A Discrete-time Framework for Stability Analysis of Nonlinear Networked Control Systems

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Abstract

In this paper we develop a prescriptive framework for the stabilising controller design based on approximate discrete-time models for nonlinear Networked Control Systems (NCS) with time-varying sampling intervals, potentially large time-varying delays and packet dropouts. As opposed to emulation-based approaches where the effects of sampling-and-hold and delays are ignored in the phase of controller design, we propose an approach in which the controller design is based on approximate discrete-time models constructed for a nominal (non-zero) sampling interval and a nominal delay. Subsequently, sufficient conditions for the global exponential stability and semi-global practical asymptotic stability of the closed-loop NCS are provided. These results represent extensions to the existing literature in several ways. Firstly, the results presented in this paper extend results on the controller design for nonlinear sampled-data systems (with constant sampling intervals and no delays) based on approximate discrete-time models to the case of nonlinear sampled-data systems with delays and to the case of NCS with uncertainties on the sampling intervals and delays. From a different perspective, the results in this paper extend the results on discrete-time modelling and stability analysis for linear NCS with time-varying sampling intervals, delays and packet dropouts to the realm of nonlinear systems. The results are illustrated by means of examples.

Keywords: Networked Control Systems, Approximate Discrete-time Modelling, Delays, Stability Analysis, Nonlinear Systems

1 Introduction

Networked control systems (NCSs) are control systems in which sensor data and control commands are being communicated over a wired or wireless communication network. The recent increase of interest in NCSs is motivated by many benefits they offer such as the case of maintenance and installation, the large flexibility and the low cost [49,53]. Moreover, NCSs are applied in a broad range of systems, such as mobile sensor networks, remote surgery, automated highway systems and unmanned aerial vehicles, see e.g. [19,42,49]. However, still many challenges need to be faced before all the advantages of wired and wireless networked control systems can be exploited to their full extent. One of the major challenges is related to guaranteeing the robustness of stability (and performance) of the control system in the face of imperfections and constraints imposed by the communication network, see e.g. the survey papers [19,42,49,53]. Typically, the following types of network-induced imperfections and constraints can occur: variable sampling/transmission intervals, variable communication delays and packet dropouts caused by the unreliability of the network, so-called communication constraints caused by the sharing of the network by multiple nodes and quantization-related errors.

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Most of the work on NCSs has been focussing on the stability analysis of linear NCS. In the scope of linear NCSs, different approaches towards the modelling and stability analysis have been developed. In [14, 51] a continuous-time modelling approach is taken leading to NCS models in terms of (infinite-dimensional) delay-differential equations (DDEs) and stability analysis results based on the Lyapunov-Krasovskii functional method. A related approach is to model the NCS as a delay-impulsive differential equation [25–27, 43] where the piece-wise continuous nature of the control system induced by sample-and-hold is taken into account exactly. Emulation-based approaches (also applicable to nonlinear systems) have been reported in [2, 8, 16, 17, 28, 29, 31, 41, 45, 46], where the control design is based on the continuous-time plant, ignoring the effect of sampling-and-hold and the network, and stability analysis is performed on the basis of a hybrid systems model of the NCS. Finally, discrete-time approaches, based on the exact discretisation of the linear plant (typically on the sampling instants) have been developed in [6, 7, 12, 13, 15, 18, 20, 35, 38, 43, 47, 48, 52, 53] and others.

Results on the stability analysis and controller design for nonlinear NCS have also been pursued in the literature. In [1, 50] a continuous-time approach leading to NCS models in terms of delay-differential equations (DDEs) and stability analysis results based on the Lyapunov-Krasovskii functional method is pursued for certain classes of nonlinear systems. Results on the stabilisation of nonlinear systems with limited-capacity communication channels (i.e. quantisation-related issues) have been reported in [4, 22, 39]. Model predictive control strategies for nonlinear NCS can be found in [23], focussing on the effect of measurement delays being multiple of the constant sampling interval, and [9], focussing on stabilisation in the face of data losses. In [2, 3, 16, 17, 28, 29, 31, 41, 45], a comprehensive emulation-based framework for the stability analysis of nonlinear NCS has been developed. These results consider network-induced effects such as time-varying sampling intervals, delays, packet dropouts, communication constraints and quantisation; however, the results are limited to the small delay case (delays smaller than the sampling interval).

Results on discrete-time approaches for nonlinear NCS are rare. Some extensions of the discrete-time approach for sampled-data systems as developed in [30, 33] towards NCS-related problem settings have been pursued in [36, 37], in [36], an extension towards multi-rate sampled-data systems is proposed. In [37], results for NCS with time-varying sampling intervals and delays for a specific predictive control scheme and matching protocol are presented. However, in these results the delays are always assumed to be a multiple of the sampling interval and delays are artificially elongated to match a ‘worst-case’ delay. Firstly, artificially elongating the delay to a ‘worst-case’ delay being a multiple of the sampling interval may be detrimental to the stability and performance of the NCS, especially when the uncertainty on the delay is large. Moreover, although some early works on NCS also adopted such rather restrictive assumptions on the delays [24], in many subsequent works on linear NCS, see e.g. [7, 15, 20, 48, 53], it has been pointed out that a realistic delay modelling should allow for delays to be different from (multiples) of the sampling interval and should incorporate such delay modelling by accounting for the fact that multiple different controller commands may be active within one sampling interval. The latter realistic modelling of delays in NCS drastically increases the complexity of the stability analysis and controller design problem at hand.

In this paper we develop a prescriptive framework for the stabilising controller design based on approximate discrete-time models for NCS with time-varying sampling intervals, potentially large and time-varying delays, not being limited to multiples of the sampling interval, and packet dropouts. Although an emulation-based approach is powerful in its simplicity since, in the phase of controller design, ignores sampled-data and network effects, an approach towards stability analysis and controller design based on approximate discrete-time models may exhibit several advantages over an emulation-based approach. Firstly, in the emulation approach one typically designs the controller for the case of fast sampling (and no delay) and subsequently investigates the robustness of the resulting closed-loop NCS with respect to uncertainties in the sampling intervals (and delays), see [16, 17, 28, 31]. In the context of networked control one generally faces the situation in which sampling intervals exhibit some level of jitter (uncertainty) around a nominal (non-zero) sampling interval and the delays exhibit some uncertainty around a nominal delay. It appeals to our intuition, which is supported by earlier results for nonlinear sampled-data systems in [21, 30, 33], that it is beneficial to design a discrete-time controller based on a nominal (non-zero) sampling interval and a nominal delay. Secondly, it has been shown in [21, 33] for the case of nonlinear sampled-data systems with fixed sampling intervals (and no delays) that controllers based on approximate discrete-time models may provide superior performance (in terms of the domain of attraction and convergence speed). Finally, we would like to note that, for the case of linear NCSs, it has been shown in [10, 11], that the discrete-time approach may provide less conservative bounds on sampling intervals and delays.

The contributions of this paper can be summarised as follows. Firstly, the results in this paper extend the results of [30, 33] on the stabilisation of nonlinear sampled-data systems based on approximate discrete-time models to the case with delays (and uncertainties in the sampling intervals and delays). Secondly, we develop a prescriptive framework for the robustly stabilising discrete-time controller design for nonlinear NCS with time-varying sampling intervals, large time-varying delays and packet dropouts. In this sense it extends results on the discrete-time approach
The outline of the paper is as follows. In Section 2, an (approximate) discrete-time modelling approach for nonlinear NCS will be discussed. Based on the resulting approximate discrete-time models, parametrised by the nominal sampling interval and delay, and discrete-time controllers designed to stabilise these approximate models, we propose sufficient conditions for the global exponential stability of the closed-loop sampled-data NCS in Section 3. In Section 4, we relax these conditions to obtain sufficient conditions for semi-global practical asymptotic stability. The results are illustrated by means of examples in Section 5. Finally, concluding remarks are given in Section 6. The proofs can be found in Appendix B.

