formulates the problem, and Section III describes the new rollout ETC method. Our main results addressing the performance properties of the proposed method are presented in Section IV. Section V discusses how to extend the main ideas to other networked control configurations. A numerical example is given in Section VI while Section VII provides concluding remarks. The proofs of the main results are given in Section VIII.

\textbf{Notation} : The \(n \times m\) zero matrix is denoted by \(0_{n \times m}\) and the \(n\)-dimensional identity matrix is denoted by \(I_n\). When clear from the context, we omit the subscripts and write \(0\) and \(I\). The trace of a square matrix \(A\) is denoted by \(\text{tr}(A)\).

\section{Problem Formulation}

Consider a continuous-time plant modeled by the following stochastic differential equation

\[ dx_C = (A_C x_C + B_C u_C)dt + B_\omega d\omega, \quad x_C(0) = x_0, \quad t \in \mathbb{R}_{\geq 0}, \tag{1} \]

where \(x_C(t) \in \mathbb{R}^{n_x}\) is the state and \(u_C(t) \in \mathbb{R}^{n_u}\) is the control input at time \(t \in \mathbb{R}_{\geq 0}\), and \(\omega\) is an \(n_\omega\)-dimensional Wiener process with incremental covariance \(I_{n_\omega}\) (cf. [43]). Performance is measured by the discounted cost

\[ \int_0^\infty \mathbb{E}[e^{-\alpha t} g_C(x_C(t), u_C(t))]dt, \tag{2} \]

where \(g_C(x, u) := x^T Q_C x + u^T R_C u\), for positive semi-definite matrices \(Q_C\) and \(R_C\), and \(\alpha_C \in \mathbb{R}_{\geq 0}\). To guarantee that (2) is bounded we assume that \(\alpha_C\) may only take the value \(\alpha_C = 0\) if \(B_\omega = 0\). For the undiscounted case \(\alpha_C = 0\) in which (1) is disturbed by Gaussian noise \((B_\omega \neq 0)\) performance is measured by the following average cost

\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}[g_C(x_C(t), u_C(t))]dt. \tag{3} \]

Performance indexes (2) and (3) are widely used in control problems. In particular, when \(\alpha_C = 0\) the problems of designing a feedback strategy for the control input \(u_C\) to minimize (2) and (3) can be seen as versions of the well-known LQR and LQG problems, respectively. The LQR and LQG problems are also considered in the context of sampled-data systems [45], in which case \(u_C\) is a staircase signal updated \textit{periodically} and designed to minimize discrete-time equivalents of (2) and (3), respectively. The main motivation of the present work is to show that, by properly choosing the actuation update times (which shall coincide with transmission times in networked control settings) in a non-periodic fashion, one can achieve better performance indexes as considered in the LQR and LQG problems, using the same average actuation (or transmission) rate.\footnote{In fact, while the case \(\alpha_C > 0\) is interesting in its own right, here we consider a discounted cost (2) mainly for convenience and we shall be mostly interested in \(\alpha_C = 0\) (cf. Section III-D).}

For ease of exposition, we assume that a scheduler-controller pair is collocated with the plant sensors and that it is connected to the actuators by a communication network. The scheduler-controller periodically samples the state of the plant \(x_C\) and decides whether or not to compute and transmit control and measurement data over a network to the actuators, as it is common in so called periodic event-triggered control (see, e.g., [24]). The setup is depicted in Figure 1, where the scheduler-controller is denoted by event-triggered controller (ETC). While we consider this setup for concreteness, the ideas of our proposed methods can be applied in a straightforward manner also to other configurations (cf. Section V).

![Fig. 1. Setup: the plant operates in continuous-time (continuous-time connections are indicated by thick solid lines); the event-triggered controller operates at discrete times \(\{t_k\}_{k \in \mathbb{N}_0}\) (discrete-time connections are indicated by thin solid lines); transmissions over the communication network occur only at times \(\{t_k | \sigma_k = 1, k \in \mathbb{N}_0\}\) (connections are indicated by thin dashed lines). The event-triggered controller periodically samples the state of the plant and decides the transmission times \(\{t_k | \sigma_k = 1, k \in \mathbb{N}_0\}\) at which it computes the control input and transmits it to the actuators; at these times the actuators receive the control input enforcing it in the plant.](image-url)
We denote the sampling times by $t_k$, $k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, spaced by a baseline period $\tau \in \mathbb{R}_{>0}$, i.e., $t_k = k\tau$, $k \in \mathbb{N}_0$. We assume that the network is always available for transmitting data at times $t_k$, $k \in \mathbb{N}_0$, that the transmission delays are small with respect to $\tau$, and that the probability of a packet drop is small. These assumptions are reasonable in shared networks using Time-Division Multiple Access (TDMA) protocols [46] or if the communication between the scheduler-controller and actuators is made via a point-to-point dedicated link. The scheduler-controller may wish to refrain from transmitting the state to the actuators at the available times $t_k$, $k \in \mathbb{N}_0$, in order to: (i) reduce power consumption [7]; (ii) allow for other (non-critical) data to be transmitted.

Assuming that the actuation is held constant between sampling times and that the transmission delays are negligible we have

$$u_C(t) = u_C(t_k), \quad \forall t \in [t_k, t_{k+1}). \quad (4)$$

Let $\{\xi_k\}_{k \in \mathbb{N}_0}$ be the transmission scheduling sequence defined as

$$\sigma_k := \begin{cases} 1, & \text{if a transmission occurs at } t_k, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, for $k \in \mathbb{N}$, let $x_k := x_C(t_k)$ and $\hat{u}_k := u_C(t_k-1)$, and let $\hat{x}_0 \in \mathbb{R}^{n_x}$ and $\hat{u}_0 \in \mathbb{R}^{n_u}$ be given initial conditions. Furthermore, let $\xi_k := [x_k^\top \hat{u}_k^\top]^\top \in \mathbb{R}^n, k \in \mathbb{N}_0, n := n_x + n_u$, and $u_k$ be the control input sent by the controller to the actuators at times $t_k$, $k \in \mathbb{N}_0$, that satisfy $\sigma_k = 1$; at times $t_k$, $k \in \mathbb{N}_0$, that satisfy $\sigma_k = 0$ we use the notation $u_k := \emptyset$, also used in [29], to denote that $u_k$ is not transmitted. Then, we can write

$$\xi_{k+1} = \begin{cases} A_t \xi_k + B_t \hat{u}_k + w_k, & \text{if } \sigma_k = 1 \\ A_0 \xi_k + w_k, & \text{if } \sigma_k = 0, \quad k \in \mathbb{N}_0. \end{cases} \quad (5)$$

where, for $j \in \{0, 1\}$

$$A_j := \begin{bmatrix} A_t & (1-j)B_t \\ 0 & (1-j)I_{n_u} \end{bmatrix}, \quad B_1 := \begin{bmatrix} B_t \\ I_{n_u} \end{bmatrix},$$

$$A_T := e^{A_C \tau}, \quad \hat{B}_T := \int_0^\tau e^{A_C s} dB_C, \quad (6)$$

and $w_k, k \in \mathbb{N}_0$, is a sequence of zero-mean independent random vectors with covariance $\mathbb{E}[w_k w_k^\top] = \Phi^w_k, k \in \mathbb{N}_0$, where

$$\Phi^w := \begin{bmatrix} \Phi^w_T & 0_{n_x \times n_u} \\ 0_{n_u \times n_x} & \Phi^w_T \end{bmatrix}, \quad \Phi^w := \int_0^\tau e^{A_C s} \Phi^w e^{A_C^\top s} ds.$$
imaginary parts that differ by an integral multiple of \( \frac{2\pi}{q r} \) (cf. [45, p. 45]).

Then, from standard optimal control arguments (cf. [42, 43]), we can obtain the optimal control law, which results in the combined scheduling and control policy, both for the average and discounted cost problems, \( \gamma = \{(\mu_0^\gamma, \mu_0^{\gamma}), (\mu_1^\gamma, \mu_1^{\gamma}), \ldots \} \), given for \( k \in \mathbb{N}_0 \) by

\[
(\mu_k^\gamma(l_k), \mu_k^{\gamma}(l_k)) = \begin{cases} 
(1, K_\delta x_k), & \text{if } k \text{ is an integer multiple of } q, \\
(0, \varnothing), & \text{otherwise},
\end{cases}
\]

where

\[
K_\delta := -(\bar{R}_\delta + \alpha_\delta \bar{B}_3^\top \bar{P}_\delta \bar{A}_\delta)^{-1}(\alpha_\delta \bar{B}_3^\top \bar{P}_\delta \bar{A}_\delta + \bar{S}_3^\top),
\]

and \( \bar{P}_\delta \) is the unique positive definite solution to the algebraic Riccati equation

\[
\bar{P}_\delta = \alpha_\delta \bar{A}_3^\top \bar{P}_\delta \bar{A}_\delta + \bar{Q}_\delta -
(\alpha_\delta \bar{A}_3^\top \bar{P}_\delta \bar{B}_\delta + \bar{S}_3^\top)(\bar{R}_\delta + \alpha_\delta \bar{B}_3^\top \bar{P}_\delta \bar{B}_\delta)^{-1}(\alpha_\delta \bar{B}_3^\top \bar{P}_\delta \bar{A}_\delta + \bar{S}_3^\top).
\]

This policy has an average transmission rate (11) of \( \frac{1}{\delta} \), a discounted cost

\[
J^d_{\text{per, } \delta} := x_0^\top \bar{P}_\delta x_0 + \frac{\alpha_\delta}{1 - \alpha_\delta} \text{tr}(\bar{P}_\delta \bar{P}^w_\delta),
\]

and an average cost

\[
J^u_{\text{per, } \delta} := \frac{1}{\delta} \text{tr}(\bar{P}_\delta \bar{P}^w_\delta)
\]

(cf. [42, Vol. II, p. 142 and 273]). The main focus of this paper is to design combined scheduling/control policies which achieve (strictly) better performance than traditional periodic control using the same average transmission rate. This design problem can be formally written as follows.

**Problem 2:** Given a desirable transmission rate \( \frac{1}{q r} \), for some \( q \in \mathbb{N} \), find a policy \( \pi \) for which \( R_\pi = \frac{1}{q r} \) and

\[
J^u_\pi < J^u_{\text{per, } q r},
\]

where \( c = d \) if the performance is measured by (2) (discounted cost problem) and \( c = a \) if the performance is measured by (3) (average cost problem).

A natural additional challenge after designing a policy that guarantees (17) is to quantify how much is the performance improvement expressed in (17). In Section IV-D, we address this challenge for the average cost problem \( (c = a) \).

