Squaring the Circle: An Algorithm for Generating Polyhedral Invariant Sets from Ellipsoidal Ones

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Abstract
This paper presents a new (geometrical) approach to the computation of polyhedral (robustly) positively invariant sets for general (possibly discontinuous) nonlinear discrete-time systems possibly affected by disturbances. Given a \( \beta \)-contractive ellipsoidal set \( E \), the key idea is to construct a polyhedral set that lies between the ellipsoidal sets \( \beta E \) and \( E \). A proof that the resulting polyhedral set is contractive and thus, positively invariant, is given, and a new algorithm is developed to construct the desired polyhedral set. The problem of computing polyhedral invariant sets is formulated as a number of quadratic programming (QP) problems. The number of QP problems is guaranteed to be finite and therefore, the algorithm has finite termination. An important application of the proposed algorithm is the computation of polyhedral terminal constraint sets for model predictive control based on quadratic costs.

Key words: Positively invariant sets, Contractive sets, Model predictive control, Stability, Robust stability.

1 Introduction
Positively invariant sets and contractive sets have been used in many control theoretic problems, such as the synthesis of stabilizing controllers and the computation of domains of attraction, e.g., see (Kolmanovsky and Gilbert, 1998) and (Blanchini, 1999) for comprehensive overviews. In particular, positively invariant sets play a very important role in the design of stabilizing model predictive controllers (MPC). For example, the terminal cost and constraint set approach in MPC (Mayne et al., 2000) requires that the terminal set is positively invariant under some appropriate local feedback. The most utilized types of invariant sets are the ellipsoidal ones, which have a simple representation, but can be less flexible than polyhedral invariant sets, which can be arbitrarily complex. Polyhedral invariant sets are preferred in various cases due to the fact that they are often derived from physical constraints on state and control variables, which makes them a better approximation of domains of attraction. Moreover, a polyhedral set is more suitable for usage in an optimization problem. For instance, in case of MPC based on quadratic costs, to guarantee recursive feasibility of the MPC optimization problem and stability, one often constrains the terminal state to a terminal set. This set can be naturally chosen as an ellipsoidal sublevel set of a constructed (local) quadratic Lyapunov function, which is needed as terminal cost for the MPC algorithm. However, if an ellipsoidal set is used as the terminal set, then the MPC optimization problem becomes a quadratically constrained quadratic programming (QCQP) problem in case linear prediction models are used (or a mixed integer QCQP problem, if piecewise affine prediction models are used), which is usually not tackled by standard solvers. Note that QCQP problems cannot be solved by QP solvers, but rather require semi-definite programming solvers (Cannon et al., 2001), which are computationally much more demanding. In (Lobo et al., 1998), the authors show how to reduce a QCQP problem into a second order cone program, which can be solved via a primal-dual interior-point method (Nesterov and Nemirovsky, 1994).

If a polyhedral invariant set is employed instead, then the
MPC optimization problem is a standard QP (or mixed integer QP) problem. Since most MPC algorithms with an a priori stability guarantee are based on quadratic costs, e.g., see the survey (Mayne et al., 2000) for an overview, a lot of effort has been put in developing new approaches for computing polyhedral positively invariant sets, see, for example, (Raković et al., 2005), (Pluymers et al., 2005), (Lazar et al., 2006).