The following notational conventions will be used in this paper. \( \mathbb{R} \) denotes the field of all real numbers and \( \mathbb{N} \) denotes all nonnegative integers. By \( | \cdot | \) we denote the Euclidean norm. A function \( \alpha : [0, \infty) \to [0, \infty) \) is said to be of class-\( \mathcal{K} \) if it is continuous, zero at zero and strictly increasing. It is of class-\( \mathcal{K}_\infty \) if it is of class-\( \mathcal{K} \) and unbounded. A continuous function \( \beta : [0, \infty) \times [0, \infty) \to [0, \infty) \) is said to be of class-\( \mathcal{K}_\mathcal{L} \) if \( \beta(., t) \) is of class-\( \mathcal{K} \) for each \( t \geq 0 \) and \( \beta(s, .) \) is monotonically decreasing to zero for each \( s > 0 \). We denote the transpose of a matrix \( A \) by \( A^T \). For a symmetric positive definite matrix \( P = P^T > 0 \), \( \lambda_{\max}(P) \) denotes the maximum eigenvalue of \( P \). For a locally Lipschitz function \( f(x) \), \( \partial f(x) \) denotes the generalised differential of Clarke [5].

## 2 Discrete-time Modelling of Nonlinear NCS

Consider a NCS as depicted schematically in Figure 1. The NCS consists of a nonlinear continuous-time plant

\[
\dot{x} = f(x, u),
\]

(1)

where \( f(0, 0) = 0, \ x \in \mathbb{R}^n \) is the state and \( u \in \mathbb{R}^m \) is the continuous-time control input, and a discrete-time static time-invariant controller, which are connected over a communication network that induces network delays (\( \tau^{sc} \) and \( \tau^{ca} \)). The state measurements of the plant are being sampled by a time-driven sampler at the sampling instants \( s_k \), where the related sampling intervals \( h_k = s_{k+1} - s_k \) are possibly time-varying such that \( h_k \in [\underline{h}, \overline{h}] \), \( \forall k \in \mathbb{N} \), with \( 0 < \underline{h} \leq \overline{h} \). Let us denote \( x_k := x(s_k) \). Moreover, \( u_k \) denotes the discrete-time controller command corresponding to \( x_k \). In the model, both the varying computation time (\( \tau^{sc}_k \), needed to evaluate the controller, and the time-varying network-induced delays, i.e. the sensor-to-controller delay (\( \tau^{sc}_k \)) and the controller-to-actuator delay (\( \tau^{ca}_k \)), are taken into account. As indicated above, the sensor acts in a time-driven fashion and we assume that both the controller and the actuator act in an event-driven fashion (i.e. responding instantaneously to newly arrived data). Under these assumptions, all three delays can be captured by a single delay \( \tau_k := \tau^{sc}_k + \tau^{ca}_k \), see also [53]. Furthermore, we consider that not all the data may be used due to message rejection, i.e. the effect that more recent control data is available before the older data is implemented and therefore the older data is neglected. We assume that the time-varying delays are bounded according to \( \tau_k \in [\underline{\tau}, \overline{\tau}] \), \( \forall k \in \mathbb{N} \), with \( 0 \leq \underline{\tau} \leq \overline{\tau} \). Note that the delays may be both smaller and larger than the sampling interval. Define \( \overline{d} := \lceil \tau/\overline{h} \rceil \), the largest integer smaller than or equal to \( \tau/\overline{h} \) and \( \overline{\overline{d}} := \lceil \tau/\overline{h} \rceil \), the smallest integer larger than or equal to \( \tau/\overline{h} \). Finally, the zero-order-hold (ZOH) function (in Figure 1) is applied to transform the discrete-time control input \( u_k \) to a continuous-time control input \( u(t) = u_k(\cdot) \), where \( k^*(t) := \max\{k \in \mathbb{N} | s_k + \tau_k \leq t \} \). More explicitly, in the sampling interval \([s_k, s_{k+1}]\), \( u(t) \) can be described by

\[
u(t) = u_{k+j-\overline{d}} \quad \text{for} \quad t \in [s_k + t_j^k, s_k + t_{j+1}^k],
\]

(2)

where the actuation update instants \( t_{j}^k \in [0, h_k] \) are defined as:

\[
t_j^k = \min \left\{ \max\{0, \tau_{k+j-\overline{d}} - \sum_{l=k+j-\overline{d}}^{k-1} h_l\}, \max\{0, \tau_{k+j-\overline{d}+1} - \sum_{l=k+j+1-\overline{d}}^{k-1} h_l\}, \ldots, \max\{0, \tau_{k-\overline{d}} - \sum_{l=k-\overline{d}}^{k-1} h_l\}, h_k \right\}
\]

(3)

\[\text{Figure 1. Schematic overview of the networked control system.}\]
In general the exact discrete-time model is unknown since the plant is nonlinear and, consequently, we can not explicit expressions for

\[ \psi_j^k = [\tau_k - \bar{d} + j, \ldots, \tau_k - \bar{d} + 1, h_k - \bar{d} + j, \ldots, h_k - \bar{d} + 1] \]

containing all past delays and sampling intervals defining \( t^k_j \), i.e. we can write (3) as \( t^k_j = t^k_j(\psi^k_j) \). Note that \( \psi_j^k \in \Psi_j := [\tau, \bar{d} - j + 1] \times [\bar{d} - j + 1] \) for all \( k \in \mathbb{N} \) and \( j \in \{0, 1, \ldots, \bar{d} - d\} \).

Next, let us consider the exact discretisation of (1), (2), (3) at the sampling instants \( s_k \):

\[ x_{k+1} = x_k + \int_{s_k}^{s_{k+1}} f(x(s), u(s)) \, ds = x_k + \sum_{j=0}^{\bar{d} - d} \int_{s_k + t^k_j}^{s_{k+1} + t^k_j} f(x(s), u_{k+j-\bar{d}}) \, ds =: F^u_{\theta_k}(x_k, \bar{u}_k, u_k) \tag{4} \]

with \( \theta_k := [h_k, t^k_1, t^k_2, \ldots, t^k_{\bar{d} - d}]^T \in \mathbb{R}^{\bar{d} - d + 1} \), all \( k \in \mathbb{N} \), the vector of uncertainty parameters consisting of the sampling interval \( h_k \) and the control update instants within the interval \([s_k, s_k+1]\). Moreover, \( \bar{u}_k := [u^T_{k-1}, u^T_{k-2}, \ldots, u^T_{k-\bar{d}}]^T \) represents a vector containing past control inputs. The uncertain parameter vector \( \theta_k \) is taken from the uncertainty set \( \Theta \) with

\[ \Theta = \{ \theta \in \mathbb{R}^{\bar{d} - d + 1} \mid h \in [\bar{d} - \bar{d}], t_j \in [\bar{l}_j, \bar{t}_j], 1 \leq j \leq \bar{d} - d, 0 \leq t_1 \leq \ldots \leq t_{\bar{d} - d} \leq h \} \tag{5} \]

where \( \bar{l}_j \) and \( \bar{t}_j \) denote the minimum and maximum values of \( t^k_j \), \( j = 1, 2, \ldots, \bar{d} - d \), respectively, given by

\[ \bar{l}_j = \min_{\psi_j \in \Psi_j} t_j(\psi_j), \quad 1 \leq j \leq \bar{d} - d \] \quad and \quad \bar{t}_j = \max_{\psi_j \in \Psi_j} t_j(\psi_j), \quad 1 \leq j \leq \bar{d} - d \] \tag{6}