### III. Rollout Event-Triggered Control

The proposed method is a receding horizon algorithm. At a given step of the algorithm, \( m \) transmission decisions over a horizon of \( h \) possible scheduling decisions are chosen, based on which transmission pattern would lead to a lower cost, assuming that after the horizon an optimal periodic control policy would be used, also using \( m \) transmissions in each block of \( h \) scheduling decisions (see Figure 2).

Since periodic transmission belongs to the options of the optimization procedure at each step, we will be able to prove in the sequel that this strategy outperforms periodic control.

We formalize the algorithm by (i) defining the admissible transmission scheduling decisions over the horizon \( h \) in Section III-A; (ii) determining the optimal control policy and associated cost for each of these scheduling sequences in Section III-B; and (iii) specifying the execution of the algorithm in Section III-C. We consider a discounted cost framework with \( \alpha_C > 0 \) for convenience in Sections III A-C and in Section III-D we consider the case \( \alpha_C = 0 \) which includes the average cost problem. The implementation of the proposed method is discussed in Section III-E.

#### A. Admissible scheduling sequences

Let \( \mathcal{T} \) denote the set of transmission scheduling sequences with \( m \) transmissions in the first \( h \) time steps \( 0, \tau, \ldots, (h - 1)\tau \), where \( h \) is an integer multiple of \( m \), and that conform with periodic transmission with period \( q \tau \). \( q := \frac{h}{m} \), in the subsequent time steps \( h \tau, (h + 1)\tau, \ldots \), starting with a transmission at \( h \tau \) (see Figure 2). The parameters \( h \) and \( q \) can be viewed as tuning knobs of the proposed ETC algorithm. Formally, there are

\[
n_\mathcal{T} := \frac{h!}{(h - m)!m!}
\]

scheduling sequences \( \{\sigma_k^h\}_{k \in \mathbb{N}_0} \subseteq \mathcal{T}, i \in \mathcal{M}, \mathcal{M} := \{1, \ldots, n_\mathcal{T}\} \), characterized by

\[
\sigma_k^h = \nu_k^h, \quad k \in \{0, 1, \ldots, h - 1\}, \quad i \in \mathcal{M},
\]

where \( \nu^i = (\nu^i_0, \ldots, \nu^i_{h-1}) \in \mathcal{I}, i \in \mathcal{M} \), with

\[
\mathcal{I} := \{\nu \in \{0, 1\}^h \mid \sum_{k=0}^{h-1} \nu_k = m\}
\]

and by

\[
\sigma_k^h = \begin{cases} 
1, & \text{if } k \text{ is an integer multiple of } q, \\
0, & \text{otherwise,}
\end{cases}
\]

for \( 0 \leq k \leq h - 1 \). The associated scheduling sequence in \( \mathcal{T} \) corresponds to periodic transmission with period \( q \tau \).

#### B. Optimal policy and cost for each scheduling sequence

Our proposed method is based on solutions to optimal control subproblems in which the transmission scheduling sequence is fixed and belongs to the set \( \mathcal{T} \). Here, we describe the optimal control input policy that minimizes (8) for a fixed scheduling sequence in \( \mathcal{T} \), labeled by \( i \in \mathcal{M} \), which can
be derived by standard optimal control arguments (cf. [42], [43]), under Assumption 1.

Let $P_\delta$ be the first matrix $W_{0,i}$ of the backward recursion

$$W_{h,i} = \begin{bmatrix} P_{\delta} & 0 \\ 0 & 0_{n_x \times n_x} \end{bmatrix},$$

$$W_{i,j} = F_{\delta,i} (W_{k+1,i}), \quad 0 \leq k \leq h - 1,$$

where $P_{\delta}$ can be obtained as the solution to (14) (with $\delta$ replaced by $q$) and

$$F_0(P) := \alpha_\tau A_1^T PA_1 + Q_0,$$

$$F_1(P) := \alpha_\tau A_1^T PA_1 + Q_1 - (S_1 + \alpha_\tau A_1^T PB_1)(R_1 + \alpha_\tau B_1^T PB_1)^{-1}(\alpha_\tau B_1^T PA_1 + S_1^T).$$

Then the optimal control input policy corresponding to the scheduling sequence in $T$ is given by

$$u_k = \begin{cases} K_{k,i} x_k, & \text{if } \nu_k^i = 1, \\ \varnothing, & \text{otherwise}, \end{cases}$$

for $k \in \{0, 1, \ldots, h - 1\}$, where for $\nu_k^i = 1$ the gains $K_{k,i}$ are given by

$$K_{k,i} := - (R_1 + \alpha_\tau B_1^T W_{k+1,i}B_1)^{-1}(\alpha_\tau B_1^T W_{k+1,i}A_1 + S_1^T)[0_{n_x \times n_x}]_1,$$

and for $k \in \mathbb{N}_{\geq h}$,

$$u_k = \begin{cases} \tilde{K}_{\delta} x_k, & \text{if } k \text{ is an integer multiple of } q, \\ \varnothing, & \text{otherwise}, \end{cases}$$

where $\tilde{K}_{\delta}$ is described by (13). The discounted cost (8) for this policy is given by

$$\xi^T \Phi_0 P_\delta \xi_0 + c_i + b, \quad i \in \mathcal{M},$$

where $b := \frac{\alpha_\delta}{1 - \alpha_{\delta}} \text{tr}(\tilde{P}_{\delta} \tilde{W}_{\delta})$ and

$$c_i := \sum_{k=1}^{h} \alpha_{\delta} \text{tr}(W_{k,i} \Phi_{\delta})$, \quad i \in \mathcal{M}.$$

Note that when $i \in \mathcal{M}$ corresponds to periodic control, $i = 1$, cost (24) equals (15), which implies that

$$P_1 = \begin{bmatrix} P_{\delta} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$c_1 = \sum_{k=1}^{m} \alpha_{\delta} \text{tr}(\tilde{P}_{\delta} \tilde{W}_{\delta}).$$

C. Algorithm

The proposed rollout method, described next, finds at each scheduling time, in a receding horizon fashion, the scheduling sequence in $T$ that would optimize (8) if this scheduling sequence would be used thereafter, along with a corresponding optimal policy for the control input.

**Algorithm 3:**

(i) At scheduling times $\ell := jh$, $j \in \mathbb{N}_0$, compute

$$\nu(\ell) = \argmin_{i \in \mathcal{M}} \xi^T \Phi_\ell \xi + c_i.$$

(ii) For times $k \in \{jh, jh + 1, \ldots, (j + 1)h - 1\}$ pick the schedules $\sigma_k = \nu(\ell)$ and the control inputs

$$u_k = \begin{cases} K_{k-jh,i} x_k, & \text{if } \nu(\ell) = 1, \\ \varnothing, & \text{otherwise}. \end{cases}$$

Repeat (i) and (ii) at scheduling time $(j + 1)h$.

Note that at time $jh$ step (i) fixes the scheduling actions and the control policy (the feedback gains) to be taken in the interval $k \in \{jh, jh + 1, \ldots, (j + 1)h - 1\}$, but not the control actions. The latter are computed from (28) based on the actual state $x_k$ of the plant at times $k \in \{jh, jh + 1, \ldots, (j + 1)h - 1\}$ with $\sigma_k = 1$.

In the terminology of Section II, Algorithm 3 corresponds to a family of policies described by $\rho = \{(\mu_{\kappa}^x, \mu_{\kappa}^u), (\mu_{\kappa}^x, \mu_{\kappa}^u), \ldots\}$,

$$(\mu_k^x(l_k), \mu_k^u(l_k)) = (\nu(\ell), K_{k-jh,i} x_k),$$

$$j h \leq k < (j + 1)h, \quad j \in \mathbb{N}_0.$$

D. Average cost problem, $\alpha_C = 0$

Considering $\alpha_C > 0$ in the previous section was convenient since for the average cost problem, the costs (24) are unbounded (the constant $b$ tends to infinity as $\alpha_C \downarrow 0$). However, since Algorithm 3 does not depend on $b$, we can still consider the algorithm for the average cost problem ($\alpha_C = 0$ and $B_{\omega} \neq 0$). In this case, Algorithm 3 can be viewed as a suboptimal method for designing a combined scheduling and control policy for the average cost problem, obtained by taking the limit as $\alpha_C$ tends to zero of the suboptimal method derived for the discounted cost problem. Note that in the case $B_{\omega} = 0$ and $\alpha_C = 0$ in (24) we have $b = 0$ and $c_i = 0$, $\forall i \in \mathcal{M}$, and one can also consider Algorithm 3.

E. Implementation

Although Algorithm 3 relies on receding horizon ideas, it does not require any on-line optimization. This resembles explicit model predictive control [47]. In fact, Algorithm 3 requires only computing the explicit functions (27) and (28). For each recursion of the algorithm, computing (28) requires at most $mn_{i,i}$ multiplications, whereas computing (27) for

3We arbitrate that if the minimum argument in (27) is achieved by two or more indexes $i_1, i_2 \in \mathcal{M}$ the smallest index is selected, although this is not relevant in the results that follow. Hence, by the argmin function in (27) we mean $\argmin_{i \in \mathcal{M}} \xi^T \Phi_\ell \xi + c_i := \min(h^{-1}((\min_{i \in \mathcal{M}} b(i))$, where $h(i) = \xi^T \Phi_\ell \xi + c_i.$
a state \( v = \xi_\ell \) with components \( v_i \), \( 1 \leq i \leq n \), requires at most \( (n_T + 1)^{n(n+1)} \) multiplications since each of the \( n_T \) quadratic functions can be computed in terms of linear combinations of the \( \binom{n_T}{2} \) products \( v_i v_j \), \( 1 \leq i, j \leq n \) (these products are computed once at each scheduling decision time and the \( n_T \) linear combinations are computed after). In the numerical example of Section VI, we consider the following parameters: \( h = 6 \) \( m = 2 \), which results from (18) in \( n_T = 15 \). Note that additions and other operations as taking the minimum in (27) typically have a negligible computational burden with respect to multiplications.

IV. MAIN RESULTS

In Section IV-A we establish that the proposed rollout algorithm (Algorithm 3) performs no worse than periodic control both for the average cost and the discounted cost problems. Obtaining strict performance improvement results requires additional technical assumptions and these results are presented in Section IV-B. In Section IV-C we discuss the stability properties of our proposed method and in Section IV-D we quantify the performance improvements for the average cost problem. The proofs are deferred to Section VIII.

A. Performance improvement

We start with the following performance improvement result which requires only the basic Assumption 1. Let \( J^{\rho, \tau}_c \), \( c \in \{a, d \} \), denote the discounted cost (2) of the policy \( \rho \), described by (29), when \( c = d \) and the average cost (3) of the policy \( \rho \) when \( c = a \).

