Brief paper

Model reduction for delay differential equations with guaranteed stability and error bound

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ABSTRACT

In this paper, a structure-preserving model reduction approach for a class of delay differential equations is proposed. Benefits of this approach are, firstly, the fact that the delay nature of the system is preserved after reduction, secondly, that input–output stability properties are preserved and, thirdly, that a computable error bound reflecting the accuracy of the reduction is provided. These results are applicable to large-scale linear delay differential equations with constant delays, but also extensions to a class of nonlinear delay differential equations with time-varying delays are presented. The effectiveness of the results is evidenced by means of an illustrative example.

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1. Introduction

Complex dynamical system models in terms of delay differential equations appear naturally in a wide variety of problems in for example engineering, biology and control theory (Altintas, 2000; Erneux, 2009; Gu, Kharitonov, & Chen, 2003; Michiels & Niculescu, 2007; Stepan, 1989). In support of the dynamic analysis, optimization or controller design for such systems, we often desire to employ methods for model complexity reduction. Model order reduction is a tool for the order reduction of high-order dynamical systems in pursuit of complexity reduction. A wide range of results are available for the model order reduction of models in terms of ordinary differential equations, see e.g. Antoulas (2005), Bai (2002), Craig (2000), deKlerk, Rixen, and Voormeeren (2008), Freund (2003), Gallivan, Grimme, and Van Dooren (1999) and Gugercin and Antoulas (2004).

Also for delay differential equations (DDEs) different approaches for model reduction are available, albeit to a more limited extent. Methods for the finite-dimensional approximation of delay systems through rational approximations have been proposed in Märkl and Partington (1999a,b), see also Glover, Curtain, and Partington (1988). Recently, a technique based on the dominant pole algorithm has been proposed to obtain a rational approximation of an input–output transfer function representing second-order delay differential equations (Saadvandi, Meerbergen, & Jarlebring, 2012). A Krylov-based model reduction approach leading to finite-dimensional (delay-free) model approximations has been proposed in Michiels, Jarlebring, and Meerbergen (2011). In Harkort and Deutscher (2011), Krylov methods for infinite-dimensional systems, applicable to delay systems, have been proposed also leading to finite-dimensional approximations. The above methods have the common property that the resulting models are of a finite-dimensional nature; hence the inherent delay nature of the original system is lost.

In this paper, we aim at constructing reduced-order models which preserve the delay nature of the system dynamics (i.e. the reduced-order model is also a delay differential equation, though of a reduced order). The desire to preserve the delay nature in the reduced-order model is motivated by, firstly, the fact that, for a given order of the reduced model, a reduced model in the form of
A delay differential equation is in general more accurate than a reduced model in the form of a delay free system, see e.g. Saadvandi et al. (2012), and, secondly, the fact that by preserving the delay nature also related system properties (such as e.g. the infinite-dimensional system character and the infinite number of eigenvalues) are preserved. Such structure-preserving model reduction techniques for delay differential equations, yielding reduced-order delay models, are needed as, on the one hand, powerful simulation and controller synthesis techniques for such systems have become available in the recent past (Bellen, Maset, Zennaro, & Guglielmi, 2009; Gu et al., 2003; Michiels & Niculescu, 2007; Shampine & Thompson, 2001), while, on the other hand, the main bottleneck of these methods is that in most cases they require the order of the delay differential equation to be moderate. In Beattie and Gugercin (2009), interpolatory projection methods based have been proposed, which are also applicable to delay systems and preserve the delay nature in the reduced-order model. In Jarlebring, Dam, and Michiels (2013), a structure preserving model reduction technique for delay differential equations has been proposed, which extends the notion of position balancing from second-order systems to time-delay systems and relies on solving delay Lyapunov equations (Kharitonov, 2013).

In this paper, we propose a structure-preserving model order reduction strategy for a class of delay differential equations, based on balancing techniques, which, firstly, preserves the delay nature of the model, secondly, guarantees the preservation of both internal and input–output stability properties and, thirdly, comes with a computable error bound on the reduced-order model. We note that the latter two aspects (stability preservation and an error bound) are lacking in the existing results in the literature mentioned above. Error bounds have been proposed for finite-dimensional rational approximations, see Glover et al. (1988). Moreover, error bounds and the preservation of stability are also guaranteed in the works (Lam, Gao, & Wang, 2005; Xu, Lam, Huang, & Yang, 2001), in which an $H_\infty$ model reduction approach for linear time-delay systems has been proposed.