Explicit expressions for \( \bar{l}_j \) and \( \bar{t}_j \) are given in [6]: \( \bar{l}_j = \min\{\tau - \bar{d} - h, h\} \) for \( j = \bar{d} - d \), \( \bar{l}_j = 0 \) for \( 1 \leq j < \bar{d} - d \), and \( \bar{t}_j = \min\{\tau - (\bar{d} - j) \bar{d} - h\} \) for \( 1 \leq j \leq \bar{d} - d \). Additionally, \( t^k_0 := 0 \) and \( t^k_{\bar{d} - d + 1} := h_k \), which gives for the minimum and maximum bound \( t^k_j \in [\bar{d} - d, \bar{d} - d + 1] \) for all \( k \in \mathbb{N} \).

Let us now introduce the extended (lifted) state vector \( \xi_k := [x^T_k, u^T_{k-1}, u^T_{k-2}, \ldots, u^T_{k-\bar{d}}]^T = [x^T_k, \bar{u}^T_k]^T \in \mathbb{R}^{n + \bar{d}m} \).

Then, the exact discrete-time plant model can be written as:

\[ \xi_{k+1} = \left[ x^T_{k+1}, u^T_k, u^T_{k-1}, \ldots, u^T_{k-\bar{d}+1} \right]^T = \left[ F^u_{\theta_k}(x_k, \bar{u}_k, u_k) u^T_{k-1}, u^T_{k-2}, \ldots, u^T_{k-\bar{d}+1} \right]^T =: F^{\bar{u}}_{\theta_k}(\xi_k, u_k). \tag{7} \]

In general the exact discrete-time model is unknown since the plant is nonlinear and, consequently, we can not explicitly compute the exact model. In order to design a stabilising discrete-time controller we construct an approximate discrete-time plant model based on a nominal choice for the uncertain parameters \( \theta_k \) given by \( \theta^* = [h^*, t^*_1, t^*_2, \ldots, t^*_{\bar{d} - d}]^T \in \Theta \subset \mathbb{R}^{\bar{d} - d + 1} \), where \( h^* \in [\bar{d} - \bar{d}] \) is a nominal sampling interval and \( t^*_j \in [\bar{l}_j, \bar{t}_j] \), \( j \in \{1, 2, \ldots, \bar{d} - d\} \) are nominal control update instants. Note that arbitrarily choosing the nominal parameter vector

\[ F^{\bar{u}}_{\theta^*}(\xi^*_k, u^*_k) \]

\[ \begin{array}{c}
\begin{array}{cccccc}
\hline
j = 0 & j = 1 & j = 2 & j = 3 & j = \bar{d} - d \\
\hline
s_k & s_k + t^0_1 & s_k + t^1_2 & s_k + t^2_3 & s_k + t^3_{\bar{d} - d} = s_{k+1} \\
\hline
\end{array}
\end{array} \]

Figure 2. Graphical interpretation of \( t^k_j \).
\( \theta^* = [h^*, t^*_1, t^*_2, \ldots, t^*_d]^{T} \in \Theta \subseteq \mathbb{R}^{|d|+1} \), such that \( h^* \in [\underline{h}, \bar{h}] \) and \( t^*_j \in [\underline{t}_j, \bar{t}_j], j \in \{1, 2, \ldots, \bar{d} - d\} \), may lead to sequences of control update instants that, when repeated for each sampling interval, represent unfeasible sequences of control updates for the real NCS. Therefore, and for reasons we will address later in more detail, see Remark 4, we will choose \( \theta^* \) in a particular way. Let us define \( \theta^* := [h^*, t^*_1, t^*_2, \ldots, t^*_d]^{T} \in \mathbb{R}^{|d|+1} \) with

\[
h^* := \varepsilon \bar{h} + (1 - \varepsilon)\bar{h}, \quad 0 \leq \varepsilon < 1 \quad \text{and} \quad t^*_j :=\begin{cases} 0, & j \in \{0, 1, \ldots, \bar{d} - d^* - 1\} \\ \tau^* - d^* h^*, & j = \bar{d} - d^* \\ h^*, & j \in \{\bar{d} - d^* + 1, \ldots, \bar{d} - 1\} \end{cases},
\]

where \( \tau^* \in [\underline{\tau}, \bar{\tau}], d^* := \lceil \tau^*/h^* \rceil \). Note that \( \theta^* \) now only depends on two nominal parameters; namely \( h^* \), which represents the nominal sampling interval, and \( \tau^* \), which represents the nominal delay. Hence, the nominal control update instants \( t^*_j \) correspond to this nominal sampling interval \( h^* \) and nominal delay \( \tau^* \). See Figure 3 for a graphical explanation of the meaning of the nominal control update instants \( t^*_j \).

![Graphical interpretation of \( t^*_j \).](image)

By exploiting a discretisation scheme we can now formulate the approximate discrete-time plant model as:

\[
x_{k+1} = F_{\theta^*}^0(x_k, \bar{u}_k, u_k),
\]

which leads to

\[
\xi_{k+1} = \left[ F_{\theta^*}^T(x_k, \bar{u}_k, u_k) u_k^T \quad u_k^T \quad \ldots \quad u_k^T \right]^{T} := \bar{F}^0_{\theta^*}(\xi_k, u_k)
\]

and corresponds to the nominal parameter vector \( \theta^* \) defined in (8). Next, we design a controller given by \( u_{\theta^*}(\xi) \) for a nominal distribution of the (past) control inputs over the sampling interval \( [s_k, s_{k+1}] \) corresponding to the nominal parameter vector \( \theta^* \) defined in (8). The discrete-time controller

\[
u_k = u_{\theta^*}(\xi_k)
\]

will now be designed to stabilise this approximate discrete-time plant model for a nominal parameter vector \( \theta^* \). In fact, since \( \theta^* \) only depends on \( h^* \) and \( \tau^* \), \( u_{\theta^*}(\xi) \) is a controller that is designed to stabilise the system for the nominal sampling interval \( h^* \) and nominal delay \( \tau^* \). Let us now define the set

\[
\Theta_0^* = \Theta_0^{0,\bar{h},\bar{d}} \left\{ \theta^* \in \mathbb{R}^{|d|+1} \mid h^* \in (0, \bar{h}], \tau^* \in [dh^*, \bar{d}h^*], t^*_j := 0, \text{ for } j \in \{0, 1, \ldots, \bar{d} - d^* - 1\}, \right. \\
\left. t^*_j := \tau^* - d^* h^*, \text{ for } j = \bar{d} - d^*, t^*_j := h^*, \text{ for } j \in \{\bar{d} - d^* + 1, \ldots, \bar{d} - 1\} \right\},
\]

where \( \bar{h} \geq \bar{h} \). Note that the set \( \Theta_0^* \) is defined such that:

1. \( h^* \) resulting from (8) is always covered by the range of sampling intervals in \( \Theta_0^* \);
2. the definition of \( \Theta_0^* \) in (12) allows \( h^* \) to be taken arbitrarily close to zero;
3. any \( \tau^* \in [\underline{\tau}, \bar{\tau}] \) resulting from (8) is always covered by the range of nominal delays in \( \Theta_0^* \). Namely, for \( \tau^* \in [\underline{\tau}, \bar{\tau}] \) as in (8) it holds that \( \tau^* \leq \tau \leq \bar{d}h \leq \bar{d}h^* \) and it holds that \( \tau^* \geq \tau \geq \underline{d}h \geq \underline{d}h^* \).
In Sections 3 and 4, we will require the approximate discrete-time plant model \( \tilde{F}_{\theta^*}(\xi, u) \), the controller \( u_{\theta^*}(\xi) \) and the resulting approximate discrete-time closed-loop system \( \tilde{F}_{\theta^*}(\xi, u_{\theta^*}(\xi)) \) to exhibit certain properties for \( \theta^* \in \Theta^* \subseteq \Theta_0^* \) that will be used to guarantee certain stability properties for the exact discrete-time closed-loop system \( F_{\theta^*}(\xi, u_{\theta^*}(\xi)) \).