**Theorem 4**: Consider Algorithm 3 for \( \tau \in \mathbb{R}_{>0}, q \in \mathbb{N}, m \in \mathbb{N} \), and \( \alpha_C \geq 0 \), and suppose that Assumption 1 holds. Then

\[
J^{d, \tau}_\rho \leq J^{d, \tau}_{\text{per}},
\]

Moreover, if \( \alpha_C = 0 \),

\[
J^{a, \tau}_\rho \leq J^{a, \tau}_{\text{per}}.
\]

It is clear, from the construction in Algorithm 3, that policy (29) yields an average transmission rate (11) equal to \( \frac{1}{q_\tau} \). Thus Theorem 4, establishes that policy (29) performs no worse than the traditional periodic strategy with a corresponding transmission rate \( \frac{1}{q_\tau} \). In fact, in most situations policy (29) performs strictly better, thus providing a solution to Problem 2. However, this is often hard to guarantee formally [42, p.338]. In the next section we will prove formally that, under given assumptions, policy (29) performs strictly better than the traditional periodic strategy. Still, such assumptions do not encompass important cases (e.g. \( B_\omega = 0 \) and \( \alpha_C = 0 \)) captured by Theorem 4, which is hence interesting in its own right.

B. Strict performance improvement

Consider the following assumptions:

**Assumption 5**: (i) \((A_C, B_\omega)\) is controllable.

(ii) The following matrix has full rank

\[
\alpha_s A_s^T P_\tau B_s + S_s
\]

for every \( s \in \{k \mid k \in \{1, \ldots, (h - m + 1)\}\} \).

Assumption 5(i) guarantees that all the states of (1) are affected by the disturbance input. Assumption 5(ii) is a mild technical assumption to simplify the proof of our main results and, along with Assumption 5(i), it is used to guarantee that (5) driven by policy (29) (described by (35) below) is not concentrated in some lower dimensional subset of the state space \( \mathbb{R}^n \) (see Remark 15 below). Assumption 5(ii) is rather mild. In Lemma 11 we prove that Assumption 5(ii) always holds for sufficiently small \( \tau \). Moreover, as discussed in Remark 15 below, Assumption 5(ii) is not necessary for the theorems stated in the sequel to hold (Theorems 7, 9, and 10).

In addition to Assumption 5, to obtain strict performance improvement of the rollout ETC method for the discounted cost problem when \( \alpha_C > 0 \) we need the following assumption.

**Assumption 6**: \( \bar{K}_q \neq \bar{K}_{q_\tau} (A_{q_\tau} + B_{q_\tau} \bar{K}_{q_\tau}) \).

Assumption 6 is equivalent to the optimal periodic control inputs \( u_k = \bar{K}_{q_\tau} x_k, k \in \mathbb{N}_0 \) (see (12)) not being equal to a constant signal, which may occur (pathologically) for a given \( \alpha_C > 0 \). Note that, since \((\bar{A}_{q_\tau} + B_{q_\tau} \bar{K}_{q_\tau})\) is Hurwitz when \( \alpha_C = 0 \) (cf. [42]), Assumption 6 holds for \( \alpha_C = 0 \), which is the case we are mostly interested in.

We state next the strict performance improvement result.

**Theorem 7**: Consider Algorithm 3 for \( \tau \in \mathbb{R}_{>0}, q \in \mathbb{N}, m \in \mathbb{N} \) and \( \alpha_C \geq 0 \). Then, if Assumptions 1, 5, and 6 are satisfied, the following holds

\[
J^{a, \tau}_\rho < J^{a, \tau}_{\text{per}}.
\]

Moreover, if Assumptions 1 and 5 are satisfied and \( \alpha_C = 0 \) then

\[
J^{a, \tau}_\rho < J^{a, \tau}_{\text{per}}.
\]

C. Stability

Here we investigate the implications of the performance improvement results Theorems 4, 7 for the stability of the closed-loop when the rollout ETC method is used. We restrict ourselves to \( \alpha_C = 0 \) since if \( \alpha_C > 0 \) it might be the case that (2) is bounded but the state grows unbounded even for the optimal periodic controller.

In this setting (\( \alpha_C = 0 \)), consider first that no disturbances act on the plat \( (B_\omega = 0) \) and hence stability is simply defined as the state converging to zero. Then, as shown in the next result, (exponential) stability follows readily from the performance improvement result (30).

**Theorem 8**: Consider Algorithm 3 for \( \tau \in \mathbb{R}_{>0}, q \in \mathbb{N}, m \in \mathbb{N} \) and \( \alpha_C = 0 \), and suppose that Assumption 1 holds and \( B_\omega = 0 \). Then there exists \( c \in \mathbb{R}_{>0} \) and \( \alpha \in \mathbb{R}_{>0} \), \( 0 < \alpha < 1 \), such that \( \|\xi_k\| \leq c \alpha^k \|\xi_0\|, \forall k \in \mathbb{N}_0 \).
We consider next the case \( B_w \neq 0 \) (and \( \alpha_C = 0 \)). We shall establish a stability property (ergodicity) for the Markov chain (see (48)) obtained by considering (5) driven by policy (29) along a period \( h \). In fact, let \( \xi_t := \xi_{\ell h} \) and
\[
\tilde{w}_t := [w^T_{\ell h}, w^T_{\ell h+1}, \ldots, w^T_{\ell h+(h-1)}]
\]
Then (5) driven by policy (29) along a period \( h \) can be described by the Markov chain
\[
\xi_{t+1} = \Phi_{\ell t}(\xi_t)\xi_t + \Psi_{\ell t}(\xi_t)\tilde{w}_t, \quad \ell \in \mathbb{N}_0,
\]
where \( \ell(\xi_t) \) is described by (27), and the matrices \( \Phi_j \) and \( \Psi_j \), \( j \in \mathcal{M} \), are given by
\[
\Phi_j := \Pi^0_{r=h-1}\Theta_{s,j} = \Theta_{h-1,j}\Theta_{h-2,j} \ldots \Theta_{0,j}.
\]
and
\[
\Psi_j := \Pi^1_{r=h-1}\Theta_{s,j} \Pi^2_{r=h-1}\Theta_{s,j} \ldots \Theta_{h-1,j} I_n,
\]
where for \( 0 \leq \kappa \leq h - 1 \) and \( j \in \mathcal{M} \),
\[
\Theta_{\kappa,j} = \begin{cases} A_0, & \text{if } \nu_j^\kappa = 0, \\ A_1 + B_1[K_{\kappa,j}0_{n_a \times n_a}], & \text{if } \nu_j^\kappa = 1. \end{cases}
\]
Let
\[
P^f(y, A) := \text{Prob}[\xi_{\ell h} \in A | \xi_0 = y]
\]
be the probability that the chain (35) is in a set \( A \) at \( \ell \in \mathbb{N} \) given that it starts at time zero in state \( y \in \mathbb{R}^n \). In addition, recall that a probability measure \( \chi_{\text{inv}} : \mathcal{B}(\mathbb{R}^n) \to [0,1] \), where \( \mathcal{B}(\mathbb{R}^n) \) denotes the collection of Borel sets in \( \mathbb{R}^n \), is said to be an invariant probability distribution for (35) if \( \int_{\mathcal{B}(\mathbb{R}^n)} P^1(\xi, A)\chi_{\text{inv}}(d\xi) = \chi_{\text{inv}}(A) \) for every \( A \in \mathcal{B}(\mathbb{R}^n) \) (cf. [48, Ch.10]). We state next that when Algorithm 3 corresponds to \( \alpha_C = 0 \), the Markov chain (35) is ergodic [48, Ch. 13].

**Theorem 9:** Consider Algorithm 3 for \( \tau \in \mathbb{R}_{>0}, q \in \mathbb{N}, \) \( m \in \mathbb{N} \), and \( \alpha_C = 0 \), and suppose that Assumptions 1 and 5 hold. Then, there exists a unique invariant measure for the Markov chain (35), denoted by \( \chi_{\text{inv}} \), and (35) is ergodic, i.e.,
\[
\lim_{\ell \to \infty} \sup_{A \in \mathcal{B}(\mathbb{R}^n)} |P^f(y, A) - \chi_{\text{inv}}(A)| = 0
\]
for every \( y \in \mathbb{R}^n \).

Ergodicity is a crucial property to quantify the performance improvements obtained with the rollout method for the average cost problem.

**D. Quantifying the performance improvements**

In the following result we explicitly quantify the performance improvement obtained with the rollout method over optimal periodic control for average cost problems. Due to the difficulty is obtaining such results (cf. [42, Ch. 6]), the following is one of the main results of the paper.

**Theorem 10:** Consider Algorithm 3 for \( \tau \in \mathbb{R}_{>0}, q \in \mathbb{N}, \) \( m \in \mathbb{N} \) and \( \alpha_C = 0 \). Then, if Assumptions 1 and 5 are satisfied, the following holds
\[
J^n_{\mu,\tau} = J^n_{\text{opt},\tau} - g_a,
\]
where \( g_a \) is a strictly positive constant given by
\[
g_a = \frac{1}{\tau h} \int_{\mathbb{R}^n} f(\xi)\chi_{\text{inv}}(d\xi)
\]
with
\[
f(\xi) := \xi^T(P_1 - P_1(\xi))\xi + c_1 - c_1(\xi),
\]
and \( \chi_{\text{inv}} \) is the unique invariant measure of the Markov chain (35).

Note that \( g_a \) is nonnegative since the integrand \( f(\xi) \) is nonnegative due to (27). Theorem 10 states that \( g_a \) is actually strictly positive. The integrand \( f(\xi) \) should be seen as the performance gain at state \( \xi \) obtained by performing a single step optimization over the horizon \( h \) assuming periodic control is used after the horizon \( h \), i.e., the gain obtained at a single scheduling time in Algorithm 3 by making decision (27). The overall gain \( g_a \), described by (40), is obtained by repeating the process at every scheduling time step, according to Algorithm 3. It has the following interpretation: it is the (scaled) expected value of these single step optimization gains \( f(\xi) \) with respect to the invariant probability measure (also a limiting measure according to (38)) of the Markov chain (35), induced by using Algorithm 3. Thus, if Algorithm 3 picks options different from that corresponding to periodic control (\( \iota = 1 \) in (27)) with large single step optimization gains \( f(\xi) \), for states \( \xi \) likely to be visited asymptotically, then one may expect a large overall gain \( g_a \). Contrarily, if \( \iota = 1 \) in (27) for a large region (likely to be visited asymptotically) in the state-space, then \( g_a \) is small. Numerical methods to estimate \( \chi_{\text{inv}} \) can be found, e.g., in [49], [50]. In the example of Section VI we obtain a good approximation of \( g_a \) by running Monte-Carlo simulations.

**V. OTHER NETWORKED CONTROL CONFIGURATIONS**

Our ideas can be adapted to other network configurations. In this section we briefly discuss two examples.