In this paper we consider the problem of constructing a polyhedral positively invariant set for discrete-time systems when an ellipsoidal one is already available, which is the case for MPC based on quadratic costs, as mentioned before. Given a $\beta$-contractive $^1$ ellipsoidal set $\mathcal{E}$, the key idea is to construct a polyhedral set that lies between the ellipsoidal sets $\beta \mathcal{E}$ and $\mathcal{E}$. We prove that the resulting polyhedral set is contractive and thus, positively invariant. Next, the problem of fitting a polyhedral set between two ellipsoidal sets is solved by treating the ellipsoidal sets as sublevel sets of quadratic functions and approximates the “outer” quadratic function well enough. A solution to the original problem is then obtained by constructing a piecewise affine (PWA) function that approximates the “outer” quadratic function well enough. A function $f : \mathbb{R}^n \to \mathbb{R}$ is a quadratic function if $f(x) \triangleq x^\top P x + C x + \alpha$ for some $P \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{1 \times n}$ and $\alpha \in \mathbb{R}$. A quadratic function $f$ is strictly convex if and only if $P > 0$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called a piecewise quadratic (PWQ) function, if there exists a polyhedral partition $\Omega_1, \ldots, \Omega_N$ of $\mathbb{R}^n$, if $\Omega_i \cap \mathbb{R}^n$ is a polyhedron (not necessarily closed) for all $i = 1, \ldots, N$, and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, int($\Omega_i$) $\neq \emptyset$ for all $i = 1, \ldots, N$, and $\Omega_{i_1} \cap \cdots \cap \Omega_{i_l} = \emptyset$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is a piecewise polyhedral (PWP) function, if there exists a polyhedral partition $\Omega_1, \ldots, \Omega_N$ of $\mathbb{R}^n$ such that $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = H_i x + a_i$ when $x \in \Omega_i$, for some $H_i \in \mathbb{R}^{1 \times n}$, $a_i \in \mathbb{R}$, $i = 1, \ldots, N$.

An ellipsoid (or an ellipsoidal set) $\mathcal{E}$ is defined as a sublevel set (corresponding to some constant level $f_0 \in \mathbb{R}_+$) of a strictly convex quadratic function, i.e. $\mathcal{E} \triangleq \{ x \in \mathbb{R}^n \mid f(x) \leq f_0 \}$. A piecewise ellipsoidal set is defined in this paper as a sublevel set of a piecewise quadratic function with matrices $P_i > 0$ for all $i = 1, \ldots, N$. Note that the sublevel set of PWA function is a piecewise polyhedral set.

2 Problem Statement and Proposed Solution

Consider the discrete-time perturbed nonlinear system:

$$x_{k+1} = G(x_k, w_k, v_k), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{W} \subset \mathbb{R}^p$ and $v_k \in \mathbb{V} \subset \mathbb{R}^q$ are the state and an unknown parametric uncertainty and disturbance inputs, respectively, and $\mathbb{W}$ and $\mathbb{V}$ are known, bounded sets. $G : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$ is an arbitrary, possibly discontinuous, nonlinear function. We assume that the origin is an equilibrium of (1) for zero disturbance input, meaning that $G(0, w, v) = 0$ for all $w \in \mathbb{W}$.

1 A set $\mathcal{E}$ is a $\beta$-contractive set for an arbitrary discrete-time system, if for all initial conditions in $\mathcal{E}$, the state obtained after one discrete-time step lies in the set $\beta \mathcal{E}$.
Definition 1 For a given $0 \leq \lambda \leq 1$, a set $P \subseteq \mathbb{R}^n$ with $\lambda P \subseteq P$ and $0 \in \text{int}(P)$ is called a (robustly) $\lambda$-contractive set for system (1) if for all $x \in P$, it holds that $G(x, w, v) \in \lambda P$ for all $w \in W$ and all $v \in V$. For $\lambda = 1$ a (robustly) $\lambda$-contractive set is called a (robustly) positively invariant set.

For a set $P \subseteq \mathbb{R}^n$, let $Q_1(P) \triangleq \{x \in \mathbb{R}^n \mid G(x, w, v) \in P, \forall w \in W, \forall v \in V\}$ denote the (robustly) one-step controllable set for system (1), with respect to $P$.

Problem 2 Suppose that a (piecewise) ellipsoidal $\beta$-contractive set with $\beta \in [0, 1]$ is known for system (1). Construct a (piecewise) polyhedral $\lambda$-contractive set with $\lambda \in [0, 1]$ for system (1).

Systematic solutions to obtain $\beta$-contractive (piecewise) ellipsoidal sets are available in the literature for many relevant subclasses of (1), such as linear systems subject to input saturation (Hu et al., 2002), perturbed linear systems (Kolmanovsky and Gilbert, 1998), piecewise affine systems (Ferrari-Trecate et al., 2002), et cetera. Typically, they are obtained as sublevel sets of quadratic (PWQ) Lyapunov functions, which can be calculated efficiently via semi-definite programming.