**Remark 1**
The usage of an extended state feedback controller as in (11) places some restrictive assumptions on the delays that may occur, which can be avoided by considering pure state feedback controllers of the form \( u_k = u_{\theta^*}(x_k) \), see [6].

**Remark 2**
Packet dropouts can be directly incorporated in the above model, see [6] for the appropriate expressions for \( t_j^k \) in the case of packet dropout assuming that there exists a bound on the maximal number of subsequent packet dropouts. Alternatively, the occurrence of packet dropouts can be modelled as an extension of the sampling interval, see e.g. [15, 44]. For the sake of clarity, we refrain from making such extensions explicit in the current paper.

The problem considered in the paper can now be formulated as follows. Given a nonlinear plant and a (family of) discrete-time controllers, parametrised by and designed for a nominal sampling interval and a nominal delay, we aim to provide sufficient conditions for the robust stability of the resulting sampled-data NCS in the face of uncertainties in the sampling interval and delays.

### 3 Global Exponential Stability of the NCS

In this section we aim to formulate conditions under which the closed-loop sampled-data system (1), (2), (3), (11) is globally exponentially stable (GES). A Lyapunov characterisation of GES, exploited in the next section, for uncertain discrete-time nonlinear systems is given in Appendix A.1.

#### 3.1 Sufficient conditions for GES

Let us adopt the following assumptions.

**Assumption 1**
There exist a parametrised family of functions \( V_{\theta^*}(\xi) \), a parametrised family of controllers \( u_{\theta^*}(\xi) \), \( a_i > 0, i = 1, 2, 3 \), and \( \overline{h} > \overline{h}^* \) such that the following inequalities hold for some \( 1 \leq p < \infty \):

\[
a_1|\xi|^p \leq V_{\theta^*}(\xi) \leq a_2|\xi|^p, \quad \text{and} \quad \frac{V_{\theta^*}(\xi)}{\overline{h}^*} - \frac{V_{\theta^*}(\xi)}{h^*} \leq -a_3|\xi|^p, \quad \forall \xi \in \mathbb{R}^{n+\overline{d}m}, \forall \theta^* \in \Theta^*
\]

with \( \Theta^* \subseteq \Theta_0^* \) and \( \Theta_0^* \) as in (12).

This assumption requires that the control law \( u_{\theta^*}(\xi) \) globally exponentially stabilises the approximate discrete-time plant (formulated for the nominal parameter set \( \theta^* \)), see Theorem 3 and Remark 8 in Appendix A.1. Note that this assumption does not guarantee the stability of the exact closed-loop plant model for time-varying \( \theta_k \in \Theta \).

**Assumption 2**
The parametrised family of functions \( V_{\theta^*}(\xi) \) is locally Lipschitz and satisfies the following condition uniformly over \( \theta^* \): there exist \( L_v > 0 \), \( 1 \leq p < \infty \) (with \( p \) coinciding with Assumption 1) and \( \overline{h}^* \geq \overline{h} \), such that \( \sup_{\xi \in \partial V_{\theta^*}(\xi)} |\xi| \leq L_v|\xi|^{p-1}, \forall \xi \in \mathbb{R}^{n+\overline{d}m}, \text{ and } \forall \theta^* \in \Theta^* \) with \( \Theta^* \subseteq \Theta_0^* \) and \( \Theta_0^* \) as in (12).

Note that Assumption 2 is a reasonable assumption that holds for a broad class of (possibly non-smooth) Lyapunov functions (e.g. for \( p = 1 \), \( L_v \) reflects a global Lipschitz constant and, for the case of quadratic Lyapunov functions \( V = \frac{1}{2}\xi^T P \xi \) \( (p = 2) \), \( L_v = \lambda_{\max}(P) \)).

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\(^4\) We will call \( \tilde{F}(\xi, u) \) a plant model by which we indicate the discrete-time dynamics \( \dot{\xi}_{k+1} = \tilde{F}(\xi_k, u_k) \).
Assumption 3
The parametrised family of approximate nominal discrete-time plant models \( \hat{F}^n_\theta(x, u) \) is one-step consistent with the parametrised family of exact nominal discrete-time plant models \( F^n_\theta(x, u) \) uniformly over \( \theta^* \), i.e. there exists \( \hat{h} \in K^\infty \) and \( \tilde{h} \geq \hat{h} \) such that \( |\hat{F}^n_\theta(x, u) - F^n_\theta(x, u)| \leq \hat{h} \rho(h^*) ((|\xi| + |u|), \forall \xi \in \mathbb{R}^{n+\tilde{d}m}, u \in \mathbb{R}^m \) and \( \forall \theta^* \in \Theta^* \) with \( \Theta^* \subseteq \Theta_0^* \) and \( \Theta_0^* \) as in (12).

The notion of consistency is commonly used in the numerical analysis literature, see e.g. [40], to address the closeness of solutions of families of models (obtained by numerical integration). Moreover, the notions of one-step consistency (used here) and multi-step consistency have been used before in the scope of the stabilisation of nonlinear sampled-data systems based on approximate discrete-time models [30, 33]. In Section 3.2, we introduce a one-step consistent integration scheme with which approximate discrete-time plant models satisfying Assumption 3 can be constructed.

Assumption 4
The right-hand side \( f(x, u) \) of the continuous-time plant model is globally Lipschitz, i.e. there exists \( L_f > 0 \) such that \( |f(x_1, u_1) - f(x_2, u_2)| \leq L_f (|x_1 - x_2| + |u_1 - u_2|), \forall x_1, x_2 \in \mathbb{R}^n, u_1, u_2 \in \mathbb{R}^m \).

Assumption 5
The parametrised family of discrete-time control laws \( u_\theta^* (\xi) \) is linearly bounded uniformly over \( \theta^* \), i.e. there exist \( L_u > 0 \) and \( \overline{\theta} \geq \tilde{\theta} \) such that \( |u_\theta^* (\xi)| \leq L_u |\xi|, \forall \xi \in \mathbb{R}^{n+\tilde{d}m}, \) and \( \forall \theta^* \in \Theta^* \) with \( \Theta^* \subseteq \Theta_0^* \) and \( \Theta_0^* \) as in (12).