**A. Remote controller**

Consider first the configuration depicted in Figure 3, in which a remote controller sends control inputs and receives state measurements from the plant through a communication...
network. To guarantee that the controller can make scheduling decisions at times $jh$, the plant must transmit the state to the controller at times $jh$, which can be directly used to compute $u_{jh}$. Thus, the free transmission times to be chosen at scheduling time $jh$ are restricted to the interval $\{jh + 1, \ldots, (j + 1)h - 1\}$, i.e., the set $\mathcal{I}$, described in (20), is adapted to

$$\mathcal{I} = \{\nu \in \{0, 1\}^{h-1} \sum_{k=0}^{h-1} \nu_k = m \text{ and } \nu_0 = 1\},$$

where we assume now that $m \in \mathbb{N}_{\geq 2}$. Considering that the network-induced delays are negligible when compared to the baseline period $\tau$ we can assume that at scheduling times $jh$ the controller receives state measurements, makes scheduling decisions for the next $h-1$ possible transmission times $\{jh + 1, \ldots, (j + 1)h - 1\}$, characterized by $\iota$ and computed according to (27), and sends these scheduling decisions along with the control input at time $jh$ to the plant according to (28). At times $k \in \{jh + 1, \ldots, (j + 1)h - 1\}$ such that $\nu_k - jh = 1$, the plant sends again state measurements to the controller, the controller computes the control input according to (28) and sends it to the actuators. In this manner the scheme works for the setup of Figure 3 as well, and can be easily implemented in networks based on TDMA.

B. Model-based predictor at the actuators

In the setup considered in Section II, while the actuators can update $u_k = u_C(t_k)$ at the sampling rate $\frac{1}{\tau}$, this only occurs if a new transmission occurs at time $t_k$. An alternative configuration, considered in several works (see, e.g., [13], [29]), is to assume that the actuators use a predictor to update the control input even if no transmission occurs. Consider the following predictor-based control update

$$\hat{x}_{k+1} = \bar{A}_r \hat{x}_k + \bar{B}_r \hat{u}_k, \quad \hat{u}_k = \bar{K}_r \hat{x}_k, \quad \forall k \in \mathbb{N}_0, \quad (42)$$

where $\hat{x}_k$ starts at time zero with an initial estimate of the state, denoted by $\hat{x}_0$, and resets its state to the transmitted state each time a transmission occurs, i.e.,

$$\hat{x}_k = x_k, \quad \text{when } \sigma_k = 1, \quad \forall k \in \mathbb{N}_0. \quad (43)$$

Note that we assume here that the full state is sent from the event-triggered controller collocated with the sensors to the actuators, which run (42). Moreover, since the control policy is already determined by (42), only the scheduling decisions (to send the state) need to be determined. A base policy for the scheduling is to transmit periodically with period $q\tau$ for some $q \geq 1$. An alternative rollout method, which is a straightforward adaption of the ideas presented in Section III is described next.

The equations for the process and predictor take now the form

$$\eta_{k+1} = L_\sigma \eta_k + \omega_k, \quad k \in \mathbb{N}_0$$

where $\eta_k = [\hat{x}_k^\top, x_k^\top]^\top$ and

$$L_0 = \begin{bmatrix} A_r & B_r \bar{K}_r \\ 0 & (A_r + B_r \bar{K}_r) \end{bmatrix}, \quad L_1 = \begin{bmatrix} (A_r + B_r \bar{K}_r) & 0 \\ (A_r + B_r \bar{K}_r) & 0 \end{bmatrix}$$

and the covariance matrix of $\omega_k$ is given by

$$\Psi_r^w := \begin{bmatrix} \Phi_r^w & 0_{n_x \times n_x} \\ 0_{n_x \times n_x} & 0_{n_x \times n_x} \end{bmatrix}.$$  

The discounted cost (2) takes the form

$$\sum_{k=0}^{\infty} \mathbb{E}[\alpha_k^w \eta_k^\top X_{\sigma_k} \eta_k], \quad (44)$$

apart from an additive constant factor, where

$$X_0 = \begin{bmatrix} \bar{Q}_r & \bar{S}_r \bar{K}_r \\ \bar{K}_r^\top \bar{S}_r^\top & \bar{K}_r^\top \bar{R}_r \bar{K}_r \end{bmatrix}, \quad X_1 = \begin{bmatrix} \bar{Q}_r + \bar{S}_r \bar{K}_r + \bar{K}_r^\top \bar{S}_r^\top + \bar{K}_r^\top \bar{R}_r \bar{K}_r & 0 \\ 0 & 0 \end{bmatrix}.$$  

Using similar arguments to the ones used in Section III-B, the discounted cost (44) for a scheduling sequence taken from the set $\mathcal{T}$, labeled by $i \in \mathcal{T}$, can be shown to be given by

$$\eta_0^\top Z_i \eta_0 + z_i + d$$

where $Z_i, i \in \mathcal{T}$, are positive semi-definite matrices and $z_i, i \in \mathcal{T}$, and $d$ are positive constants. The expressions are omitted for the sake of brevity. Scheduling decisions at each step $\ell = jh, j \in \mathbb{N}_0$ are obtained by computing

$$\iota(\eta_i) = \arg\min_{\iota \in \mathcal{M}} \eta_0^\top Z_i \eta_i + z_i$$

which determine the scheduling decisions in the interval $\{jh, \ldots, (j + 1)h - 1\}$, given by

$$\{\nu_0^{\iota(\xi)}, \ldots, \nu_{j-1}^{\iota(\xi)}\},$$

for $\nu^i \in \mathcal{T}, i \in \mathcal{M}$. Note that the scheduler needs also to run the model-based estimator (42) to make decisions based on $\hat{x}_k$. Similar performance improvements results can be obtained paralleling the ones in Section IV.

VI. EXAMPLE

Consider two unitary masses on a frictionless surface connected by an ideal spring and moving along a one-dimensional axis. The control input is a force acting on the first mass. The state vector is $x_C = [x_1 \ x_2 \ v_1 \ v_2]^\top$, where $x_i, v_i$ are the displacements and velocities of the mass $i \in \{1, 2\}$, respectively, and the plant model (1) is described by

$$A_C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\kappa_m & \kappa_m & 0 & 0 \\ \kappa_m & -\kappa_m & 0 & 0 \end{bmatrix}, \quad B_C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad (45)$$

where $\kappa_m$ is the spring coefficient. We set the initial state to $x_0 = [-1100]^\top$, meaning that the masses start with zero velocity and at opposite distances from their equilibrium values. The matrix $A_C$ has two eigenvalues at zero and two complex conjugates eigenvalues at $\pm \sqrt{28}\kappa_m$. The free response hence has oscillations with a period $\frac{2\pi}{\sqrt{28}\kappa_m}$. We normalize time so that one time unit $t = 1$ corresponds to one period of these oscillations, which results in $\kappa_m = 2\pi^2$. This implies that the sampling period must be different from
the pathological sampling periods $0.5\kappa$, $\kappa \in \mathbb{N}$, so that the discretization of the plant remains controllable [45].

We start by assuming that there are no disturbances acting on the plant, i.e.,

Case I: $B_w = 0$,

and by considering the following cost

\begin{equation}
\text{Case I: } \int_0^\infty x_1(t)^2 + x_2(t)^2 + 0.1u_C(t)^2 \, dt,
\end{equation}

which takes the form (2) with $\alpha_C = 0$. Using standard optimal control theory (cf., e.g., [44, Ch. 3]) we can compute the optimal continuous-time feedback law that minimizes (46) which is a state-feedback law $u_C(t) = K_C x_C(t)$ yielding a cost (46) given by $x_0^TP_C x_0$ where $P_C$ is the solution to the Riccati equation (76) given in Section VIII. For the numerical values given above this gives

\begin{equation}
x_0^TP_C x_0 = 5.7411
\end{equation}

and the eigenvalues of $A_C + B_C K_C$ are given by $-0.1775 \pm 6.2857$, $-1.0564 \pm 1.0566$, resulting is a lightly damped closed-loop system. Figure 4 plots the (normalized) performance (46) obtained with the traditional periodic control strategy and with the rollout ETC strategy described by Algorithm 3 in the setup of Figure 3 with parameters $h = 6$, $m = 2$, $q = 3$, for several values of the average transmission period $q \tau$ in the range $[0, 0.5]$. The performance (46) for the rollout event-triggered control method is obtained via simulating (5) for (7), (29) for a large time ($t \in [0, 500]$) and computing the cost (8) resulting from the parameters in (46). This method can also be used to obtain the cost of the optimal control strategy (12) to confirm the expression (15), which is used to plot the values of Figure 5 pertaining to periodic control. The performance values in Figure 4 are normalized with respect to the optimal LQR performance achievable by a continuous-time controller (47). The time evolution of the actuation $u_C$ and the position $x_1$ of the first mass for the considered initial state and for $t \in [0, 30]$ are shown in Figure 6 when the average transmission rate is 0.4. Note that a faster convergence to zero of these signals is obtained for the rollout method, due to the extra degree of freedom of choosing different actuation pattern than periodic update times.

We consider next the case in which disturbances are acting on the plant characterized by the injection matrix

Case II: $B_w = [0 \ 0 \ 0.5 \ 0]^T$.

Performance is measured by the following cost

\begin{equation}
\text{Case II: } \lim_{T \to \infty} \frac{1}{T}E \left[ \int_0^T [x_1(t)^2 + x_2(t)^2 + 0.1u_C(t)^2] \, dt \right]
\end{equation}

which takes the form (3). Figure 5 plots the (normalized) performance (48) obtained with the traditional periodic control strategy and with the same rollout ETC method as in Case I. The cost (48) is estimated via Monte-Carlo simulations with 300 trials simulating (5) for (7), (29) for a large time ($t \in [0, 1500]$) and computing the cost (10) resulting from the parameters in (48). This method can also be used to obtain the cost of the optimal periodic control strategy (12) to confirm the expression (15), which is used to plot the values of Figure 5 pertaining to periodic control. The performance values in Figure 5 are normalized with respect to the optimal LQG performance achievable by a continuous-time controller, which is given by $tr(P_C B_w B_w^T)$ where $P_C$ is the solution to the Riccati equation (76) given in Section VIII. For the numerical values given above $tr(P_C B_w B_w^T) = 0.06170$.

Both Figure 4 and Figure 5 show that for small average transmission periods the methods perform very closely. In fact, this is natural as periodic control approaches the optimal performance (2) achievable by a continuous-time controller when the sampling period tends to zero. As such, there is little room for improvements. However, for larger transmission periods the rollout strategy in Case I obtains significant performance improvements over traditional periodic control. This is a clear illustration of the main theorems in this paper and shows the effectiveness of the novel ETC strategy proposed in this paper. On the other hand, for Case II the gains are less pronounced. A possible explanation is the fact that we have considered Wiener disturbances. As discussed in [51] the performance gains of ETC strategies with respect to periodic control may be much larger considering classes of
Fig. 6. Time evolution of state $x_1$ and control input $u_C$ for the periodic and rollout methods when the average transmission rate is 0.4.