An alternative solution to Problem 2 can be obtained via the existing algorithms for computing maximal $\lambda$-contractive (positively invariant) sets (also named maximal admissible sets (MAS) in some works). The articles (Blanchini, 1994), (Blanchini et al., 1995), (Kolmanovsky and Gilbert, 1998), (Dorea and Hennet, 1999) provide recursive algorithms for calculating polyhedral invariant sets for linear systems. Although these algorithms do not require that an ellipsoidal contractive set is known, existence of a quadratic Lyapunov function is often used to prove finite termination of the algorithms.

Remark 3 The algorithm developed in the present paper does not produce a maximal positively invariant set, but just a polyhedral positively invariant set. However, while the above-mentioned methods for computing MAS are only applicable to linear systems, the procedure presented here is independent of the system dynamics and can be applied to a wider class of systems, including linear systems, piecewise linear systems and nonlinear systems that are quadratically stabilizable. Furthermore, the polyhedral PI set obtained via the developed algorithm can be used as a starting point for computing the MAS, using the backward procedure of (Blanchini et al., 1995).

In this paper we generalize results from (Blanchini, 1995) to obtain a novel solution to Problem 2. In Lemma 4.1 and Lemma 4.2 in (Blanchini, 1995), where perturbed linear systems are considered, it was shown that a polyhedral set contained in between two convex sublevel sets

2 Notice that $\lambda = 1$ corresponds to a positively invariant set.

of a Lyapunov function is invariant and $\lambda$-contractive. The result of (Blanchini, 1995) is extended in the theorem presented next to a wide class of systems, which includes, for example, any stable (closed-loop) system allowing a PWQ Lyapunov function.

Theorem 4 Consider system (1) and let $E \subseteq \mathbb{R}^n$ be a $\beta$-contractive set for system (1), for some $\beta \in (0, 1)$, that contains the origin in its interior. Let $\lambda P \subseteq P \subseteq E$ for some $\lambda \in (0, 1]$. Then, $P$ is a (robustly) $\lambda$-contractive set for system (1) and $0 \in \text{int}(P)$. Moreover, $Q_1(\lambda P)$ is a (robustly) $\lambda$-contractive set for system (1) and $E \subseteq Q_1(\lambda P)$.

**Proof.** For any $x \in P \subseteq E$ it follows that $G(x, w, v) \in \beta E \subseteq \lambda P$ for any $w \in W$ and any $v \in V$ due to the fact that $E$ is a $\beta$-contractive set for system (1). Hence, $P$ is a (robustly) $\lambda$-contractive set for system (1) and $0 \in \text{int}(P)$ as $\beta E \subseteq \lambda P$. Moreover, from the fact that for any $x \in E$ it holds that $G(x, w, v) \in \beta E \subseteq \lambda P$ for any $w \in W$ and any $v \in V$, it follows that $E \subseteq Q_1(\lambda P)$. Since $P \subseteq E$, we have that $P \subseteq Q_1(\lambda P)$ and thus, $\lambda P \subseteq \lambda Q_1(\lambda P)$. Then, for any $x \in Q_1(\lambda P)$ we have that $G(x, w, v) \in \lambda P \subseteq \lambda Q_1(\lambda P)$ for any $w \in W$ and any $v \in V$. Hence, $Q_1(\lambda P)$ is a (robustly) $\lambda$-contractive set for system (1) and $E \subseteq Q_1(\lambda P)$. \hfill $\Box$

From the above theorem we get the following corollary for the case that $\lambda = 1$ which is related to (robust) positive invariance.

Corollary 5 Consider system (1) and let $E \subseteq \mathbb{R}^n$ be a $\beta$-contractive set for system (1), for some $\beta \in (0, 1)$, with $0 \in \text{int}(E)$. Suppose there exists a set $P \subseteq \mathbb{R}^n$ that satisfies $\beta E \subseteq P \subseteq E$. Then, $P$ is a (robustly) PI set for system (1) and $0 \in \text{int}(P)$. Moreover, $Q_1(P)$ is a (robustly) PI set for system (1) and $E \subseteq Q_1(P)$.

The hypothesis of Theorem 4 requires the satisfaction of the triple inclusion $\beta E \subseteq \lambda P \subseteq E$. The corollary below provides a way to reduce it to a double inclusion at the price of having tighter inclusions.