The benefits of the approach proposed in the current paper in comparison with the approach in Lam et al. (2005) and Xu et al. (2001) are twofold. Firstly, by the grace of the fact that we employ balancing-type techniques as a basis, which use the solution to two algebraic Lyapunov equations, the approach proposed here is applicable to systems up to order $O(10^5)$ using standard (Bar- tels–Stewart) algorithms and to systems up to order $O(10^6)$ using tailored algorithms, see e.g. Benner and Saak (2013). On the other hand, the approach in Lam et al. (2005) and Xu et al. (2001) of reformulating the model reduction problem as a $H_\infty$-norm minimization problem of the ‘error system’, induced by the reduction, leads to an (non-convex) optimization problem constrained by a set of matrix inequalities. The latter fact makes such an approach more computationally complex and hence obstructs applicability to systems of high order. Secondly, we propose a natural approach of decomposing the delay system dynamics in terms a feedback interconnection between a finite-dimensional linear part and a delay-operator part. This approach is natural in many applications, in which the delay only affects certain outputs, see e.g. models for high-speed milling processes (Altintas, 2000; Faassen, van de Wouw, Oosterling, & Nijmeijer, 2003; Insperger & Stepan, 2000) and drilling processes (Germann, Denoë, & Detournay, 2009; Ger- man, van de Wouw, Sepulchre, & Nijmeijer, 2009). Moreover, such a decomposition allows to employ incremental $L_2$-gain properties of the systems in the feedback interconnection to guarantee the preservation of stability and to provide an error bound. The latter analysis strategy is also instrumental in supporting the extension of the model reduction approach to systems with nonlinearities and (uncertain) time-varying delays. Finally, we provide an expression for an a priori error bound depending on (1) the properties of the high-order system, (2) the delay and (3) the order of the reduced-order system.

The structure of the paper is as follows. Section 2 specifies in detail the problem formulation and the class of delay systems considered. Next, in Section 3 the model reduction approach is introduced as applicable to a class of linear delay differential equations with constant delays. Section 4 presents the results on the preservation of stability properties and a bound on the reduction error. Moreover, in this section also the extension to nonlinear systems with time-varying delays is highlighted. Finally, Section 5 presents an illustrative example and Section 6 presents concluding remarks.

**Notation.** The field of real numbers is denoted by $\mathbb{R}$. For a vector $x \in \mathbb{R}^n$, $|x|^2 = x^T x$. The space $L_2^\infty$ consists of all functions $x : [0, \infty) \to \mathbb{R}^n$ which are bounded using the norm $\|x\|^2_2 := \int_0^\infty |x(t)|^2 \, dt$.

### 2. Problem formulation

Consider a generic class of linear delay differential equations (with point-wise delay) that can be formulated in the following form:

$$
\begin{align*}
\dot{x}(t) &= A_0 x(t) + A_1 x(t - \tau) + B u(t), \\
y(t) &= C_0 x(t) + D_0 u(t)
\end{align*}
$$

(1)

with $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$ and $u(t) \in \mathbb{R}^p$. Alternatively, the dynamics in (1) can be written in the following form, to be used in the remainder of this paper:

$$
\begin{align*}
\dot{x}(t) &= A_0 x(t) + A_1 (x(t) - x(t - \tau)) + B u(t), \\
y(t) &= C_0 x(t) + D_0 u(t)
\end{align*}
$$

(2)

with $A_0 = A_0 + A_1 \text{ and } A_1 = -\hat{A}_1$.

We study the problem of model reduction for delay differential equations of the form (2) and later comment on extensions to certain classes of nonlinear systems and the case of (uncertain) time-varying delays. Let us explicate what we mean by model reduction for a delay differential equation as in (2). Here, we recall the fact that the model in (2) is infinite-dimensional, i.e. the initial condition for system (2) is the function segment $\phi \in C([-\tau, 0], \mathbb{R}^n)$ with $C([-\tau, 0], \mathbb{R}^n)$ the Banach space of continuous functions mapping the interval $[-\tau, 0]$ to $\mathbb{R}^n$. In fact, we aim to preserve the infinite-dimensional nature of the system in the model reduction approach to be proposed. Still, we can speak of the order of the delay differential equation (2) in terms of the number of equations in the first equality in (2), which in this case is $n$. Now, we aim at constructing a reduced-order model in terms of a linear delay differential equation of order $\tilde{n}$ (i.e. with 'state' $\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$) such that,

- the reduced-order model is also a delay differential equation similar in form to (2), i.e. the delay-nature of the system is preserved;
- $\tilde{n} < n$, i.e. model (order) reduction is achieved;
- if (2) is asymptotically stable (for $u = 0$) and hence finite $L_2$-gain stable with respect to the input/output pair $(u, y)$, then the reduced-order model is also asymptotically stable (for $u = 0$) and $L_2$-gain stable with respect to the same input/output pair $(\tilde{u}, \tilde{y})$, where $\tilde{y}$ is the output of the reduced-order system;
- there exists a computable error bound reflecting the accuracy of the reduction.

Clearly, in the above problem statement we aim at the preservation of asymptotic stability for zero inputs and $L_2$-gain stability with respect to the input/output pair $(u, y)$, the latter of which is defined below (see also Fridman & Shaked, 2006).

\[\text{2 For a definition of asymptotic stability for functional differential equations, we refer to Gu et al. (2003) and Hale and Verduyn Lunel (1993).}\]
Definition 1. System (2) is called \( L_2 \)-gain stable with respect to the input/output pair \((u, y)\) with finite gain \( \gamma \) if for solutions of (2) corresponding to the zero initial condition \((\phi = 0)\) it holds that 
\[ \|y\|_2 \leq \gamma \|u\|_2. \]

Remark 1. We foresee that the results in this paper can be extended towards systems of the form (2) with multiple delays. For the sake of transparency and to alleviate the burden of notation, we do not pursue this extension explicitly in this paper.