VII. CONCLUSIONS AND FUTURE WORK

In this paper, we proposed a novel ETC strategy called rollout ETC, that guarantees a performance improvement over traditional periodic control. The key to our method is to select at given scheduling times control and scheduling decisions over a given horizon assuming that periodic optimal control is used afterwards. Under mild assumptions, for the new class of ETC strategies, we showed that strict performance improvements could be formally guaranteed with respect to the performance of periodic controllers with the same average transmission rate. We illustrated by a numerical example that the proposed ETC strategy can significantly outperform periodic control.

While we have focused on basic models for the process and for the communication network, the obtained numerical results encourage pursuing various research directions for extending such models. These directions include scenarios in which (i) the full state of the plant is not available; (ii) multiple control loops are closed over the communication network; (iii) the noise model is different from Wiener processes; (iv) packet drops are taken into account in the model of the communication network.

stochastic disturbances different from Wiener disturbances. A topic for future research is to incorporate such models in the setting of the present paper.

VIII. PROOFS

Theorems 4 and 8 are proved in Section VIII-A. The proof of Theorem 8 builds upon some of the statements used in the proof of Theorem 4. Theorems 7, 9 and 10 are proved in Section VIII-C, building upon two key lemmas established in Section VIII-B.

A. Proof of Theorems 4 and 8

Before we prove Theorem 4, we note that we can think of $\lambda := \{\iota(\xi_0), \iota(\xi_1), \ldots \}$ as a stationary policy for (35). We can then write (8) and (10) when $\pi = \rho$, where $\rho$ is the rollout policy (29), as

$$ J^d_\lambda(\xi_0) := \sum_{\ell=0}^{\infty} \mathbb{E}[\alpha^h_{\iota(\xi_\ell)} g(\xi_\ell, \iota(\xi_\ell), \bar{w}_\ell)], \tag{49} $$

and

$$ J^a_\lambda = \lim_{L \to \infty} \frac{1}{\tau h L} \sum_{\ell=0}^{L-1} \mathbb{E}[\tilde{g}(\xi_\ell, \iota(\xi_\ell), \bar{w}_\ell)], $$

respectively, where for $\bar{w} = (w_0, \ldots, w_{h-1})$,

$$ \tilde{g}(\xi, i, \bar{w}) := \sum_{\kappa=0}^{h-1} \alpha^c_{\xi, i} y_\kappa, \tilde{K}_{\kappa, i} y_\kappa, \nu_\kappa, $$

the $y_\kappa$ are defined recursively

$$ y_{\kappa+1} = \Theta_{\kappa, i} y_\kappa + w_\kappa, \quad y_0 = \bar{\xi}, \quad \kappa \in \{0, 1, \ldots, h-1\}. $$

and

$$ \tilde{K}_{\kappa, i} = \begin{cases} [K_{\kappa, i} I_{n_x \times n_u}], & \text{if } \nu_\kappa = 1, \\ [0_{n_x \times n_u}], & \text{otherwise}. \end{cases} \tag{50} $$

That is, $J^a_\rho = J^\lambda_\rho$ and $J^d_{\rho, \iota \tau}(\xi_0) = J^d_\lambda(\xi_0)$, for every $\xi_0 \in \mathbb{R}^n$.

Proof: (of Theorem 4)

To establish (30) we start by defining the following policies $\psi^r(\iota(\xi) \in \mathbb{N}_0$ for (35),

$$ \psi^r_{j}(\xi_\ell) = \begin{cases} \iota(\xi_\ell), & \text{if } 0 \leq j < r, \\ 1, & \text{if } j \geq r. \end{cases} $$

obtained by applying policy (27) to (35) until iteration $r$ and afterwards using always periodic control ($\iota = 1$). Note that $\lim_{t \to \infty} J^d_t(\xi_0) = J^a_\rho(\xi_0) = J^d_{\rho, \iota \tau}(\xi_0)$ for every $\xi_0 \in \mathbb{R}^n$ and $J^d_t(\xi_0) = J^d_{\rho, \iota \tau}(\xi_0)$ for every $\xi_0 \in \mathbb{R}^n$. From the definition of $t$ in (27) we have that

$$ J^d_t(\xi_0) \leq J^d_{\psi^r}(\xi_0), \quad \text{for every } \xi_0 \in \mathbb{R}^n. \tag{51} $$

Since the cost (49) is additive along stages, we can write

$$ J^d_\rho(\xi_0) = \sum_{\ell=0}^{r-1} \mathbb{E}[\alpha^h_{\iota(\xi_\ell)} g(\xi_\ell, \iota(\xi_\ell), \bar{w}_\ell)] + \alpha^h \mathbb{E}[J^d_{\psi^r}(\xi_\ell)], \tag{52} $$

for $r \in \mathbb{N}$, and

$$ J^d_{\psi^r}(\xi_0) = \sum_{\ell=0}^{r-1} \mathbb{E}[\alpha^h_{\iota(\xi_\ell)} g(\xi_\ell, \iota(\xi_\ell), \bar{w}_\ell)] + \alpha^h \mathbb{E}[J^d_{\psi^r}(\xi_\ell)], \tag{53} $$
for \( r \in \mathbb{N}_0 \). From (51), we conclude that
\[
\mathbb{E}[J_{r}^d(\xi_c)] \leq \mathbb{E}[J_{r}^d(\xi_c)] \tag{54}
\]
for every \( r \in \mathbb{N} \), where the expectations are taken with respect to \( \tilde{\omega}_0, \ldots, \tilde{\omega}_{r-1} \) that determine \( \xi_c \) by (35). Using this latter inequality in (52) and taking into account (53) we conclude that
\[
\tilde{J}^d_{r+1}(\xi_0) \leq \tilde{J}^d_{r}(\xi_0) \tag{55}
\]
for every \( r \in \mathbb{N}_0 \) and \( \xi_0 \in \mathbb{R}^n \). Thus, for a given \( \xi_0 \in \mathbb{R}^n \),
\[
J_{\rho,q_T} = \lim_{r \to \infty} \tilde{J}^d_r \leq \cdots \leq \tilde{J}^d_2 \leq \tilde{J}^d_1 \leq \tilde{J}^d_0 = J_{\rho,q_T} \tag{56}
\]
establishing (30).

To establish (31) for the average cost \( (c = a) \), we define
\[
V(\xi) := \xi^T P_1 \xi, \xi \in \mathbb{R}^n, \tag{57}
\]
and take the limit as \( \alpha_C \downarrow 0 \) in (26) (see Section III-D) obtaining
\[
c_1 = \text{tr}(P_q \tilde{w}_q). \tag{58}
\]
Taking into account (25) and the definition of \( P_t \) in Section III-B one can conclude that at iteration \( \ell \)
\[
\mathbb{E}[V(\xi_{\ell+1}) + \bar{g}(\xi_{\ell}, t(\xi_{\ell}), \bar{w}_\ell)|\xi_{\ell}] = \bar{\xi}_t^T P_{\ell}(\xi_{\ell}) \bar{\xi}_t + c_1(\bar{\xi}_t) \tag{59}
\]
Thus,
\[
\mathbb{E}[V(\xi_{\ell+1}) + \bar{g}(\xi_{\ell}, t(\xi_{\ell}), \bar{w}_\ell)|\xi_{\ell}] - V(\xi_{\ell}) = \bar{\xi}_t^T (P_{\ell}(\xi_{\ell}) - P_{\ell-1}) \bar{\xi}_t + c_1(\bar{\xi}_t) \tag{60}
\]
where \( f \) is described by (41). Adding (60) for \( \ell = 0, 1, \ldots, L-1 \), dividing by \( \tau hL \), and taking expectations we obtain
\[
\frac{1}{\tau hL} \mathbb{E}\sum_{\ell=0}^{L-1} \bar{g}(\xi_{\ell}, t(\xi_{\ell}), \bar{w}_\ell) = \frac{c_1}{\tau h} \frac{1}{\tau hL} \mathbb{E}\sum_{\ell=0}^{L-1} f(\xi_{\ell})] + \frac{1}{\tau hL} (V(\xi_0) - \mathbb{E}[V(\xi_L)|\xi_0]) \tag{61}
\]
Provided that we prove that \( \mathbb{E}[V(\xi_L)|\xi_0] \) remains bounded as \( L \to \infty \) we can take the limit as \( L \to \infty \) in (61), use the fact that the left-hand side converges to \( \bar{J}_\alpha = J^{\alpha}_{\rho,q_T} \), and use (58) to obtain
\[
J^{\alpha}_{\rho,q_T} = \frac{1}{q_T} \text{tr}(P_q \tilde{w}_q) - \lim_{L \to \infty} \frac{1}{\tau hL} \mathbb{E}\sum_{\ell=0}^{L-1} f(\xi_{\ell}). \tag{62}
\]
Then, (31) follows from (16) and the fact that \( f \), described by (41), is a nonnegative function due to \( i = t(\xi_{\ell}) \) and (27).

To prove that \( \mathbb{E}[V(\xi_L)|\xi_0] \) remains bounded as \( L \to \infty \), we use the fact that
\[
\mathbb{E}[\bar{g}(\xi_{\ell}, t(\xi_{\ell}), \bar{w}_\ell)|\xi_{\ell}] \geq a_1 \bar{x}_\ell^T \bar{x}_\ell \tag{63}
\]
for some sufficiently small \( a_1 > 0 \), where we used the decomposition \( \xi_{\ell} = [\bar{x}_\ell^T \bar{u}_\ell^T]^T \), \( \bar{x}_\ell := x_{\ell h}, \bar{u}_\ell := \bar{u}_{\ell h} \).

Equation (63) can be proved using the positive semi-definite assumption on \( Q_C \), the assumption that the pair \((A_C, Q_C^{-1}) \) is observable, and the assumption that \( R_C \) is positive definite. Moreover, choosing \( b_1 \) such that \( b_1 > a_1 > 0 \) and \( \bar{P}_{q_T} < b_1 I_n \), and taking into account (25) we conclude that
\[
V(\xi) \leq b_1 \bar{x}^T \bar{x} \tag{64}
\]
for \( \xi = [\bar{x}^T \bar{u}^T]^T \). Using (60), (63), (64) we conclude that for \( \ell \in \mathbb{N}_0 \),
\[
\mathbb{E}[V(\xi_{\ell+1})|\xi_{\ell}] \leq d_1 V(\xi_{\ell}) + c_1,
\]
where \( d_1 := 1 - \frac{a_1}{b_1} < 1 \), which in turn implies that for \( L \in \mathbb{N} \) and \( d_2 = \sum_{\ell=0}^{L-1} d_1^\ell c_1 \),
\[
\mathbb{E}[V(\xi_L)|\xi_0] \leq d_2^L V(\xi_0) + d_2,
\]
leading to the conclusion that \( \mathbb{E}[V(\xi_L)|\xi_0] \) is bounded as \( L \to \infty \).