We note that these assumptions are natural extensions of the assumptions used in the scope of the stabilisation of nonlinear sampled-data systems (with constant sampling intervals and no delays), see [33]. Assumption 3 bounds the difference between the approximate and exact nominal discrete-time plant models. In the notion of one-step consistency used in Assumption 3, the integration error is bounded by a term only depending on the nominal sampling interval \( h^* \). Less conservative formulations may take into account the dependency of such error bounds on all parameters in \( \theta^* \) (see e.g. (B.28) in the proof of Proposition 1, which proposes a one-step consistent integration scheme in the sense of Assumption 3). For the sake of transparency, we refrain from using more complex definitions of one-step consistency here. Assumption 4 is typically needed to bound the intersample behaviour, which, in turn is needed to bound the difference between the nominal and uncertain exact discrete-time plant models. Moreover, the satisfaction of Assumption 1 guarantees GES of the approximate discrete/time plant model, for any fixed \( \theta^* \in \Theta^* \), and avoids non-uniform bounds on the overshoot and non-uniform convergence rates for the solutions of the approximate nominal discrete-time plant model, whereas Assumption 5 avoids non-uniform bounds on the controls. Finally, Assumption 2 implies continuity of the Lyapunov function. It has been shown in [30, 33] that if Assumptions 1, 2 and 5 are not satisfied then the approximate closed-loop discrete-time system does not exhibit sufficient robustness to account for the mismatch between the approximate and exact discrete-time models. Moreover, note that Assumption 5 allows for the exploitation of discontinuous control laws.

Based on these assumptions we can formulate a result that provides sufficient conditions under which the closed-loop uncertain exact discrete-time system (7), (11) is GES. Hereto, consider the following definitions: \( L_c := (2 + L_u) (1 + (d - \tilde{d} + 1)(e^{L_f \|\tilde{h}\|} - 1)) \) and \( L_a := (L_c + h^* \hat{h} \rho(h^*) (1 + L_u)) \).

Theorem 1
Consider the exact discrete-time plant model (7) with \( \theta_k \in \Theta, \forall k \in \mathbb{N} \) and \( \Theta \) as in (5). Consider nominal parameter vector \( \theta^* = [h^* \ t_1^* \ t_2^* \ldots t_{\tilde{d}}^*] \) chosen according to (8), for some fixed \( 0 \leq \varepsilon < 1 \), which guarantees that \( \theta^* \in \Theta_0^* \) with \( \Theta_0^* \) as in (12), and a corresponding discrete-time controller (11). The following two statements hold:

- If Assumptions 1-5 are satisfied for \( \Theta^* = \{ \theta^* \} \), for some \( \theta^* \in \Theta_0^* \), and if there exists \( 0 < \beta < 1 - \varepsilon \) such that

\[
(\beta - (1 - \varepsilon))a_3 + \frac{L_v (L_a + L_c)^{p-1}}{h} (h^* \hat{h} \rho(h^*) (1 + L_u) + \rho_0 \left(h^*, M_h, M_i, \ldots, M_{\varepsilon-, \varepsilon} \right)) \leq 0, \tag{14}
\]

where the function \( \hat{h} \rho \) follows from Assumption 3 and

\[
\rho_0 \left(h^*, M_h, M_i, \ldots, M_{\varepsilon-, \varepsilon} \right) := e^{L_f h^*} \left((e^{L_f M_h} - 1) (1 + \max(1, L_u)) + \sum_{j=1}^{\overline{\varepsilon} - \tilde{d}} \left(e^{L_f M_i} - 1 \right) (3 + \max(1, L_u)) \right).
\]
with $M_h := \max_{h \in [\underline{h}, \bar{h}]} |h - h^*|$, $M_{t_j} := \max_{j \in [\underline{t}_j, \bar{t}_j]} |t_j - t^*_j|$, $j = 1, 2, \ldots, \bar{d} - \underline{d}$, and $\underline{t}_j$ and $\bar{t}_j$ defined in (6), then the closed-loop uncertain exact discrete-time system (7), (11) is globally exponentially stable for $\theta_k \in \Theta$, $\forall k \in \mathbb{N}$.

- If Assumptions 1-5 are satisfied for $\Theta^* = \Theta^*_0$, then the nominal sampling interval $h^*$ and the uncertainty intervals $\bar{h} - \underline{h}$ and $\bar{\tau} - \underline{\tau}$ on the sampling intervals and the delays, respectively, can always be chosen small enough such that there exists $0 < \beta < 1 - \varepsilon$ satisfying (14). In other words, under these assumptions, there exists an $h^*_{\text{max}} \leq \bar{h}$ such that for all $h^* \in (0, h^*_{\text{max}})$, there exist $\underline{h}, \bar{h}, \underline{\tau}, \bar{\tau}$ and $0 < \beta < 1 - \varepsilon$ satisfying (14). Consequently, the family of closed-loop uncertain exact discrete-time system (7), (11), parametrised by $\theta^* \in \Theta^*_0(h^*_{\text{max}}, \underline{d}, \bar{d})$, is globally exponentially stable for $\theta_k \in \Theta$, $\forall k \in \mathbb{N}$, where $\Theta$ may depend on $\theta^*$.

**Proof** The proof is given in Appendix B.1. \qed

The first statement of the theorem can be interpreted as follows. If Assumptions 1-5 hold for a fixed $\theta^* \in \Theta^*$ (i.e. for a fixed nominal sampling interval $h^*$ and nominal delay $\tau^*$) and condition in (14) is satisfied for that fixed $\theta^*$, then system (7), (11) is GES for $\theta_k \in \Theta$, $\forall k \in \mathbb{N}$ (i.e. for $h_k \in [\underline{h}, \bar{h}]$ and $\tau_k \in [\underline{\tau}, \bar{\tau}]$, $\forall k \in \mathbb{N}$). Note that the condition in (14) involves two distinct terms:

1. $\frac{L_v(L_o + L_v)^{-1}}{h^*} \rho(h^*) (1 + L_o)$, which reflects the effect of approximately discretising the nonlinear plant using a nominal parameter vector $\theta^*$ (i.e. corresponding to a nominal sampling interval $h^*$ and a nominal delay $\tau^*$);
2. $\frac{L_v(L_o + L_v)^{-1}}{h^*} \rho_0 \left( h^*, M_h, M_{t_1}, \ldots, M_{\underline{d} - \underline{d}} \right)$, which reflects the effect of the uncertainty in the sampling interval and delay.

In this case, only one Lyapunov function $V_{\theta^*}(\xi)$ and a single controller $u_{\theta^*}(\xi)$ need to be found, which is a relatively simple task. Note, however, that for a priori fixed $\theta^*$ there is no guarantee that condition (14) will be satisfied, because the discretisation error (expressed by the term under point 1) above) may be too large. If condition (14) is not satisfied one has to resort to designing a Lyapunov function $V_{\theta^*}(\xi)$ and a controller $u_{\theta^*}(\xi)$ for a smaller nominal sampling interval $h^*$ (and corresponding $\theta^*$) and, subsequently, checking whether condition (14) is satisfied. Although this approach is beneficial in the sense that one only needs the existence of a Lyapunov function and controller for a fixed $\theta^*$, it may lead to an iterative design procedure for Lyapunov functions and controllers. Therefore, we formulated the second statement of Theorem 1, which makes explicit that we can always choose the nominal sampling interval $h^*$, the uncertainty on the sampling interval $\bar{h} - \underline{h}$ and the uncertainty on the delay $\bar{\tau} - \underline{\tau}$ sufficiently small such that (14) is satisfied. To validate such a statement, we required in the second statement of Theorem 1 that Assumptions 1, 2, 3 and 5 hold for all $\theta^* \in \Theta^*_0$. Here, in turn, we need to design a parametrised family of controllers $u_{\theta^*}$ and construct a parametrised family of Lyapunov functions $V_{\theta^*}$. In order to design (families of) control laws and Lyapunov functions satisfying such an assumption, one may exploit, for instance, (extensions of) the results presented in [32] on backstepping designs for Euler approximate discrete-time models.