3. Model reduction approach

In support of the pursuit of the model reduction of system \( \Sigma \) in (2), let us transform this system into a feedback interconnection of a finite-dimensional linear system \( \Sigma_1 \) and an operator \( \Sigma_2 \) related to the delay (we will denote this feedback interconnection by \((\Sigma_1, \Sigma_2)\)):
\[
\begin{align*}
    \dot{x}(t) &= A_0 x(t) + B_1 v(t) + B_2 u(t), \\
    w(t) &= C_0 x(t) + D_{uv} v(t) + D_{uu} u(t), \\
    y(t) &= C_1 x(t) + D_{vu} u(t), \\
    \Sigma_1 : v(t) &= \int_{t-\tau}^{t} w(s) ds, \\
    \Sigma_2 : (y(t)) &= (\Sigma_1) (x(t)) + \Sigma_2 (u(t, \tau)).
\end{align*}
\]

where \((v(t)), (w(t)) \in \mathbb{R}^q\) and we employed a (rank revealing) decomposition of the matrix \( A_1 \) in (2) in the form \( A_1 = B_0 C_0 \). In other words, the latter decomposition should be performed such that \( B_0 \) has a minimum number of columns in order to make the model reduction pursued henceforward most effective. Moreover, in (3) we defined \( B_0 := B_0 C_0, D_{uv} := C_1 B_1, D_{uu} := C_1 B_2 \). In interpreting how (3)-(4) represent (2), it helps to realize that \( v(t) = C_1 x(t) + x(t - \tau) \) and \( w(t) = \dot{x}(t) \) with \( z(t) = C_0 x(t) \).

In many engineering applications in which models are formulated as delay differential equations, such as e.g. models for high-speed milling processes (Altintas, 2000; Faassen et al., 2003; Insperger & Stepan, 2000) and drilling processes (Germay, Denuel, & Detournay, 2009; Germay, van de Wouw et al., 2009), the matrix \( A_1 \) indeed has low rank. Namely, in such models the high-order related dynamics typically corresponds to models of the structural dynamics of the spindle-tool dynamics in high speed milling or the drill-string dynamics in drilling, while the delay-related terms related to localized cutting processes depending on low-dimensional variables. Similarly, in the context of boundary control of partial differential equations, feedback delays affect control inputs localized at the boundary also leading to models with the matrix \( A_1 \) having low rank.

The system decomposition as a feedback interconnection of a finite-dimensional linear system and a delay-dependent term, see (3), (4), is schematically depicted in Fig. 1. Clearly, with such decomposition we pursue a delay-dependent approach towards the analysis of the delay system involved, see e.g. Gu et al. (2003). Moreover, the form of the system decomposition in (3), (4) naturally supports a model reduction strategy in which the order of \( \Sigma_1 \) is reduced, while \( \Sigma_2 \) is left unchanged. In this way, we meet the objectives, as put forward in Section 2, of achieving order reduction while preserving the delay nature of the system. In particular, we show in Section 3.1 that with a particular reduction approach a reformulation of the reduced system as a DDE is possible.

Let us adopt the following assumption on system (3).

Assumption 1. \( \Sigma_1 \) is asymptotically stable (i.e. \( A_0 \) is Hurwitz).

Remark 2. Note that, due to the asymptotic stability of \( \Sigma_1 \) (Assumption 1), there exist input-output operators \( \mathcal{F}_y : \mathcal{L}_2^q \times \mathcal{L}_2^q \rightarrow \mathcal{L}_2^q \) and \( \mathcal{F}_w : \mathcal{L}_2^q \times \mathcal{L}_2^q \rightarrow \mathcal{L}_2^q \) defined as \( y = \mathcal{F}_y(u, v) \) and \( w = \mathcal{F}_w(u, v) \), respectively. These operators generate the outputs \( y \) and \( w \) of the finite-dimensional linear system \( \Sigma_1 \) for given inputs \( u \) and \( v \) and zero initial condition \( x(0) = 0 \). Linearity and asymptotic stability of \( \Sigma_1 \) together imply a bounded incremental \( L_2 \) gain property, such that the above input–output operators satisfy
\[
\|\mathcal{F}_y(u_1, v_1) - \mathcal{F}_y(u_2, v_2)\|_2 \leq \gamma_y \|u_1 - u_2\|_2 + \gamma_v \|v_1 - v_2\|_2,
\]
for all \( u_1, u_2 \in \mathcal{L}_2^q, v_1, v_2 \in \mathcal{L}_2^q \), and some bounded \( \gamma_y, \gamma_v \geq 0 \) with \( i \in \{y, v\} \). Due to linearity, the incremental \( L_2 \) gain is equivalent to the (non-incremental) \( L_2 \) gain, such that the gains \( \gamma_y \) in (5) can be chosen as the \( H_{\infty} \)-norm of the corresponding transfer functions.

Later, we will use the following lemma on an incremental gain property of the operator \( \Sigma_2 \).

Lemma 1. The operator \( \Sigma_2 \) satisfies the following incremental gain property:
\[
\|v_2 - v_1\|_2 \leq \tau \|w_2 - w_1\|_2, \quad \forall \, w_1, w_2 \in \mathcal{L}_2^q.
\]

Proof. The proof for the non-incremental version of (6), i.e. \( \|v\|_2 \leq \tau \|w\|_2 \), for all \( w \), is given in Fridman and Shaked (2006) and Michiels, Fridman, and Niculescu (2009). Due to linearity, the operator \( \Sigma_2 \) this fact also implies the validity of the incremental gain property in (6).

Let us now adopt the following assumption on the feedback interconnection (\( \Sigma_1, \Sigma_2 \)) given by (3), (4).