Corollary 6 Consider system (1) and let $E \subseteq \mathbb{R}^n$ be a $\beta$-contractive set for system (1), for some $\beta \in (0, 1)$, with $0 \in \text{int}(E)$. Suppose there exists a set $P \subseteq \mathbb{R}^n$ that satisfies $\sqrt{\beta} E \subseteq P \subseteq E$. Then, $P$ is a (robustly) $\lambda$-contractive set for system (1) for $\lambda = \sqrt{\beta}$ and $0 \in \text{int}(P)$.

**Proof.** From $\sqrt{\beta} E \subseteq P \subseteq E$ one obtains $\beta E \subseteq \sqrt{\beta} P \subseteq P \subseteq E$.

3 The result also holds when $\beta = 0$ and $\lambda = 0$ except that in this case $P$ does not necessarily contain the origin in its interior.
Notice that the above results apply to certain types of non-convex sets $E$ and $P$, i.e. piecewise ellipsoidal and piecewise polyhedral sets, respectively (see Section 4 for an illustrative example).

**Remark 7** The fact that $E \subset Q_1(\lambda P)$ in Theorem 4 is relevant when the matrix of system (1) is constrained in a compact polyhedral set $\bar{X} \subset \mathbb{R}^n$ with $0 \in \text{int}(\bar{X})$. Then, given the largest $\lambda$-contractive piecewise ellipsoidal set contained in $\bar{X}$, we construct a larger $\lambda$-contractive set, i.e. $Q_1(\lambda P) \cap \bar{X}$.

The case of interest in this paper is, as stated in Problem 2, when $E$ is a piecewise ellipsoidal set and $P$ is a piecewise polyhedral set. By Theorem 4, it is sufficient to construct a piecewise polyhedral set $P$ that lies between the piecewise ellipsoidal sets $\beta E$ (or $\sqrt{\beta}E$) and $E$ to obtain a positively invariant or $\sqrt{\beta}$-contractive solution, respectively to Problem 2. In the next section we present an algorithm for solving this problem of computational geometry and also indicate in Remark 8 how one can solve the triple inclusion $\beta E \subset \lambda P \subset \mathcal{P} \subset E$ of Theorem 4.

### 3 “Squaring the circle”

In this section we present a solution to the problem of fitting a piecewise polyhedral set $\mathcal{P}$ between two piecewise ellipsoidal sets where one is contained in the interior of the other, i.e. $\beta E \subset E$, with $\beta$ a real number $^{4}$ in $(0, 1)$. In case $E$ is an ellipsoid, the main idea is to treat the sets $E$ and $\beta E$ as sublevel sets of two quadratic functions $f_E(x)$ and $f_{\beta E}(x)$, respectively, that correspond to the same constant (level) $f_0 \in \mathbb{R}_+$, i.e. $E \triangleq \{x \in \mathbb{R}^n \mid f_E(x) \leq f_0\}$ and $\beta E \triangleq \{x \in \mathbb{R}^n \mid f_{\beta E}(x) \leq f_0\}$. Then, we compute a PWA function $\bar{f}$ that satisfies $f_{\beta E}(x) \geq f(x) \geq f_E(x)$ for all $x \in \mathbb{R}^n$. The desired piecewise polyhedral set is obtained as $\mathcal{P} \triangleq \{x \in \mathbb{R}^n \mid f(x) \leq f_0\}$.

In case of a piecewise ellipsoidal set $E$ we assume that the polyhedral partitioning $\{\Omega_j \mid j \in S\}$ ($S$ is a finite set of indexes) consists of cones, which ensures that $\beta \Omega_j \subset \Omega_j$. We write $E$ as:

$$E = \bigcup_{j \in S} (\Omega_j \cap \bar{E}_j) \triangleq \{x \in \mathbb{R}^n \mid f_{E_j}(x) \leq f_0\},$$

where $f_{E_j}(x) \triangleq x^T P_j x + C_j x + \alpha_j$ is a strictly convex quadratic function for all $j \in S$. Then, we construct for each $j \in S$ a PWA function $f_{E_j}(x)$, as in the quadratic (ellipsoidal) case mentioned above, such that

where $f_{\beta E_j}(x) > f_j(x) \geq f_{E_j}(x)$ for all $x \in \mathbb{R}^n$. Then, a piecewise polyhedral set $\mathcal{P}$ that satisfies $\beta E \subset \mathcal{P} \subset E$ is simply obtained as

$$\mathcal{P} = \bigcup_{j \in S} (P_j \cap \Omega_j) \text{ with }$$

$$P_j \triangleq \text{Co}(\bar{P}_j) = \text{Co}(\{x \in \mathbb{R}^n \mid f_j(x) \leq f_0\}).$$