**Remark 3**
In Theorem 1, one can freely choose the parameter $0 \leq \varepsilon < 1$. By choosing a particular value for $\varepsilon$ one fixes the positioning of the uncertainty interval $[\underline{h}, \bar{h}]$ with respect to $h^*$, see (8). For example, $\varepsilon = \frac{1}{2}$ gives a symmetric uncertainty interval around $h^*$.

**Remark 4**
Note that by the grace of the particular choice of $\theta^*$ as in (8) directly corresponding to the case of a constant delay $\tau^*$ and a constant sampling interval $h^*$, we have that the term $\frac{L_v(L_o + L_v)^{-1}}{h^*} \rho_0 \left( h^*, M_h, M_{t_1}, \ldots, M_{\underline{d} - \underline{d}} \right)$ vanishes when all uncertainty in the sampling interval and the delay vanishes. If we would have chosen $\theta^* \in \Theta$ such that it would not have corresponded to a sequence of control update instants induced by a constant sampling interval $h^*$ and delay $\tau^*$, the term $\frac{L_v(L_o + L_v)^{-1}}{h^*} \rho_0 \left( h^*, M_h, M_{t_1}, \ldots, M_{\underline{d} - \underline{d}} \right)$ would not have necessarily vanished for vanishing uncertainties.

**Remark 5**
In order to further enhance the accuracy between the approximate and exact discrete-time models, one can introduce the integration interval of the integration scheme used to construct the approximate plant model as an independent parameter (i.e. use integration intervals unequal to, typically smaller than, the sampling interval as opposed to...
the intersample behaviour is linearly globally uniformly bounded over the maximum sampling interval \( h^* \) but also on the integration interval and can be made arbitrarily small by decreasing the integration interval. Such extensions would avoid the need to decrease the nominal sampling interval in order to satisfy (14) and may enable us to guarantee stability for larger levels of uncertainty in the sampling period \((\bar{\tau} - h)\) and delay \((\tau - \tau)\). For the sake of brevity and transparency, we do not pursue such extensions in this paper and refer to [30] for further details on the exploitation of integration period as an independent parameter in constructing one-step consistent approximate discrete-time models.

3.2 A one-step consistent discretisation scheme

Of course, one may wonder how to construct an approximate discrete-time model \( \hat{F}_\theta^a(\xi, u) \) such that Assumption 3 on the one-step consistency of the approximate nominal discrete-time model with the exact nominal discrete-time model is satisfied. This question becomes even more pressing given the fact that within one sampling interval a multitude of different control commands are active, see Figure 2. In this section we will present a discretisation scheme, which does comply with Assumption 3, i.e. which is one-step consistent with the exact nominal discrete-time model \( \hat{F}_\theta^a(\xi, u) \). Such a discretisation scheme is useful from two perspectives: firstly, it can be used to construct a one-step consistent approximate discrete-time model and, secondly, it allows us to employ other discretisation schemes that, when one-step consistent with this known discretisation scheme we will propose here, will also be one-step consistent with the exact discrete-time model.

We propose the following adapted Euler-like discretisation scheme to discretise the sampled-data system (1), (2), (3) at the sampling instants \( s_k \) for a fixed \( \theta = [h t_1 t_2 \ldots t_{n-1}]^T \in \Theta_0^* \):

\[
x_{k+1} = x_k + \sum_{j=0}^{n-1} (t_{j+1} - t_j) f(x_k, u_{k+j-\theta}) =: F_{\theta}^{Euler}(x_k, \bar{u}_k, u_k).
\]

The corresponding discrete-time model in terms of the extended state \( \xi \) can then be formulated as

\[
\xi_{k+1} = [F_{\theta}^{EulerT}(x_k, \bar{u}_k, u_k) \ u_k^T \ u_{k-1}^T \ldots u_{k-T_{\bar{d}}+1}^T]^T =: \hat{F}_{\theta}^{Euler}(\xi_k, u_k).
\]

**Proposition 1**

Let us adopt Assumption 4. Then, the following statements are valid:

- The approximate discrete-time plant model \( \hat{F}_{\theta}^{Euler}(\xi, u) \) in (17), (16) is one-step consistent with the exact discrete-time model \( \hat{F}_{\theta}^a(\xi, u) \), i.e. there exists a function \( \rho^* \in K_\infty \) such that \( |\hat{F}_{\theta}^{Euler}(\xi, u) - \hat{F}_{\theta}^a(\xi, u)| \leq h\rho^*(h)(|\xi| + |u|) \), \( \forall \xi \in \mathbb{R}^{n+\bar{d}m}, u \in \mathbb{R}^m \).
- Let us consider an approximate discrete-time plant model \( \hat{F}_{\theta}^a(\xi, u) \) and suppose that \( \hat{F}_{\theta}^{Euler}(\xi, u) \) is one-step consistent with the discrete-time model \( \hat{F}_{\theta}^{Euler}(\xi, u) \) given in (17), (16), i.e. there exists a function \( \rho \in K_\infty \) such that \( |\hat{F}_{\theta}^a(\xi, u) - \hat{F}_{\theta}^{Euler}(\xi, u)| \leq \hat{\rho}(h)(|\xi| + |u|) \), \( \forall \xi \in \mathbb{R}^{n+\bar{d}m}, u \in \mathbb{R}^m \). Then, the approximate discrete-time plant model \( \hat{F}_{\theta}^a(\xi, u) \) is one-step consistent with the exact discrete-time model \( \hat{F}_{\theta}^{Euler}(\xi, u) \), i.e. there exists a function \( \hat{\rho} \in K_\infty \) such that \( |\hat{F}_{\theta}^a(\xi, u) - \hat{F}_{\theta}^{Euler}(\xi, u)| \leq \hat{\rho}(h)(|\xi| + |u|) \), \( \forall \xi \in \mathbb{R}^{n+\bar{d}m}, u \in \mathbb{R}^m \).

**Proof** The proof is given in Appendix B.2.

3.3 Stability of the Sampled-data NCS

In this section, we will study the stability of the sampled-data NCS (1), (2), (3), (11) based on the results on the global exponential stability of the exact uncertain discrete-time model in Section 3.1. Let us first show that the intersample behaviour is linearly globally uniformly bounded over the maximum sampling interval \( \bar{\tau} \). Solutions of the sampled-data NCS (1), (2), (3) satisfy \( |x(t)| \leq |x_k| + \sum_{j=0}^{n-1} f(x_k + j\tau_{\bar{d}}) \int_{s_{k+j} + j\tau_{\bar{d}}}^{s_{k+j+1} + j\tau_{\bar{d}}} |f(x(s), u_{k+j+1} + j\tau_{\bar{d}})| ds, \; \forall t \in [s_k, s_{k+1}] \).
Using Assumptions 4 and 5, we can exploit the reasoning in the proof Theorem 1 (especially the developments from (B.4) to (B.6)) to obtain that the solutions of the closed-loop sampled-data NCS (1), (2), (3), (11) satisfy
\[ |x(t)| \leq \left(1 + (2 + L_u)(d - d + 1) \left(e^{L_f \bar{h}} - 1 \right) \right)|\xi_k|, \quad \forall t \in [s_k, s_{k+1}). \]
So, the intersample behaviour is linearly globally uniformly bounded over the maximum sampling interval \( \bar{h} \). Now, we can use the results in [34] to conclude that the closed-loop sampled-data NCS (1), (2), (3), (11) is globally exponentially stable.