Assumption 2. The feedback interconnection (\( \Sigma_1, \Sigma_2 \)) satisfies the small-gain condition
\[
\gamma_{yu} \tau < 1.
\]

Remark 3. Due to the asymptotic stability of \( \Sigma_1 \) (Assumption 1), \( \gamma_{yu} \) always exists (i.e. is bounded) and hence (7) can always be satisfied for small enough delay \( \tau \).

Lemma 2. Consider system (3), (4) satisfying Assumptions 1 and 2. Then the feedback interconnection (\( \Sigma_1, \Sigma_2 \)) is
- \( L_2 \) gain stable with respect to the input/output pair \((u, y)\);
- asymptotically stable for \( u = 0 \).

Proof. Under Assumption 1, there exist bounded \( \gamma_{wu} \) and \( \gamma_{uw} \), such that \( \|u\|_2 \leq \gamma_{wu} \|u\|_2 + \gamma_{uw} \|v\|_2 \). Using (7) and the non-incremental version of Lemma 1, we conclude that
\[
\|w\|_2 \leq \frac{\gamma_{wu}}{1 - \gamma_{wu} \tau} \|u\|_2.
\]
Using (8) and the non-incremental version of Lemma 1 in \( \|y\|_2 \leq \gamma_{wu} \|u\|_2 + \gamma_{wu} \|v\|_2 \) gives
\[
\|y\|_2 \leq \gamma_{wu} \|u\|_2 + \gamma_{wu} \tau \|w\|_2 \leq \frac{\gamma_{wu} + \gamma_{wu} \tau \gamma_{uw}}{1 - \gamma_{wu} \tau} \|u\|_2,
\]
which shows that (\( \Sigma_1, \Sigma_2 \)) is \( L_2 \) gain stable with respect to the input/output pair \((u, y)\). Now, using the fact that system \( \Sigma_1 \) is
an asymptotically stable linear time-invariant system, \( \Sigma_2 \) has a finite impulse response and the feedback interconnection \((\Sigma_1, \Sigma_2)\) satisfies a small gain condition, we can conclude that \((\Sigma_1, \Sigma_2)\) is also asymptotically stable for \( u = 0 \) (see also Huang & Zhou, 2000; Tits & Balakrishnan, 1998). This completes the proof.

In pursuing model reduction of (3), (4), we construct a reduced-order model \( \hat{\Sigma}_1 \) for the finite-dimensional system \( \Sigma_1 \) in the following form:

\[
\begin{align*}
\hat{\Sigma}_1: \quad \dot{\hat{x}}(t) &= \hat{A}_\hat{\Sigma}\hat{x}(t) + \hat{B}_\hat{u}\hat{u}(t), \\
\hat{\dot{y}}(t) &= \hat{C}_\hat{\Sigma}\hat{x}(t) + \hat{D}_{\hat{u}\hat{y}}\hat{u}(t) + \hat{D}_{\hat{y}u}\hat{u}(t),
\end{align*}
\]

with \( \hat{x}(t) \in \mathbb{R}^n \) and \( \hat{n} < n \). For an efficient reduction of the system in (3) to the system in (10), the number of inputs and outputs should be small. For approaches based on balanced truncation, this can be understood from the fact that in such a case the decay rate of the Hankel singular values is fast (Antoulas, Sorensen, & Zhou, 2002). In (3), the number of inputs is determined by the dimension of \( u(t) \) and the dimension of \( v(t) \), the latter of which stems from a feedback interconnection interpretation of the delayed term, see Fig. 1. Hence, it is important to keep the size of \( v(t) \) (and \( u(t) \)) as small as possible. This can be done by starting from a rank revealing decomposition of \( A_t \), i.e., such that the dimension of \( v(t) \) is equal to rank(\( A_t \)).

Let us adopt the following assumption on the reduced-order linear system \( \hat{\Sigma}_1 \).

**Assumption 3.**

- \( \hat{\Sigma}_1 \) is asymptotically stable;
- An (incremental) error bound on reduction of the linear subsystem exists of the form

\[
\|\delta_s(u_1, v_1) - \delta_s(u_2, v_2)\|_2 
\leq \varepsilon_u \|u_1 - u_2\|_2 + \varepsilon_v \|v_1 - v_2\|_2, \tag{11}
\]

for all \( u_1, u_2 \in \mathbb{L}^p_{\varepsilon}; v_1, v_2 \in \mathbb{L}^q_{\varepsilon} \), with \( \varepsilon_u, \varepsilon_v \geq 0 \) and \( 0 < \varepsilon < 1 \).

If we employ balanced truncation (Moore, 1981), optimal Hankel norm approximation (Glover, 1984), or balanced residualization,\(^3\) then the resulting reduced-order linear system is of the form \( \hat{\Sigma}_1 \) and satisfies **Assumption 3**. Note in this respect that the incremental error bound in (11) is, due to linearity, directly implied by an ordinary (i.e. non-incremental) error bound. In Section 3.1, we show that if balanced residualization is used to reduce \( \Sigma_1 \), then the delay-structure of the original system can be preserved in the reduced-order system.

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\(^3\) By balanced residualization, we indicate the singular perturbation approximation of balanced realizations as proposed in Fernando & Nicholson (1982) and Liu and Anderson (1989).