We prove Theorem 8 next.

Proof: (of Theorem 8) If we consider the case \( B_\omega = 0 \) and \( a_C = 0 \), we conclude from (60) that
\[
V(\xi_{\ell+1}) - V(\xi_{\ell}) \leq \bar{g}(\xi_{\ell}, t(\xi_{\ell}), 0) \tag{65}
\]
where \( \bar{V} \) is described by (57) and we used the fact that \( c_1 = 0 \) in this case and \( f \) is a nonnegative function. As in (63) we can conclude that
\[
\bar{g}(\xi_{\ell}, t(\xi_{\ell}), 0) \geq a_2 \bar{x}_\ell^T \bar{x}_\ell \tag{66}
\]
for sufficiently small \( a_2 > 0 \), where again we used the decomposition \( \xi_{\ell} = [\bar{x}_\ell^T \bar{u}_\ell^T]^T \). From (65) and (66) and taking into account (25) we can conclude that
\[
\bar{x}_\ell^T \bar{P}_{q_T} \bar{x}_{\ell+1} - \bar{x}_\ell^T \bar{P}_{q_T} \bar{x}_\ell \leq -a_2 \bar{x}_\ell^T \bar{x}_\ell \tag{67}
\]
Thus
\[
\bar{x}_\ell^T \bar{P}_{q_T} \bar{x}_\ell \leq (1 - \frac{a_2}{c})^\ell \bar{x}_0^T \bar{P}_{q_T} \bar{x}_0
\]
where \( c \) is a sufficiently large constant such that \( \bar{P}_{q_T} < c l_{n_\ell} \) and \( (1 - \frac{a_2}{c}) \) is positive. Since, under Assumption 1, \( P_{q_T} \) is positive definite, this implies that \( \bar{x}_\ell \) converges to zero exponentially fast, which in turn implies that \( x_{\ell h} \) converges to zero exponentially fast. Moreover, since the control input is a hold version of (28) this implies that the control input also converges to zero exponentially fast and hence also \( \xi_{\ell} \).

B. Two key lemmas

We need two preliminary lemmas to prove Theorems 7, 9, and 10. For each option \( i \in \mathcal{M} \) for the scheduling vector \( \nu_k \), \( k \in \{0, 1, \ldots, h-1\} \) in (19), let
\[
\bar{k}^i \in \{m - 1, m, m + 1, \ldots, h - 1\} \tag{68}
\]
be the largest \( k \) such that \( \nu_{k+1}^i \) equals one, i.e., \( \bar{k}^i \) is uniquely determined by \( \nu_{\bar{k}^i + 1}^i = 1 \) and \( \nu_{\bar{k}^i}^i = 0 \), if \( k \in \{ \bar{k}^i + 1, \ldots, h - 1 \} \).

Lemma 11: Suppose that Assumption 1 holds and consider Algorithm 3 for \( r \in \mathbb{R}_{>0}, q \in \mathbb{N} \) and \( m \in \mathbb{N} \). Then:
(i) Assumption 5(ii) holds for sufficiently small \( \tau \).
(ii) if Assumption 5(ii) holds, then \( \bar{k}_{p,i} \), obtained from (23), has full rank for every \( i \in \mathcal{M} \).

4In the proof of Theorem 7 we shall establish that the limit in the right-hand side of (62) exists. Then the limit in the left-hand side of (62), described in (10), also exists.
(iii) if Assumption 6 holds, there exist \( \xi \in \mathbb{R}^n \) and \( i \in \mathcal{M} \setminus \{1\} \) such that
\[
\xi^\top P_i \xi + c_i < \xi^\top P_{\bar{i}} \xi + c_{\bar{i}}.
\] (69)

Note that (iii) assures that for at least one state the choice in (27) is different from (21), which corresponds to periodic scheduling, i.e., there always exists a state in \( \mathbb{R}^n \) for which the periodic scheduling option is not chosen. The proof of Lemma 11(iii) needs the following proposition.

**Proposition 12:** Suppose that Assumptions 1 and 6 hold and consider the unique solutions \( P_r \) and \( P_{r+1} \) to (14) when \( \delta \) is replaced by \( \tau \) and \( q\tau \) respectively, \( \tau \in \mathbb{R}, q \in \mathbb{N}, q \geq 2 \). Then
\[
P_r \preceq P_{r+1}
\] (70)
and
\[
\exists x \in \mathbb{R}^n : \ x^\top P_r x < x^\top P_{r+1} x
\] (71)

**Proof:** By construction \( P_{r+1} \) is such that \( x_0^\top P_{r+1} x_0 \) is the cost of the following optimal control problem
\[
\min_{\{u_k, k \in \mathbb{N}_0\}} \int_0^\infty e^{-\alpha c t} g(x_C(t), u_C(t)) dt
\] (72)
s.t. \( x_C(0) = x_0 \), where \( x_C \) and \( u_C \) satisfy (1) for \( B_{r+1} = 0 \), and \( u_C \) is given by
\[
u_C(t) = u_k, \quad t \in [t_k, t_{k+1}),
\] (73)
for \( t_k = j\tau k, k \in \mathbb{N}_0 \), when \( j = q \). Let \( u_k \) denote the optimal solution corresponding to \( j\tau \), for a given \( j \in \mathbb{N} \), which equals
\[
u_k := \bar{K}_{j\tau} (\hat{A}_{j\tau} + \bar{B}_j \bar{K}_{j\tau}) ^k x_0
\] (74)
since the control input is described by (12) and there are no disturbances acting on the plant. If \( j = 1 \), we make \( u_k \) in (73) emulate the optimal control input corresponding to \( q\tau, q \geq 2 \), i.e.,
\[
u_k = u_{k,q}^q
\] (75)
where \( \lfloor a \rfloor \) denotes the floor of \( a \) (largest integer less or equal than \( a \)), then the cost (72) for these (not necessarily optimal) control inputs equals \( x_0^\top P_{r+1} x_0 \). Then the (optimal) control inputs \( u_k^{1, q+1} \) will yield a cost \( x_0^\top P_{r+1} x_0 \) smaller than \( x_0^\top P_{r+1} x_0 \) for every \( x_0 \in \mathbb{R}^n \) which implies (70).

To prove (71) it suffices to prove that there exists one initial condition \( x_0 \) for the problem (72) with \( j = 1 \) for which (75) is not the optimal solution, since the optimal solution to the problem (72) is unique (cf. [42]) and hence will lead to a strictly smaller cost. To this effect, suppose that for a given initial condition \( x_0 \), (75) is the optimal solution. In particular, the first \( q \) controls are the same \( u_0 = u_1 = \cdots = u_{q-1} \). Due to Bellman's principle of optimality [42], if the system would start at time \( k = 1 \) with initial condition \( x_0 = x_1 = \bar{A}_r x_0 + \bar{B}_r u_0 \) the optimal control inputs would be shifted, i.e., the first control would be \( u_1 \), the second \( u_2 \), etcetera. However, such optimal control input does not take the form (75), unless (74) is constant, which is excluded by Assumption 6. Hence, for such initial condition \( x_0 \) the optimal control input is different than (74), thereby concluding the proof.

**Proof:** (of Lemma 11) We start by recalling that the following Riccati equation
\[
(AC - \frac{\alpha C}{2} I) P_C + P_C (AC - \frac{\alpha C}{2} I) - P_C B_C R_C^{-1} B_C^\top P_C + Q_C = 0
\] (76)
has a unique positive definite solution \( P_C \) if \( R_C \) is positive definite, and the pairs \( (AC - \frac{\alpha C}{2} I, B_C) \) and \( (AC - \frac{\alpha C}{2} I, Q_C^2) \), are controllable and observable, respectively (see [44, Ch. 3]), which holds due to the assumption that \( (AC, B_C) \) and \( (AC, Q_C^2) \), are controllable and observable, respectively. This latter fact can be seen from the characterization of controllability of the pair \( AC, B_C \) and observability using duality: \( |AC - \lambda I| B_C \) has full rank for all \( \lambda \in \mathbb{C} \) (cf. [52, p.47]). We recall also that the optimal controller that minimizes the discounted cost (2) without communication restrictions (providing a continuous-time input \( u_C(t), t \in \mathbb{R}_{\geq 0} \), based on full access to the state \( x_C(t), t \in \mathbb{R}_{\geq 0} \) ) yields a cost \( x_C(0)^\top P_C x_C(0) \) (see [44, Ch. 3]). Then, it is clear that \( \lim_{\tau \to 0} P_0 = P_C \), i.e., the optimal continuous-time performance is recovered as the sampling period of periodic control tends to zero (see [45, Sec. 9.4]). Using this latter fact, and taking into account the expressions (6), (9), we can obtain that
\[
\lim_{\tau \to 0} \frac{1}{\tau} (\alpha \tau ^2 \bar{P}_q \bar{B}_r + \bar{S}_r) = P_C B_C.
\] (77)

Since \( P_C \) is positive definite and \( B_C \) has full rank (cf. Assumption 1(i)), we can conclude that \( P_C B_C \) has full rank. Hence, in first approximation \( \alpha \tau ^2 \bar{P}_q \bar{B}_r + \bar{S}_r \) approaches a full-rank matrix \( \tau P_C B_C \), which allows to conclude (i).

To prove (ii) we use the fact that
\[
K_{k,s} = -(\bar{R}_s + \alpha s B_s \bar{P}_q \bar{B}_s)^{-1} (\alpha s B_s \bar{P}_q \bar{A}_s + \bar{S}_s)
\] (78)
where \( s = (h - k^2)\tau \), and \( k^2 \) is defined in (68). This fact can be obtained directly from (23). The derivation is straightforward but lengthy and therefore it is omitted. The matrix \( \bar{R}_s \) is positive definite (since \( R_C \) is positive definite) for every positive \( s \) and hence the inverse in (78) exists. Note that \( 1 \leq (h - k^2) \leq h - k + 1 \). Then, Assumption 5(ii) implies that \( K_{k,s,j} \) is the product of an invertible matrix and a full rank matrix and hence it is full rank.

To prove (iii) we notice that if there exists \( i \in \mathcal{M} \) such that \( c_i < c_1 \) then (69) holds for such \( i \in \mathcal{M} \) and \( \xi = 0 \). If \( c_i \geq c_1 \) for every \( i \in \mathcal{M} \), to establish (69) it suffices to prove that there exist \( \xi \) and \( i \in \mathcal{M} \) such that
\[
\xi^\top P_i \xi < \xi^\top P_i \xi
\] (79)
since then (69) holds for \( \xi = a \xi \) and sufficiently large \( a \in \mathbb{R} \).