Indeed, as $P_j$ is a polyhedral set that satisfies $\beta E_j \subset P_j \subset \mathcal{E}_j$, $j \in S$, we obtain

$$\mathcal{P} = \bigcup_{j \in S} (P_j \cap \Omega_j) \subset \bigcup_{j \in S} (E_j \cap \Omega_j) = E.$$ 

Since $\beta E_j \subset P_j$ and $\beta \Omega_j \subset \Omega_j$ for all $j \in S$, we have that:

$$\beta E = \beta \left(\bigcup_{j \in S} (E_j \cap \Omega_j)\right) = \bigcup_{j \in S} (\beta E_j \cap \Omega_j) = \bigcup_{j \in S} (P_j \cap \Omega_j) = \mathcal{P}.$$ 

As the PWQ case can be split into a finite number of quadratic instances of the problem, we consider only the ellipsoidal case, i.e. when the set $E$ is a sublevel set of a strictly convex quadratic function $f_E$.

Next, choose $P \in \mathbb{R}^{n \times n}$ (with $P > 0$) and $f_0, \alpha E \in \mathbb{R}$ (with $f_0 > \alpha E$) such that $E$ is the sublevel set of $f_E(x) \triangleq x^T Px + \alpha E$, corresponding to the level $f_0$. Then, we have that $\beta E$ is the sublevel set of $f_{\beta E}(x) \triangleq x^T Px + \alpha_{\beta E}$, corresponding to the level $f_0$, where

$$\alpha_{\beta E} \triangleq (1 - \beta^2)f_0 + \beta^2 \alpha E > \alpha E.$$ 

Consider now an initial polyhedron $P_0 \subset \mathbb{R}^n$ that contains $E$. Let $(\theta_0, \ldots, \theta_m)$, with $m \geq n$, be the vertices of $P_0$. An initial set of simplices $S^0_1, \ldots, S^0_n$ that contains these points is determined by Delaunay triangulation (Yenpremian and Falk, 2005). Then, for every simplex $S^0_i \triangleq \text{Co}(\theta_{i0}, \ldots, \theta_{im}), i = 1, \ldots, l_0$, the following operations are performed.

**Algorithm 1**

1. Let $k = 0$.

2. For each simplex $S_i^k$, $i = 1, \ldots, l_k$, construct the matrix $M_i^k \triangleq \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \theta_{i0} & \theta_{i1} & \ldots & \theta_{im} \end{bmatrix}$.

3. Set $v_i^k \triangleq \begin{bmatrix} f_E(\theta_{i0}) & f_E(\theta_{i1}) & \ldots & f_E(\theta_{im}) \end{bmatrix}^T$ and construct the function $f_i^k(x) \triangleq (v_i^k)^T (M_i^k)^{-1} \begin{bmatrix} 1 \\ x \end{bmatrix}$.

$^5$ Note that by a suitable change of coordinates we can always take $C = 0$. 

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$^4$ The case $\beta = 0$ is trivial: any $P \subset E$ with $0 \in \text{int}(P)$ works.
Algorithm 1 computes a simplicial partition of a given initial polyhedral set $\mathcal{P}_0$ that contains the ellipsoidal set $\mathcal{E}$, by splitting a single simplex $S_i^k$ into $n + 1$ simplices. This is done by fixing a new vertex $\bar{x}_i^k$ which is obtained by solving the QP problem (2), and by calculating a new PWA approximation over the new set of simplices.

The steps of Algorithm 1 are repeated for all resulting simplices, until $J_i^{k*} > 0$ for all simplices. At every iteration $k$, a tighter PWA approximation of the quadratic function $f_\mathcal{E}$ is obtained. Algorithm 1 proceeds in a typical branch & bound way, i.e. branching on a new vertex $\bar{x}_i^k$, and bounding whenever it finds a simplex $S_i^k$ for which it holds that $J_i^{k*} > 0$.