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\(^4\) Proposition 1 holds for systems \( \Sigma_1 \) in arbitrary coordinates.
Proposition 1. Consider the reduced-order system $\tilde{\Sigma}^{br} = (\tilde{\Sigma}^{br}, \Sigma_2)$. These system dynamics can be reformulated in terms of the following delay differential equation

$$
\begin{align*}
\hat{x}(t) &= \tilde{A}_1 \hat{x}(t) + \tilde{A}_2 \left( \hat{x}(t-\tau) - \hat{x}(t-\tau) \right) + \tilde{B}_u u(t), \\
\hat{y}(t) &= \tilde{C}_y \hat{x}(t) + \tilde{C}_x \left( \hat{x}(t-\tau) - \hat{x}(t-\tau) \right) + \tilde{D}_{yu} u(t)
\end{align*}
\tag{16}
$$

with

$$
\begin{align*}
\tilde{A}_1 &:= A_{11} - A_{12} A_{21} \\
\tilde{B}_u &:= B_{u1} - A_{12} A_{21} B_{u2}, \\
\tilde{C}_y &:= C_{1y} - C_{2y} A_{21}, \\
\tilde{C}_x &:= C_{1x} - C_{2x} A_{21}, \\
\tilde{D}_{yu} &:= D_{yu} - C_{2x} A_{21} B_{u2}.
\end{align*}
\tag{17}
$$

which is satisfied by the grace of (17) in the proposition. Now, we have shown that there indeed exists a matrix $\tilde{C}_\tau$ that satisfies $\hat{w} = \tilde{C}_\tau \hat{x}$.

Finally, in order to show that the output equation for $\hat{y}$ can indeed be written as in (16), we write the output equation for $\hat{y}$ as in (15) as follows:

$$
\begin{align*}
\hat{y}(t) &= (\tilde{C}_1 - C_2 A_{22}^T) \hat{x}(t) - C_2 A_{22}^T \hat{v}(t) \\
&\quad + (D_{yu} - C_2 A_{22}^T B_{u2}) u(t)
\end{align*}
\tag{24}
$$

where we used once more that $\hat{v}(t) = \int_{t-\tau}^{t} \hat{w}(s) ds = \int_{t-\tau}^{t} \tilde{C}_\tau \hat{x}(s) ds = \tilde{C}_\tau \hat{x}(t-\tau) - \hat{x}(t-\tau)$. Comparison of (24) with (16) shows that the output equation for $\hat{y}$ can indeed be written as in (16) by the grace of (17) in the proposition. This completes the proof. \hfill \square

Remark 4. It can be shown that if a reduced-order model $\hat{\Sigma}_1$ is constructed by moment matching techniques (Antoulas, 2005) to ensure matching of the moment at $s = 0$, then the resulting reduced-order model $\hat{\Sigma} = (\hat{\Sigma}_1, \Sigma_2)$ can also be formulated in terms of a delay differential equation of the form (16). However, since Assumption 3 does not hold for such moment matching techniques, while it does for balanced residulialization, and Assumption 3 will prove to be essential in proving both stability and an error bound (see the next section), we have limited the formulation of Proposition 1 to the case of balanced residulialization. Finally, we note that balanced residulialization also matches the moment at $s = 0$.

4. Stability analysis and error bound

The following result provides conditions under which, firstly, the reduced-order system inherits certain stability properties from the original system, and, secondly, an error bound can be computed reflecting the accuracy of the reduction.

Theorem 1. Suppose the system (3), (4) satisfies Assumption 1. Let $\hat{\Sigma}_1$ in (10) be a reduced-order linear system satisfying Assumption 3. Then, the following statements hold:

1. The reduced-order system $(\hat{\Sigma}_1, \Sigma_2)$ given by (10), (4) is $\mathcal{L}_2$ stable with respect to the input/output pair $(u, y)$ and asymptotically stable for $u = 0$ if

$$
(y_{uw} + \epsilon_{uw}) \tau < 1; \tag{25}
$$

2. Suppose (25) is satisfied. Then, the output error $\delta y := y - \hat{y}$ is bounded as $||\delta y||_2 \leq ||u||_2$ with

$$
\epsilon = \epsilon_y + \frac{\epsilon_{yu} \tau y_{uw}}{1 - y_{uw} \tau} + \frac{(\epsilon_{yu} + \epsilon_{yw}) \tau}{1 - (\epsilon_{yu} + \epsilon_{yw}) \tau} (\epsilon_{uw} + \frac{\epsilon_{uw} \tau y_{uw}}{1 - y_{uw} \tau}). \tag{26}
$$
Proof.} Inspired by the work in Besselink, van de Wouw, and Nijmeijer (2013), statements (1) and (2) are proven separately below.