To prove (79), we start by noticing that, by construction, \( \xi_0^\top P_i \xi_0, \xi_0 = [x_0^\top \tilde{u}_0]^\top, i \in \mathcal{M}, \) is the cost of the following
\[
\min_{(u_0, \ldots, u_{n-1}) \in \mathcal{U}} \int_0^{\tau} e^{-\alpha c t} g_C(x_C(t), u_C(t)) dt + e^{-\alpha h \tau} x_C(h \tau) \mathbf{P}_q x_C(h \tau), \quad (80)
\]

s.t. \( x_C(0) = x_0, \ x_C \) and \( u_C \) satisfy (1),
\[
\begin{align*}
\quad u_C(t) &= u_C(t_k), \ t \in [t_k, t_{k+1}), \ u_C(t_{k+1}) = 0_n, \\
\quad u_C(t_k) &= \begin{cases} u_k & \text{if } \nu_k^i = 1, \\
\quad u_C(t_k), & \text{otherwise}, \ k \in \{0, \ldots, h-1\}, \quad (81)
\end{cases}
\end{align*}
\]
where \( u_C(t_0^i) := u_0 \).

Let \( \Omega \) be the subset of \( i \in M \) such that \( \nu^i \) differs from \( \nu^1 \), described by (21), only at the first schedule, and consequently also for another schedule, e.g., if \( m = 2, \ q = 2, \nu^1 = (1, 0, 1, 0) \) and the remaining vector of schedules corresponding to \( \Omega \) are \( (0, 1, 1, 0) \), and \( (0, 0, 1, 1) \). For a given arbitrary non-zero \( x_0 \in \mathbb{R}^n \) let
\[
\xi_0^i \mathbf{P}_i \xi_0^i = x_0^T \mathbf{P}_q x_0, \quad (83)
\]
where \( \xi_0^i := \begin{bmatrix} x_0 \\ u^* \end{bmatrix} \).

Then, if \( i \in \Omega \), clearly
\[
\xi_0^i \mathbf{P}_i \xi_0^i \leq \xi_0^i \mathbf{P}_i \xi_0^i \quad (85)
\]

Since choosing (83), (84) is equivalent to solving problem (80) for optimization variables \((u_0, \ldots, u_{n-1}) \in \mathcal{U} \) in a new set containing \( m+1 \) free control inputs
\[
\mathcal{U} := \{ (u_0, \ldots, u_{n-1}) \in \mathcal{R} | u_k = 0 \text{ if } k \neq 0 \text{ and } \nu_k^i = 0 \},
\]
i.e., \( u_0 \) is also a free variable in the equivalent optimization problem. To prove that (85) cannot hold with equality for every \( x_0 \in \mathbb{R}^n \) and every \( i \in \Omega \), and therefore (79) holds for some \( i \in \mathcal{M} \) and \( \xi = \xi_0^i \), we argue by contradiction. If (85) would hold with equality for every \( i \in \Omega \), and a fixed arbitrary \( x_0 \in \mathbb{R}^n \), then by uniqueness of the optimal solution to the problem (80) (cf. [42]), and Assumption 1 this would mean that adding extra control input degrees of freedom (implied in the set \( \mathcal{U} \)) to the optimization problem (80) when \( i = 1 \) would not change the optimal control input solution. However, since the cost in the problem (80) is a quadratic function of \((u_0, \ldots, u_{n-1}) \) which must be convex due to uniqueness of the optimal solution, this would actually imply that having all the control input degrees of freedom \((u_0, \ldots, u_{n-1}) \in \mathcal{R} \) would not change the optimal control input solution. Thus, \( \xi_0^i \mathbf{P}_i \xi_0^i \) would be equal to (82) and making \( i = 1 \) in (80))
\[
\begin{align*}
\quad x_0^T \mathbf{P}_q x_0 &= e^{-\alpha h \tau} x_C(h \tau) \mathbf{P}_q x_C(h \tau) + \\
\min_{(u_0, \ldots, u_{n-1}) \in \mathcal{R}_n \times \cdots \times \mathcal{R}_n} \int_0^{\tau} e^{-\alpha c t} g_C(x_C(t), u_C(t)) dt.
\end{align*}
\]

We can use (86) to obtain an expression for \( x_C(kq \tau) \mathbf{P}_q x_C(kq \tau) \), \( k \in \mathbb{N} \) and recursively replace it in the right-hand side of (86). By doing this and taking the limit of the recursion we obtain
\[
\begin{align*}
\quad x_0^T \mathbf{P}_q x_0 &= \min_{u \in \mathcal{R}_n, k \in \mathbb{N}} \int_0^{\tau} e^{-\alpha c t} g_C(x_C(t), u_C(t)) dt.
\end{align*}
\]

But the right-hand side of (87) equals \( x_0^T \mathbf{P}_q x_0 \) and since \( x_0 \) is arbitrary this would mean \( \mathbf{P}_q = \mathbf{P}_q \) which is a contradiction due to (71).

We state next the second of the two key lemmas. Let \( B_\varepsilon(x) := \{ y \in \mathbb{R}^n | \| y - x \| < \varepsilon \} \) for \( \varepsilon > 0 \) denote the ball of radius \( \varepsilon \) around \( x \in \mathbb{R}^n \).

**Lemma 13:** Suppose that Assumptions 1 and 5 hold and consider Algorithm 3 with \( m \) transmissions along a period \( h \). Then, the following hold.

(i) If \( m \geq 2 \), then for every \( \xi \in \mathbb{R}^n \) and for every open set \( A \subseteq \mathbb{R}^n \),
\[
\mathbb{P}(A) > 0,
\]
for every \( \kappa \geq 1 \). Moreover, if \( m = 1 \), then (88) holds for every \( \kappa \geq 2 \).

(ii) For every \( \xi \in \mathbb{R}^n \) and every \( A \in B(\mathbb{R}^n) \), there exist a continuous non-negative function \( T(\cdot, A) : \mathbb{R}^n \to \mathbb{R}_0^+ \) and a constant \( \varepsilon > 0 \) such that for every \( y \in B_\varepsilon(\xi) \)
\[
\mathbb{P}(T(y, A)) > 0.
\]

**Proof:**
We start by noticing that Assumption 5(i) implies that \( \overline{\Phi}_\tau > 0 \) for every \( \tau \in \mathbb{R}_{>0} \), which in turn implies that
\[
\text{Prob}[x_{k+1} \in B(\xi_k = y) > 0 \quad (91)
\]
for every \( y \in \mathbb{R}^n \), every open set \( B \subseteq \mathbb{R}^n \), and every \( k \in \mathbb{N}_{\geq 0} \), where \( \xi_k = [x_k^T u_k^T]^T \). Suppose that \( m \geq 2 \) and fix a given \( j \geq 1 \). Notice that \( k \in \{(j-1)n+1, \ldots, jh-1\} \)
and \( \mathbb{P}(T(y, \mathbb{R}^n)) > 0 \) (since \( m \geq 2 \) there are at least two transmissions between the time steps \( (j-1)h \) and \( jh-1 \)). Due to (91) we have
\[
\text{Prob}[x_k \in B(\xi_{j-1}) \in B(\xi_{j-1}h = y) > 0 \quad (90)
\]
for every \( y \in \mathbb{R}^n \) and every open set \( B \subseteq \mathbb{R}^n \). Taking into account (28), and the fact that under Assumption 5(ii) the gain matrix \( R_{k\tau}(\xi_{j-1}) - \mathbf{I} \) is full rank (cf. Lemma 11(ii)) this implies
\[
\text{Prob}[\overline{\Phi}_k \xi_{j-1}h = y] > 0 \quad (92)
\]
for every $y \in \mathbb{R}^n$ and every open set $C \subseteq \mathbb{R}^n$, which follows directly from (5) and (28). Moreover, (91) implies that
\[
\text{Prob}[x_{jh} \in D | \xi_{(j-1)h} = y] > 0
\] (93)
for every $y \in \mathbb{R}^n$ and every open set $D \subseteq \mathbb{R}^n$. Noticing that $\xi_j = \xi_j = [x_j^T \ u_j^T]$, (92) and (93) imply that
\[
\text{Prob}[\xi_{jh} \in D \times C | \xi_{(j-1)h} = y] > 0
\]
which implies (88). A similar reasoning can be used for the case $m = 1$ and $j \geq 2$ using the fact that there are at least two transmissions between the time steps $(j - 2)h$ and $jh - 1$.

To prove (ii) we start by defining the set
\[
S := \{ \xi = (\xi_0, \xi_1, \xi_2, \ldots, \xi_{n-1}) : \xi \in \mathbb{R}^n, \xi = \xi_0, \xi = \xi_1, \text{ and } \xi = \xi_{n-1} \}
\]
The complement $S$, denoted by $S^c$, is an open set. From the linearity of the Markov chain (35) (and in particular linearity with respect to initial condition) and the fact that the noise $w_k$, $k \in \mathbb{N}_0$, is Gaussian (results from the discretization of a Wiener process) it is clear that for every $y \in S^c$, $P^1(z, A)$ is a continuous function of $z$ for $z$ in a neighborhood of $y$ which implies (89) and (90) (make $T(z, A) = P^1(z, A)$ for every $z \in \mathbb{R}^n$ and for all $A \subseteq \mathbb{R}^n$). In fact, from (35) we conclude that for $z \in S^c$, in a small neighborhood of $y$, $P^1(z, A) = \text{Prob}[\xi_{(n-1)h} = \xi_{n-1} | \xi_0 = \xi_0, \xi_1 = \xi_1, \xi_{n-1} = \xi_{n-1}]$ is a function of $y = \xi_0$ and $i = i(y)$ that may be different from $i(z)$ if $y \in S$ resulting in a discontinuity. However, if $y \in S$ more than one option $i \in \mathbb{M}$, $\mathbb{M}_y := \{ i \in \mathbb{M} : \xi \in P^1_\kappa(z, A) = \xi_i \}$ can be chosen in an arbitrarily small neighborhood of $y$. Still in this case (89) and (90) are satisfied for
\[
T(z, A) = \min\{P^1_\kappa(z, A) | \kappa \in \mathbb{M}_y \}
\]
for $z$ in a small neighborhood of $y$ where $P^1_\kappa(y, A) := \text{Prob}[\xi_{(n-1)h} = \xi_{n-1} | \xi_0 = y]$, $\kappa \in \mathbb{M}$, when $i(y) = i(\xi_0)$ is replaced by $i$ in (35), i.e., $\xi_i = \Phi_\kappa \xi_0 + \Psi_\kappa w_0$. In fact, each $P^1_\kappa(z, A)$ is a continuous function of $y = \xi_0$ and the minimum of continuous functions is continuous. Moreover, the fact that $P^1_\kappa(y, \mathbb{R}^n) > 0$ for every $y \in \mathbb{R}^n$ and $\kappa \in \mathbb{M}$ implies (90).