Suppose Algorithm 1 stops. At the $k$-th iteration$^6$ for some $k \in \mathbb{Z}_+$, the following PWA function is generated:

$$ f(x) = \begin{cases} f_i^k(x) & \text{when } x \in S_i^k, \quad i = 1, \ldots, l_k \\ H_i^k x + a_i^k & \text{when } x \in S_i^k, \quad i = 1, \ldots, l_k, \end{cases} \quad (3) $$

where $l_k$ is the number of simplices obtained at the end of Algorithm 1 and $H_i^k x + a_i^k = (x_i^k)^T (M_i^k)^{-1} \begin{bmatrix} 1 \\ x \end{bmatrix}$. The PWA function $f$ constructed via Algorithm 1 is a continuous function. Moreover, for $x = \sum_{j=0}^{n} \mu_j \theta_j$, with $\sum_{j=0}^{n} \mu_j = 1$, the corresponding functions $f_i^k$ satisfy:

$$ f_i^k(x) = \sum_{j=0}^{n} \mu_j f_{i,\theta_j}(x) = \sum_{j=0}^{n} \mu_j f_\mathcal{E}(\theta_{ij}), $$

which, by convexity of $f_\mathcal{E}$, implies that $f_i^k(x) \geq f_\mathcal{E}(x)$ for all $x \in S_i^k$ and all $i = 1, \ldots, l_k$. Hence, $f(x) \geq f_\mathcal{E}(x)$ for all $x \in \mathcal{P}_0$. Since the stopping criterion defined in Step 4 of Algorithm 1 assures that at the end of the entire procedure the optimal value $J_i^{k*}$ of the QP problem (2) will be greater than zero in every simplex $S_i^k$, $i = 1, \ldots, l_k$, it follows that

$$ f_\mathcal{E}(x) \leq \bar{f}(x) < f_\mathcal{E}(x), \quad \forall x \in \bigcup_{i=1}^{l_k} S_i^k = \mathcal{P}_0. $$

Then, the sublevel set of $\bar{f}$ given by

$$ \mathcal{P} \triangleq \bigcup_{i=1}^{l_k} \left\{ x \in S_i^k \mid H_i^k x + a_i^k \leq f_0 \right\} $$

satisfies $\beta \mathcal{E} \subset \mathcal{P} \subset \mathcal{E}$. Indeed, note that for $x \in \mathcal{P}$ it holds that

$$ \bar{f}(x) \leq f_0 \Rightarrow f_\mathcal{E}(x) \leq \bar{f}(x) \leq f_0 \Rightarrow x \in \mathcal{E}, $$

and for $x \in \beta \mathcal{E}$ it holds that

$$ f_\mathcal{E}(x) \leq f_0 \Rightarrow \bar{f}(x) < f_\mathcal{E}(x) \leq f_0 \Rightarrow x \in \mathcal{P}. $$

The desired polyhedral set $\mathcal{P}$ (see Figure 1) satisfying $\beta \mathcal{E} \subset \mathcal{P} \subset \mathcal{E}$, is obtained as the convex hull of $\mathcal{P}$. Indeed $\beta \mathcal{E} \subset \mathcal{P} \subset \text{Co} (\mathcal{P}) \triangleq \mathcal{P}$ and, by the convexity of $\mathcal{E}$, it holds that $\mathcal{P} = \text{Co} (\mathcal{P}) \subset \text{Co} (\mathcal{E})$. Notice that the computation of the vertices of $\mathcal{P}$ and of its convex hull can be performed efficiently using, for instance, the Geometric Bounding Toolbox (GBT) (Veres, 1995).

**Remark 8** The following identity

$$ \min_{x} J_i^{k*} = \min_{x \in \mathcal{P}_0} [f_\beta \mathcal{E}(x) - f_i^k(x)] = \epsilon_{\max} - \max_{x \in \mathcal{P}_0} [f_i^k(x) - f_\mathcal{E}(x)] \quad (4a) $$

is an immediate consequence of $f_\beta \mathcal{E}(x) - f_\mathcal{E}(x) = \epsilon_{\max} \triangleq \alpha_\beta - \alpha_\mathcal{E} > 0$. Hence, the error $\epsilon \triangleq \max_{x \in \mathcal{P}_0} [\bar{f}(x) - f_\mathcal{E}(x)]$
3.1 An estimate of the computational complexity