Statement (1): Lemma 2 can be employed to show that if \( \hat{\gamma}_{wy} \tau < 1 \), then statement (1) of the theorem is valid. Note that \( \hat{\gamma}_{wy} \) denotes the \( L_2 \)-gain of system \( \hat{\Sigma}_1 \) from input \( w \) to output \( v \), which is bounded by the grace of asymptotic stability of \( \hat{\Sigma}_1 \) (Assumption 3). However, the gain \( \gamma_{wy} \) is not known a priori (i.e. before actually performing the reduction). Still, we can obtain an upper bound for \( \hat{\gamma}_{wy} \) as follows. By the triangle inequality, we have that \( \| \hat{w} \|_2 \leq \| w \|_2 + \| \bar{w} \|_2 \), which implies that \( \| \hat{w} \|_2 \leq \gamma_{wy} \| v \|_2 + \gamma_{wy} \| u \|_2 + \epsilon_{uw} \| v \|_2 + \epsilon_{uw} \| u \|_2 \). Using (11) for \( i = w \). Clearly, \( \gamma_{wy} + \epsilon_{uw} \) provides an upper bound on \( \hat{\gamma}_{wy} \) and, consequently, (25) implies that \( \hat{\gamma}_{wy} \tau < 1 \), which proves, using Lemma 2, that system \( (\hat{\Sigma}_1, \Sigma_2) \) is \( L_2 \) stable with respect to the input/output pair \((u, y)\). Now, using the fact that system \( \Sigma_3 \) is an asymptotically stable linear time-invariant system, \( \Sigma_3 \) has a finite impulse response and the feedback interconnection \((\hat{\Sigma}_1, \Sigma_2)\) satisfies a small gain condition, we can conclude that \( (\hat{\Sigma}_1, \Sigma_2) \) is also asymptotically stable for \( u = 0 \) (see also Huang & Zhou, 2000; Tits & Balakrishnan, 1998).

Statement (2): By using the fact that (25) implies the satisfaction of Assumption 2 (note that \( \epsilon_{uw} \geq 0 \)), we can employ (8) in the proof of Lemma 2 to formulate a bound on \( \| v \|_2 \). Subsequently using (8) and Lemma 2, we can construct the following bound on \( \| v \|_2 \):

\[
\| v \|_2 \leq \frac{\tau \gamma_{uw}}{1 - \gamma_{wy} \tau} \| u \|_2. \tag{27}
\]

The error bound on \( v \), defined by \( \delta v := w - \hat{w} \), satisfies \( \delta w = F_y(u, v) - F_y(u, \hat{v}) \). The bounds on \( \| \delta w \|_2 \leq \| F_y(u, v) - F_y(u, \hat{v}) \|_2 \), which is bounded as follows:

\[
\| \delta w \|_2 \leq \| F_y(u, v) - F_y(u, \hat{v}) \|_2 + \| F_y(u, v) - F_y(u, \hat{v}) \|_2.
\]

Herein, we have that

\[
\| F_y(u, v) - F_y(u, \hat{v}) \|_2 \leq \epsilon_{uw} \| u \|_2 + \epsilon_{uw} \| v \|_2.
\]

which follows from (11). Moreover, we have that

\[
\| F_y(u, v) - F_y(u, \hat{v}) \|_2 \leq \gamma_{wy} \| v - \hat{v} \|_2 \leq \gamma_{wy} \| \delta w \|_2.
\]

As shown in the proof of statement (1) of the theorem, we have that \( \gamma_{wy} \leq \gamma_{wy} + \epsilon_{uw} \). Moreover, Lemma 1 implies that \( \| \delta w \|_2 \leq \tau \| \delta w \|_2 \). Exploiting these two facts in (31) gives

\[
\| \delta w \|_2 \leq \frac{1}{1 - \gamma_{wy} + \epsilon_{uw} \tau} (\epsilon_{uw} \| u \|_2 + \epsilon_{uw} \| v \|_2),
\]

where the small-gain condition in (25) guarantees the existence of the latter bound. Substituting (27) in (32) yields

\[
\| \delta w \|_2 \leq \frac{1}{1 - \gamma_{wy} + \epsilon_{uw} \tau} \left( \epsilon_{uw} + \frac{\epsilon_{uw} \tau \gamma_{wy}}{1 - \gamma_{wy} \tau} \right) \| u \|_2.
\]

We employ Lemma 1 once again to obtain a bound on \( \| \delta v \|_2 \):

\[
\| \delta v \|_2 \leq \frac{\tau}{1 - \gamma_{wy} + \epsilon_{uw} \tau} \left( \epsilon_{uw} + \frac{\epsilon_{uw} \tau \gamma_{wy}}{1 - \gamma_{wy} \tau} \right) \| u \|_2.
\]

Using (29) and (30) in (28) yields

\[
\| \delta v \|_2 \leq \epsilon_{uw} \| u \|_2 + \epsilon_{uw} \| v \|_2 + \gamma_{wy} \| \delta w \|_2.
\]

As shown in the proof of statement (1) of the theorem, we have that \( \gamma_{wy} \leq \gamma_{wy} + \epsilon_{uw} \). Moreover, Lemma 1 implies that \( \| \delta v \|_2 \leq \tau \| \delta v \|_2 \). Exploiting these two facts in (31) gives

\[
\| \delta v \|_2 \leq \frac{1}{1 - \gamma_{wy} + \epsilon_{uw} \tau} (\epsilon_{uw} \| u \|_2 + \epsilon_{uw} \| v \|_2),
\]

which guarantees the small-gain condition in (25) and that the error bound is valid. 