C. Proof of Theorems 9, 7, and 10

With the two key lemmas established in Section VIII-B available, we are ready to prove Theorem 9, which uses several results for Markov chains given in [48].

Proof: (of Theorem 9)

First, we notice that Lemma 13(i) implies that (35) is an open set irreducible Markov chain (cf [48, Sec. 6.1.2]) and also that it is an aperiodic chain (cf. [48, Sec. 5.4]). Second, we notice that Lemma 13(ii) implies that (35) is a so-called T-chain (cf. [48, Ch. 6]), which follows from [48, Props. 6.2.3,6.2.4]). Then, it suffices to find a positive coercive function $W$ (\{$\xi | W(\xi) \leq r$\} are precompact for each $r > 0$, [48, Sec.9.4]) such that, for (35),
\[
\text{E}[W(\xi_{(n+1)}) | \xi_n = y] - W(\xi_n) < -1, \ \forall \xi_n \in \mathbb{R}^n - C
\] (94)
for some compact neighborhood of the origin $C$ and such that $\text{E}[W(\xi_{(n+1)}) | \xi_n = y] - W(\xi_n)$ is bounded if $\xi_n \in C$. In fact, then we conclude that the chain (35) is a so-called Harris recurrent chain [48, Th. 9.2.2(ii) and Th. 9.4.1] which implies that there exists a unique invariant measure [48, Th. 10.0.1]. The fact that such invariant measure has finite total mass (in which case (35) is a so-called positive Harris chain) and hence can be made a probability distribution follows from (94) (see [48, Th. 11.0.1]) and ergodicity follows then from the aperiodic ergodic theorem [48, Th. 13.0.1].

We use (60) to establish (94). However we cannot make $W = V$ in (94) since $V$ in (60) is not precompact. Hence we add a regularization term considering a coercive function
\[
W(\xi) = \bar{x}^T \bar{P}_\nu \bar{x} + \nu \bar{u}^T \bar{u}
\]
for $\xi = [\bar{x}^T, \bar{u}^T]^T$ and show that such $W$ satisfies (94) for sufficiently small $\nu$. To prove this we need the fact, established below, that
\[
\text{E}[\bar{u}^T_{\ell+1} \bar{u}_{\ell+1}]^{\xi_\ell} \leq a_{\ell+1} \bar{x}_\ell + d
\] (95)
for every $\xi_\ell = [\bar{x}_\ell^T, \bar{u}_\ell^T]^T \in \mathbb{R}^n$ for given positive constants $a$ and $d$. Then, from (60), (63), (95) we conclude that
\[
\text{E}[W(\xi_{(\ell+1)}) | \xi_\ell] - W(\xi_\ell) \\
\leq -a_{\ell+1} \bar{x}_\ell + c \text{E}[\bar{u}_{\ell+1}]^{\xi_\ell} - a_{\ell+1} \bar{u}_\ell + \nu \bar{u}_\ell \\
\leq (-a_{\ell+1} + \nu) \bar{x}_\ell - a_{\ell+1} \bar{u}_\ell + c \bar{u}_\ell
\]
Picking $\nu = \frac{\alpha d}{2 \alpha}$ and $C$ equal to
\[
\{ \xi = [\bar{x}, \bar{u}]^T \in \mathbb{R}^n : a \bar{x} + \frac{\alpha d}{2 \alpha} \bar{u}, \bar{u} \}
\]
we conclude (94). It is also clear that $\text{E}[W(\xi_{(\ell+1)}) | \xi_\ell] - W(\xi_\ell)$ is bounded if $\xi_\ell \in C$.

It remains to prove (95). To this effect, we notice that
\[
\bar{u}_{\ell+1} = \left[ K_{\text{ei}(\xi_\ell),i(\xi_\ell)}^\nu \right] \bar{u}_\ell
\]
where $K_{\text{ei}(\xi_\ell),i(\xi_\ell)}$ is defined in (78) and $K_{\text{ei}(\xi_\ell),i(\xi_\ell)}$ is defined in a similar way to analogous matrices in (35). In particular $\Phi_\kappa \Psi_\kappa \in \mathbb{R}^0_{m \times k,1} - \Theta_{0,i}$ from which conclude (96) is not a function of $\bar{u}_i$ if $\nu_0 = 1$ for $i = i(\xi_\ell)$, due to the structure of $\Theta_{0,i}$ in (36). This is the case if $\xi_\ell$ lies in the set
\[
\{ \xi = [\bar{x}, \bar{u}]^T \in \mathbb{R}^n : \bar{u} \geq \gamma \bar{x} + d \}
\] (97)
for given sufficiently large positive constants $\gamma$ and $d$. In fact, one can see that the matrices resulting from (22) take the form
\[
P_i = \left[ \begin{array}{cc}
X_i & 0 \\
0 & 0
\end{array} \right], \text{ if } \nu_0 = 1,
\]
\[
P_i = \left[ \begin{array}{cc}
Y_i & Z_i \\
* & Z_i
\end{array} \right], \text{ if } \nu_0 = 0,
\]
for positive-definite matrices $X_i, Y_i, Z_i$ with dimension $n_x \times n_x, n_x \times n_z$, and $n_z \times n_u$, respectively. Positive-definiteness of these matrices can be established using Assumption 1. Then, if $\xi_\ell$ belongs to the set (97) for sufficiently large $\gamma$ and $d$, then it is clear that (27) will correspond to an option $i = i(\xi_\ell)$ such that $\nu_0 = 1$. Then (95) holds for $\xi_\ell$ in the set (97) since then (96) is not a function of $\bar{u}_\ell$ for $\xi_\ell$ in the complement of (97), the norm of $\bar{u}_\ell$ is bounded by the norm
of $\bar{x}_t$ plus a constant, which allows to obtain (95) taking into account (96).

We present next the proofs of Theorems 7, 10 which build upon the proofs of Theorems 4 and 9.

**Proof:** (of Theorem 7 and 10)

To prove (33) consider the $\xi \in \mathbb{R}^n$ and $i \in \mathcal{M}$ characterized in (69) of Lemma 11, under Assumptions 5(i) and 5(ii), and define the following

$$\bar{C} := \{y \in \mathbb{R}^n | y^T P_i y + c_i - (y^T P_i c_i + c_i) < \frac{-\bar{c}}{2}\},$$

where $\bar{c} := \xi^T P_i \xi + c_i - \xi^T P_i c_i - c_i > 0$. Note that $\bar{C}$ is an open set and Lemma 13(i) implies that $P^r_i [\xi, \bar{C}] > 0$ for every $z \in \mathbb{R}^n$ and $r \geq 2$. If Assumptions 5(i) and 5(ii) hold then (54) holds with strict inequality for $r \geq 2$ since for a fixed initial condition $\xi_0 \in \mathbb{R}^n$,

$$E[J^i_0(\xi_0) - J^i_1(\xi_0)] = E[\xi_0^T P_i \xi_0 + c_i - (\xi_0^T P_i \xi_0 + c_i)] \geq \frac{-\bar{c}}{2} P^r_i [\xi_0, \bar{C}] > 0.$$

We can then replace the inequalities in (55) and (56) for $r \geq 2$ by strict inequalities and obtain (33).

To prove (39) we note from (94) and the fact that $f(\xi) \leq V(\xi)$ for every $\xi \in \mathbb{R}^n$ we can conclude that there exists a positive constant $\alpha$ such that

$$E[W(\xi_{k+1})|\bar{\xi}_k] - W(\bar{\xi}_k) \leq -\alpha f(\bar{\xi}_k), \quad \forall \bar{\xi}_k \in \mathbb{R}^n - C \quad (98)$$

which implies that (35) is $f$-ergodic ([48, Ch. 14]). Thus, from [48, Th. 14.0.1], we conclude that

$$\lim_{\ell \to \infty} E[f(\xi_{\ell})] = \lim_{\ell \to \infty} \int_{\mathbb{R}^n} f(\xi) P^\ell_i (\xi_0, d\xi) = \int_{\mathbb{R}^n} f(\xi) \chi_{\text{inv}}(d\xi), \quad (99)$$

and

$$\lim_{L \to \infty} E[\frac{1}{L} \sum_{\ell=0}^{L-1} f(\xi_{\ell})] = \lim_{L \to \infty} \frac{1}{L} \sum_{\ell=0}^{L-1} E[f(\xi_{\ell})] = \int_{\mathbb{R}^n} f(w) \chi_{\text{inv}}(dw). \quad (100)$$

Then (39) follows from (61). Moreover, due to Lemma 13(i) we have that $\chi_{\text{inv}}(A) > 0$, for every open set $A$. This fact that can be proved from the characterization of the unique invariant distribution given in [48, Th. 10.0.1], whose interpretation is the following (c.f., [48, p. 246]): for a fixed measurable set $B$ in $\mathbb{R}^n$ (which we can assume to be open), with $\chi_{\text{inv}}(B) > 0$, $\chi_{\text{inv}}(A)$ is proportional to the amount of time spent in $A$ between visits to $B$, provided that the chain starts in $B$ with a special distribution. Then, noticing that Lemma 13(i) assures that any open set is reached with positive probability from any initial state we conclude that $\chi_{\text{inv}}(A) > 0$ for every open set $A$. Then $g_u > 0$ since $\int_{\mathbb{R}^n} f(w) \chi_{\text{inv}}(dw) \geq \chi_{\text{inv}}(C) > 0$, which implies (34).

**Remark 15:** Note that Assumption 5(ii) simplified significantly the proof of Lemma 13 by guaranteeing that the gains $K_{k+1,i}$, described by (23) and (68), have full rank. Using this fact, we obtained a simple argument for (88) which enabled the proofs of Theorem 7, 9 and 10. Although Assumption 5(ii) is mild in the sense that it holds except in possible pathological cases, we make the following two remarks. First, since we only need to take into account $i \in \mathcal{M}$ such that there exists $\xi \in \mathbb{R}^n$ for which $i(\xi) = i$, i.e., scheduling decisions that can be chosen by Algorithm 3 in (27), then $K_{k+1,i}$ may only need to be full rank for a subset of $i \in \mathcal{M}$. Thus it may be the case that one does not need to test (32) for every value $k \in \{k \in \{1, \ldots, (h-m+1)\}$.

Second, and most importantly, even if Assumption 5(ii) does not hold we may still be able to prove (88). Indeed under Assumption 5, which guarantee that the noise always influences every state $x_k \in \mathbb{R}^n$ after a single step $k$, we may still be able to prove that the noise can influence $u_k$ even if Assumption 5(ii) does not hold.

\[ \square \]

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**References**