Algorithm 1 computes at every iteration $k$ a tighter PWA approximation $f^k$ of the given strictly convex quadratic function $f_\xi$. It stops when the approximation error obtained at the $k$-th iteration of the algorithm satisfies

$$0 < \min_{x \in P_0} [f_\xi(x) - f^k(x)],$$

or equivalently (due to (4)),

$$\varepsilon_k \triangleq \max_{x \in P_0} [f^k(x) - f_\xi(x)] < \varepsilon_{\text{max}}, \quad k \in \mathbb{Z}_+.$$

The algorithm builds recursively a binary tree, where in each node it stores the vertices of the current simplex $S^k$ and the pairs $(H^i_k, a^i_k)$ such that $f^k(x) = H^i_kx + a^i_k$, for all $x \in S^k, i \geq 1, k \in \mathbb{Z}_+$. If the value of $J^{k*}$ for the current simplex is less than zero, then Algorithm 1 splits $S^k$ into 2 simplices and adds a new level to the tree. The height of the tree can be easily computed once the values of the allowed maximum error $\varepsilon_{\text{max}}$ is known, which yields the following result.

**Theorem 9** Suppose that the initial polyhedral set $P_0$ and the desired final approximation error $\varepsilon_{\text{max}}$ are known. Then, Algorithm 1 has complexity:

$$O \left( 2^{\frac{n(n+1)}{2}} n \log n \left( d_0 \sqrt{\frac{\lambda_{\text{max}}(P)}{\varepsilon}} \right)^\gamma \right),$$

where $\gamma = \frac{1}{1-\log_3 2}$ and $d_0$ is the maximal length edge of the initial simplex (Alessio et al., 2005).

**Remark 10** The number of facets of the resulting polyhedron depends on the contraction factor $\beta$ and on the size of the problem data. As it depends on the number of vertices and simplices generated, in most cases the number of facets is tractable. However, as the contraction factor tends to one, the number of facets of the resulting polyhedron may become intractable.

**Remark 11** By exploiting symmetry of the problem, it is always possible to reduce the overall computational time of the algorithm. If the axes of symmetry of the ellipsoids $f_\xi(x)$ obtained at the end of Algorithm 1 is upper bounded by the allowed maximum error $\varepsilon_{\text{max}} = \max_{x \in P_0} [f_\xi(x) - f^k(x)]$. Thus, the Stop criterion of Algorithm 1 can be set as $J^{k*} > \delta$ for some $\delta \in (0, \varepsilon_{\text{max}})$, instead of just $J^{k*} > 0$, to create a gap between $P$ and $\beta E$. A larger $\delta$ will result in a smaller $\lambda \in (0, 1)$ for which it holds that $\beta E \subset \lambda P \subset P \subset E$. Note that if $\delta$ tends to $\varepsilon_{\text{max}}$, then the number of vertices of $P$ tends to infinity, $P$ recovers the ellipsoidal set $E$ and $\lambda$ tends to $\beta$.

4 Illustrative examples

In this section we present two examples that illustrate the potential of the developed algorithm.

4.1 Perturbed linear systems

Consider the perturbed discrete-time triple integrator:

$$x_{k+1} = Ax_k + Bu_k + v_k, \quad k \in \mathbb{Z}_+, \quad (6)$$

where $A = \begin{bmatrix} 1 & \frac{T_s}{2} \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} \frac{T_s^2}{2} \\ 0 \end{bmatrix}$, $T_s = 0.8$, $v_k \in \mathcal{V}$ is the additive disturbance input, and $\mathcal{V} = [-0.1, 0.1] \times [-0.1, 0.1] \times [-0.1, 0.1]$. We employed the method of (Lazar and Heemels, 2006) to calculate a robust stabilizing state-feedback control law for system (6), i.e. $u_k = Kx_k$, with $K = [-1.1739 - 2.4071 - 2.0888]$, together with a robust quadratic Lyapunov function $V(x) = x^T P x$ with $P = \begin{bmatrix} 14.4684 & 13.5850 & 4.0221 \\ 13.5850 & 17.4575 & 5.4581 \\ 4.0221 & 5.4581 & 2.5528 \end{bmatrix}$. The procedure presented in this paper was employed to calculate a polyhedral set $P$ such that $\beta E \subset P \subset E$, where $\beta$ is a perturbation factor.
\(E\) is the sublevel set of \(V\), corresponding to the level \(f_0 = 20\), and the contraction factor is \(\beta = 0.8\). The resulting set \(P\) is \(\lambda\)-contractive with \(\lambda = 0.9\) and has 56 vertices. A plot of \(P\) is given in Figure 2 together with a plot of the closed-loop system state trajectory obtained for \(x_0 = [3 \ 2 \ 2]^T\) and randomly generated additive disturbance inputs.