4 Semi-global practical asymptotic stability of the NCS

Assumptions 2-5 may in general be rather strict in the sense that these reflect global conditions, whereas Assumption 1 may be strict in the sense that GES of the nominal approximate discrete-time model is required. In this section, we will relax these assumptions and state results on semi-global practical asymptotic stability (SGPAS) of the NCS under these relaxed conditions to further widen the class of nonlinear NCS for which these stability results can be employed. A Lyapunov characterisation of SGPAS, exploited in the next section, for uncertain discrete-time nonlinear systems is given in Appendix A.2.

4.1 Sufficient conditions for SGPAS

Let us adopt the following relaxed assumptions.

Assumption 6
There exist a parametrised family of functions \( V_{\theta^*}(\xi) \), a parametrised family of controllers \( u_{\theta^*}(\xi) \), \( \alpha_1, \alpha_2 \in \mathcal{K}_\infty \), \( \alpha_3 \in \mathcal{K} \), and \( \bar{h} \geq \bar{h} \) such that the following inequalities hold:
\[
\alpha_1(|\xi|) \leq V_{\theta^*}(\xi) \leq \alpha_2(|\xi|) \quad \text{and} \quad \frac{V_{\theta^*}(\bar{F}_{\theta^*}(\xi, u_{\theta^*}(\xi))) - V_{\theta^*}(\xi)}{\bar{h}^*} \leq -\alpha_3(|\xi|), \quad \forall \xi \in \mathbb{R}^{n + m}_0, \forall \theta^* \in \Theta_0^* \quad (18)
\]
with \( \Theta_0^* \) as in (12).

This assumption requires that the control law \( u_{\theta^*}(\xi) \) globally asymptotically stabilises the approximate discrete-time plant (formulated for the nominal parameter vector \( \theta^* \)), see Theorem 3 in Appendix A.1.

Assumption 7
The parametrised family of functions \( V_{\theta^*}(\xi) \) is Lipschitz on compact sets uniformly over \( \theta^* \), i.e. for every \( \Delta > 0 \) there exist a \( L_v > 0 \) and \( \bar{h} \geq \bar{h} \), such that \( |V_{\theta^*}(x) - V_{\theta^*}(y)| \leq L_v|x - y|, \forall |x| \leq \Delta, |y| \leq \Delta, \) and \( \forall \theta^* \in \Theta_0^* \) with \( \Theta_0^* \) as in (12).

Assumption 8
The parametrised family of nominal approximate nominal discrete-time plant models \( \bar{F}_{\theta^*}(\xi, u) \) is one-step consistent with the exact nominal discrete-time plant model the parametrised family of exact nominal discrete-time plant models \( \bar{F}_{\theta^*}(\xi, u) \) on compact sets uniformly over \( \theta^* \), i.e. for every pair \( \Delta > 0, \Delta_u > 0 \), there exist \( \bar{h} \in \mathcal{K}_\infty \) and \( \bar{h} \geq \bar{h} \) such that \( |\bar{F}_{\theta^*}(\xi, u) - \bar{F}_{\theta^*}(\xi, u)| \leq h^* \bar{h}^*(h^*), \forall |\xi| \leq \Delta, |u| \leq \Delta_u \) and \( \forall \theta^* \in \Theta_0^* \) with \( \Theta_0^* \) as in (12).

Assumption 9
The right-hand side \( f(x, u) \) of the continuous-time plant model is Lipschitz on compact sets, i.e. for every pair \( \Delta_x > 0, \Delta_u > 0 \), there exists \( L_f > 0 \), such that \( |f(x_1, u_1) - f(x_2, u_2)| \leq L_f(|x_1 - x_2| + |u_1 - u_2|), \forall |x_1|, |x_2| \leq \Delta_x, \) and \( |u_1|, |u_2| \leq \Delta_u. \)

Assumption 10
The parametrised family of discrete-time control laws \( u_{\theta^*}(\xi) \) is linearly bounded on compact sets uniformly over \( \theta^* \), i.e. for every \( \Delta > 0 \), there exist \( L_u > 0 \) and \( \bar{h} \geq \bar{h} \), such that \( |u_{\theta^*}(\xi)| \leq L_u(|\xi|), \forall |\xi| \leq \Delta, \) and \( \forall \theta^* \in \Theta_0^* \) with \( \Theta_0^* \) as in (12).
Based on these assumptions (which are formulated for all \( \theta^* \in \Theta_0^* \), i.e. we are designing families of controllers and Lyapunov functions) we can formulate a result that provides sufficient conditions under which the closed-loop uncertain exact discrete-time system (7), (11) is semi-globally practically asymptotically stable. Hereafter, consider the following definitions:

\[
\delta_1 := \frac{1}{2} a_2^{-1} \left( \frac{d}{2} \right) \quad \text{and} \quad \delta_2 := a_1^{-1} \circ a_2(D) + \frac{1}{2} a_2^{-1} \left( \frac{d}{2} \right), \quad \text{with} \; d, D > 0.
\]  

**Theorem 2**

Consider the exact discrete-time plant model (7) with \( \theta_k \in \Theta, \forall k \in \mathbb{N} \) and \( \Theta \) as in (5). Consider nominal parameter vector \( \theta^* = \left[ h^* \; t^*_1 \; t^*_2 \; \ldots \; t^*_n \right]^T \) chosen according to (8), for some fixed \( 0 \leq \varepsilon < 1 \), which guarantees that \( \theta^* \in \Theta_0^* \) with \( \Theta_0^* \) as in (12), and a corresponding discrete-time controller (11). Suppose that Assumptions 6-10 are satisfied. Consider \( d, D > 0 \), with \( d \in (0, \alpha_1(D)] \) and a \( q > 1 \). Consider a \( \Delta \geq q \delta_2 \), which induces particular \( L_v, L_u > 0 \) through Assumptions 7 and 10. Consider a \( \Delta_x \geq D \) and \( \Delta_u \geq (1 + L_u)D \), which (together with a choice for \( \Delta \)) induce particular \( \hat{\rho} \in \mathcal{K}_\infty \) and \( L_f > 0 \) from Assumption 8 and 9, respectively. Define \( L_1 := \min \left\{ \frac{d}{L_v}, \frac{\alpha_2^{-1} \circ \alpha_1(D)}{2L_v} \right\}, \)

\[
L_2 := \frac{1}{L_f} \ln \left( \frac{\Delta_x + D(1 + L_u)}{(2 + L_v)D} \right) \quad \text{and} \quad L_3 := \min \left\{ \frac{(q-1)\delta_2}{D}, \frac{d}{4L_vD}, \frac{\hat{\rho} \circ \alpha_2^{-1} \circ \alpha_1(D)}{4L_vD} \right\}.
\]

If

\[
\hat{\rho}(h^*) \leq L_1,
\]

\[
\mathcal{L} \leq L_2,
\]

\[
\rho_\theta \left( h^*, M_h, M_t, \ldots, M_{t_{n-1}} \right) \leq L_3,
\]

with \( \rho_\theta \) defined by (15), then there exists a \( \mathcal{KL} \)-function \( \beta \) and a \( \delta = \alpha_1^{-1}(d) > 0 \) such that the solutions of the closed-loop exact discrete-time model (7), (11) satisfy the inequality \( |x_k| \leq \beta(|x_0|, kh) + \delta \). \( \forall k \in \mathbb{N} \). Moreover, \( d, D > 0 \), with \( d \in (0, \alpha_1(D)] \) can be chosen arbitrarily and, hence, the closed-loop exact discrete-time model (7), (11) is semi-globally practically asymptotically stable.