\textbf{Theorem 1} employs knowledge on the error bounds \( \epsilon_{ij}, i \in \{y, w\}, j \in \{u, v\} \), for the linear reduced-order system \( \Sigma_3 \), providing bounds on all relevant input-output pairs. However, existing model reduction techniques for linear systems generally provide a single error bound \( \epsilon_{uw} \), uniform for all input-output pairs. When this error bound is exploited as \( \epsilon_{uw} \leq \epsilon_{uw} \), the error bound (26) reduces to

\[
\epsilon = \epsilon_{uw} \left( 1 + \frac{\gamma_{wy} \tau}{1 - \gamma_{wy} \tau} \right) \left( 1 + \frac{\gamma_{wy} + \epsilon_{uw} \tau}{1 - \gamma_{wy} + \epsilon_{uw} \tau} \right).
\]

The small-gain condition in (25) and the error bound (26) only require knowledge on, firstly, properties of the high-order system \( \Sigma_1 \), secondly, the error bound on the linear reduced-order system \( \hat{\Sigma}_1 \), and, thirdly, the delay and can therefore be evaluated a priori (i.e. without actually performing the reduction first). However, a tighter error bound can be obtained when the gains \( \hat{\gamma}_{wy} \) and \( \gamma_{wy} \) of the reduced-order linear subsystem are computed a posteriori (i.e. after the reduction has been employed). These gains can directly be used in (31) and (35), respectively, instead of using their bounds \( \gamma_{wy} + \epsilon_{uw} \), \( i \in \{y, w\} \). Moreover, the knowledge on \( \hat{\gamma}_{wy} \) can be used for the direct evaluation of the small-gain condition via \( \hat{\gamma}_{wy} \tau < 1 \) instead of via (25), leading to less conservative results.

\textbf{Remark 5.} The results presented above can be extended to a class of nonlinear systems with (potentially uncertain) time-varying delays of the form:

\[
\begin{align*}
\dot{x}(t) &= A_{0x}(t) + B_{0x}(t)(z(t) - \tau \dot{z}(t)) + Bu(t), \\
\Sigma_{nl} : \quad z(t) &= C_{x}(t), \quad \dot{y}(t) = C_{y}(t) + D_{uy}(t), \\
\text{with } x \in \mathbb{R}^n, z \in \mathbb{R}^f, f : \mathbb{R}^f \to \mathbb{R}^n, y \in \mathbb{R}^m \text{ and } u \in \mathbb{R}^p, \text{ and typically } q \ll n. \text{ Namely, system (38) can indeed be written as a feedback interconnection } (\hat{\Sigma}_1, \Sigma_{nl}) \text{ with } \Sigma_1 \text{ as in (3)} \text{ and } \Sigma_{nl} \text{ given by }
\end{align*}
\]

\[
\Sigma_{nl} : v(t) = f \left( \int_{t - \tau}^{t} w(s) ds \right),
\]

If (1) the function \( f \) is globally Lipschitz with Lipschitz constant \( L \) and (2) the time-varying delay \( \tau + \delta(t) \) is a measurable function and satisfies the condition \( -\mu \leq \delta(t) \leq \mu \) for some \( \mu \geq 0 \) and for all \( t \geq 0 \), it can be shown (using results in Michiels et al., 2009; Shustin & Fridman, 2007) that the operator \( \Sigma_{nl} \) satisfies the following incremental gain property: \( \| x_2 - x_1 \|_2 \leq L \sigma \| w_2 - w_1 \|_2 \), for all \( w_1, w_2 \), with \( \sigma := \sqrt{\frac{\gamma_{wy} \tau}{1 - \gamma_{wy} \tau}} \). Now, under the assumption that the feedback interconnection \((\hat{\Sigma}_1, \Sigma_{nl})\) satisfies the small-gain condition \( \gamma_{wy} L \sigma < 1 \), extensions of Lemma 2 and Theorem 1 can be obtained, where in the latter \( \tau \) should be replaced by \( L \sigma \).
Systems of the form (38) are common in application fields such as high-speed milling (Altintas, 2000; Faassen et al., 2003; In-sperger & Stepan, 2000) and deep drilling (Germay, Denoel, & De-tournay, 2009; Germay, van de Wouw et al., 2009) and (without the nonlinearity) also in the scope of networked control systems.

5. Illustrative example

In order to illustrate the model reduction approach for delay differential equations discussed in Section 3 and the results on the preservation of stability and the error bound in Section 4, we consider the vibration isolation problem of a clamped flexible beam system as depicted in Fig. 3. The slender beam has the following dimensions: length × height × width = 1.3 m × 3 mm × 0.1 m. Moreover, the beam material properties are as follows: a mass density of 7746 kg/m³ and Young’s modulus of 200 GPa. Moreover, the beam is subject to a disturbance \( u \) representing an external force, which causes the beam to vibrate in the vertical plane. To attenuate the effect of these disturbances, an actuation force \( \mu \) can be applied by a controller, which acts on a measurement \( y \) at some point of the beam, see Fig. 3. The locations of the disturbance, actuation and sensor are indicated in Fig. 3. The dynamics of the beam is modelled using Euler beam elements, leading to a linear time-invariant dynamical system \( \Sigma_{\text{beam}} \) of the form \[ M\ddot{q} + D\dot{q} + Kq = b_\mu \mu + b_\nu u, \quad y = cq \] (40) with nodal coordinates \( q \in \mathbb{R}^N \) and where \( M, D, \) and \( K \) represent the mass matrix, damping matrix, and stiffness matrix, respectively. For the beam system (40), we assume that the measurement of the vertical deflection \( y \) induces a delay, such that the measurement is given as \( \tilde{y}(\tau) = y(y(\tau - \tau)) \). Then, after transforming (40) to first-order form, an \( \Phi_\infty \)-approach is taken to design a linear time-invariant controller \( \Gamma \) that minimizes (in the \( \Phi_\infty \) sense) the transfer function from the external disturbance \( u \) to the vertical deflection \( y \) by exploiting the actuation force \( \mu \) and measurements \( \tilde{y} \) of the deflection \( y \). In this design procedure, it is assumed that the measurement induces no delay (i.e., \( \tau = 0 \) and hence \( \tilde{y} = y \)), such that standard controller synthesis techniques can be applied. Moreover, as a result of the \( \Phi_\infty \) design procedure, the controller is of the same order as \( \Sigma_{\text{beam}} \), such that the closed-loop system has order \( n = 4N = 300 \). However, in the analysis of the implemented controller, the measurement delay \( \tau \) cannot be neglected, such that the closed-loop system is given as in Fig. 3. Thus, the resulting closed-loop system \( \Sigma \) comprises a linear delay differential equation\(^5\) that can be written in the form (2), with \( n = 300 \). Hence, we take the real vertical deflection \( y \) (rather than the measurement \( \tilde{y} \)) as an output of the closed-loop system.