### 4.2 Piecewise linear systems

Consider the following open-loop unstable PWL system:

\[
x_{k+1} = \begin{cases} 
A_1 x_k + B u_k & \text{if } E_1 x_k > 0 \\
A_2 x_k + B u_k & \text{if } E_2 x_k \geq 0 \\
A_3 x_k + B u_k & \text{if } E_3 x_k > 0 \\
A_4 x_k + B u_k & \text{if } E_4 x_k \geq 0
\end{cases}
\]

subject to the constraints \(x_k \in X = [-10, 10] \times [-10, 10], u_k \in U = [-1, 1],\) where \(A_1 = \begin{bmatrix} 0.5 & 0.3 \\ 0.9 & 1.35 \end{bmatrix}\), \(A_2 = \begin{bmatrix} -0.92 & 0.64 \\ -0.75 & -0.71 \end{bmatrix}\), \(B = \begin{bmatrix} 1 \end{bmatrix}\), \(A_3 = A_1\) and \(A_4 = A_2\). The state-space partition of the system is given by \(E_1 = -E_3 = \begin{bmatrix} -1 & -1 \end{bmatrix}\), \(E_2 = -E_1 = \begin{bmatrix} -1 & 1 \end{bmatrix}\). The following PWQ Lyapunov function \(V(x) = x^T P_j x\) when \(x \in \Omega_j, j = 1, 2, 3, 4,\) feedback gains and contraction factor \(\beta\) were calculated in (Lazar et al., 2005):

\[
P_1 = \begin{bmatrix} 12.9707 & 10.9974 \\ 10.9974 & 14.9026 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 7.9915 & -5.5898 \\ -5.5898 & 5.3833 \end{bmatrix},
\]

\[
P_3 = P_1, \quad P_4 = P_2,
\]

\[
K_1 = \begin{bmatrix} -0.7757 & -1.0299 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.6788 & -0.4302 \end{bmatrix},
\]

\[
K_3 = K_1, \quad K_4 = K_2, \quad \beta = 0.9378.
\]

Let \(X_U \subseteq X\) denote the set of states for which the feedback control law given in (7)-(8) satisfies the state and input constraints. A contractive piecewise polyhedral set \(P\) was computed for system (7) in closed-loop with the feedbacks given in (8) using the approach of Theorem 4 and Algorithm 1 for the sublevel sets \(E \triangleq \{x \in X \mid V(x) \leq 14\} \subseteq X_U\) and \(E \mathcal{E}\). The resulting set \(P\) is the union of four polyhedra and it is a \(\lambda\)-contractive set with \(\lambda = 0.9286\). The closed-loop state trajectories with the resulting set \(X_U\) and \(\mathcal{E}\) are plotted in Figure 3 together with a plot of the safe set \(X_U\). The trajectories of the closed-loop system remain inside \(P\) at all times and converge to zero.

### 5 Conclusions

A new method for computing (piecewise) polyhedral (robustly) positively invariant and contractive sets was developed based on a geometrical argument. The novelty of the proposed approach consists of formulating the problem of computing polyhedral invariant sets as solving a number of QP problems. This was achieved by observing that any polyhedral set that lies between two ellipsoidal sets \(\mathcal{E}\) and \(\mathcal{E}\) with \(\mathcal{E}\) \(-\)contractive for some \(\beta \in (0, 1)\) is contractive and thus, positively invariant. A new algorithm based on QP was developed to construct the desired polyhedral set. A guarantee that the number of QP problems that need to be solved is always finite was also given. This fact establishes finite termination for the algorithm. Two examples illustrated the wide applicability of the method.

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### References


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