The performance of the controller can be evaluated by means of simulations. However, to reduce the computational burden of such closed-loop performance analysis, the reduction of the closed-loop system \( \Sigma \) is of interest. We stress that the focus of this example is on facilitating numerical simulations and that the individual reduction of the controller (e.g., to enable implementation) is out of the scope of this paper. Nonetheless, we remark that controller reduction can be achieved in a similar setting by exploiting results from Besselink et al. (2013).

Before discussing the procedure to obtain the reduced-order closed-loop model, we note that the satisfaction of Assumption 1 is guaranteed by the \( \Phi_\infty \) controller design. Namely, the delay is not taken into account in this procedure, guaranteeing asymptotic stability of \( \Sigma_1 \) in (3). Next, by setting the value of the delay to \( \tau = 1 \times 10^{-5} \) s and computing the value of the gain \( \gamma_{uv} \) of \( \Sigma_1 \) as \( \gamma_{uv} = 46.70 \), we readily check that Assumption 2 is fulfilled. As a result, by Lemma 2, the closed-loop system \( \Sigma \) is \( L_2 \) gain stable (from disturbance \( u \) to measurement output \( y \)) and is asymptotically stable for \( u = 0 \).

Following the approach of Section 3, a reduced-order model is obtained by applying balanced residualization to obtain a reduced-order model for \( \Sigma_1 \) as \( \hat{\Sigma}_1 \) (see (10)) of order \( n = 12 \), where we remark that this reduction procedure guarantees the satisfaction of Assumption 3. Moreover, the use of balanced residualization ensures that the infinite-dimensional system resulting from the interconnection of \( \hat{\Sigma}_1 \) and the delay \( \Sigma_2 \) in (4) can be formulated in terms of a delay differential equation of the form (16), as guaranteed by Proposition 1. More specifically, the reduction procedure is performed on a scaled version of \( \Sigma_1 \), where the signal \( v \) is scaled with a factor \( \tilde{S}_v = \frac{\tilde{S}_v}{\tilde{S}_v} \) such that the small-gain condition of Assumption 2 reads \( (\gamma_{uv} S_v) (\gamma_{uv} S_v) < 1 \). The introduction of this scaling allows for balancing the influence of the different outputs of \( \Sigma_1 \), leading to a more accurate reduced-order model. For this scaled model, the error bound as in Assumption 3 is computed as \( \varepsilon_{uv} = 0.737 \) and it readily follows that \( (\gamma_{uv} S_v) (\gamma_{uv} S_v) < 1 \), such that condition (25) in Theorem 1 is satisfied. As a result, the reduced-order system \( \hat{\Sigma}_1 \) is \( L_2 \) gain stable from input \( u \) to output \( y \) and is asymptotically stable for \( u = 0 \). Moreover, the error bound (26) holds, which can be computed as \( \varepsilon = 33.20 \).

Finally, we compare the reduced-order closed-loop system \( \hat{\Sigma}_1 \) and the original high-order system \( \Sigma \) by means of their frequency response functions, see Fig. 4. Clearly, the reduced-order model provides a good approximation, where we recall that the use of balanced residualization guarantees the preservation of steady-state behaviour (or, stated differently, moment matching at \( s = 0 \), see also Remark 4). Moreover, the uncontrolled system \( \Sigma_{\text{beam}} \) is depicted in Fig. 4, showing the effectiveness of the controller in suppressing the first resonance peak.

6. Conclusions

We have proposed a structure-preserving model reduction approach for a class of delay differential equations. In this approach, a finite-dimensional part of the system is separated from the delay characteristics and the former part is reduced through balancing-type techniques. Benefits of this approach are, firstly, the fact that

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\(^5\) The closed-loop model has been made available in numerical form at the webpage http://twr.cs.kuleuven.be/research/software/delay-control/mor/.
the delay nature of the system is preserved after reduction, secondly, that input–output stability properties are preserved and, thirdly, that a computable error bound reflecting the accuracy of the reduction is provided. These results are applicable to large-scale linear delay differential equations with constant delays, but also extensions to a class of nonlinear delay differential equations with time-varying delays are presented. The effectiveness of the result is evidenced by means of an illustrative example of a controlled mechanical system with delay in the feedback loop.

References


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