**Proof** The proof is given in Appendix B.3. \( \square \)

Theorem 2 can be used in several ways. First and foremost it reflects a qualitative result in the sense that it states that for any performance specification in terms of the domain of attraction (characterised by \( D \)) and an ultimate bound on the error (characterised by \( \delta \)) one can find a nominal sampling interval \( h^* \) (used to construct the approximate discrete-time plant, the controller and the Lyapunov function) and uncertainties \( \tau - \bar{\tau} \) on the delay such that these performance specifications are met. As such, the formulation of this result allows to make tradeoffs between control performance requirements and requirements on the communication network. A qualitative interpretation on the inequalities (20)-(22) can be given as follows. Inequality (20) reflects a bound on the nominal sampling interval \( h^* \) needed to ensure that the difference between the exact and approximate nominal discrete-time models is small enough. Inequality (21) reflects a bound on the maximum sampling interval \( \mathcal{L} \) and is needed in the proof of Theorem 2 to bound the intersample behaviour of the sampled-data NCS such that Assumptions 6-10 may be used. Inequality (22) reflects a bound on the uncertainties \( h - h^* \) on the sampling interval and \( \tau - \bar{\tau} \) on the delay to limit the effect of these uncertainties on the evolution of the Lyapunov function \( V_\theta \). along solutions of the exact uncertain discrete-time model.

Using the results in Sections 3 and 4, we can now analyse the robustness of NCS where the controllers are designed for a nominal parameter vector \( \theta^* \) (i.e. for a nominal sampling interval \( h^* \) and a nominal delay \( \tau^* \)). Note that emulation-based controllers are actually just a subset of such controllers, which are independent of a nominal sampling interval and nominal delay and which are designed having a particular nominal sampling interval (zero) and a particular nominal delay (also zero) in mind. In this sense, the framework developed in this paper allows us to analyse and compare different controllers (both emulation-based and based on approximate discrete-time models).

The results in Sections 3 and 4 represent extensions to the existing literature in several ways. Firstly, the results presented in this paper extend the results in [30, 33] on the controller design for nonlinear sampled-data systems (with constant sampling intervals and no delays) based on approximate discrete-time models to the case of nonlinear sampled-data systems with delays (even for the case with constant delays). Moreover, the results in this paper further extend these works in the sense that we allow for time-varying uncertain sampling intervals and delays.
From a different perspective, the results in this paper extend the results on discrete-time modelling and stability analysis for linear NCS with time-varying sampling intervals, delays and packet dropouts (see Remark 2), see e.g. [6,7,12,13,15,18,20,35,38,43,47,48,52,53], to the realm of nonlinear systems.

**Remark 6**
Using an analysis similar to that in Section 3.3 it can be shown that the intersample behaviour is linearly semi-globally uniformly bounded over the maximum sampling interval $\hat{h}$. Subsequently, the results in [34] can be used to conclude that the closed-loop sampled-data NCS (1), (2), (3), (11) is semi-globally practically asymptotically stable.

**Remark 7**
Semi-global extensions of Proposition 1 are relatively straightforward and can be used to construct integration schemes that are one-step consistent with the exact discrete-time model in the sense of Assumption 8.

### 5 Illustrative Example

Let us consider a class of scalar nonlinear continuous-time plants of the form

$$\dot{x} = f(x) + u,$$  (23)

where $x \in \mathbb{R}$, $u \in \mathbb{R}$, and $f(x)$ is globally Lipschitz with Lipschitz constant $L_{fx}$. Consequently, the right-hand side of (23) satisfies Assumption 4 with $L_f = \max(1, L_{fx})$. Let us first consider the case without delays, but with uncertain time-varying sampling intervals. We use an Euler discretisation scheme (see Proposition 1) to construct the following family of approximate discrete-time plant models: $x_{k+1} = x_k + h^*(f(x_k) + u_k) =: F^a_k(x_k, u_k)$. It is straightforward to show that this family of approximate discrete-time models satisfies Assumption 3 with $h^* \rho(h^*) = \frac{L_{fx}}{2} (e^{h^*h^*} - 1 - L_f h^*)$. Moreover, consider the following controllers

$$u_k = -f(x_k) - x_k$$  \hspace{1cm} (24)

$$u_k = -f(x_k) - x_k - h^* x_k$$  \hspace{1cm} (25)

$$u_k = -f(x_k) - \frac{1}{2h^*} \left( 1 - \sqrt{1 - 4h^*} \right) x_k, \text{ with } h^* \leq \frac{1}{4}$$  \hspace{1cm} (26)

where the first controller is independent of $h^*$ and could be regarded as an example of an emulation-based controller, whereas the other two controllers are clearly parametrised by the nominal sampling interval $h^*$. Below, we will exploit the candidate Lyapunov function $V(x) = |x|$, which is independent of $\theta^*$ and which clearly satisfies Assumption 2 with $L_v = 1$ and $\rho = 1$. Note that all controllers approach each other for $h^* \downarrow 0$.

Let us first consider controller (24). This controller clearly satisfies Assumption 5 with $L_u = L_{fx} + 1$. In order to assess the satisfaction of Assumption 1, we firstly note that we can take $a_1 = a_2 = 1$ and we evaluate

$$\frac{V(F^a_k(x_k, u_k)) - V(x_k)}{h^*} = \frac{|x_k + h^*(f(x_k) + u_k)| - |x_k|}{h^*} = \begin{cases} -|x_k| & \text{if } 0 < h^* \leq 1 \\ (1 - \frac{2}{h^*}) |x_k| & \text{if } 1 \leq h^* \leq 2 \end{cases},$$  \hspace{1cm} (27)

which can be used to conclude that the approximate closed-loop discrete-time system, with controller (24), satisfies Assumption 1 with $p = 1$ for $0 < h^* < 2$.

Let us next consider controller (25). This controller clearly satisfies Assumption 5 with $L_u = L_{fx} + 1 + h^*$. In order to assess the satisfaction of Assumption 1, we evaluate

$$\frac{V(F^a_k(x_k, u_k)) - V(x_k)}{h^*} = \frac{|x_k + h^*(f(x_k) + u_k)| - |x_k|}{h^*} = \begin{cases} (-1 - h^*) |x_k| & \text{if } 0 < h^* \leq \frac{1}{2}(\sqrt{5} - 1) \\ \frac{2 + h^* + h^*^2}{h^*} |x_k| & \text{if } \frac{1}{2}(\sqrt{5} - 1) \leq h^* \leq 1 \end{cases},$$  \hspace{1cm} (28)

which can be used to conclude that the approximate closed-loop discrete-time system, with controller (25), satisfies Assumption 1 with $p = 1$ for $0 < h^* < 1$. 

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Let us finally consider controller (26), which is clearly only applicable for $0 < h^* \leq \frac{1}{4}$. This controller clearly satisfies Assumption 5 with $L_u = L_{fx} + \max_{0 < h^* \leq h^*} \frac{1}{2h^*} (1 - \sqrt{1 - 4h^*}) \leq L_{fx} + 2$. In order to assess the satisfaction of Assumption 1, we evaluate

$$V(F^*_h(x_k, u_k)) - V(x_k) = \frac{1}{h^*}(|x_k + h^*(f(x_k) + u_k)| - |x_k|) = \frac{1}{h^*} \left( \frac{1}{2}(-1 + \sqrt{1 - 4h^*})|x_k| - |x_k| \right) \leq -|x_k|,$$

which can be used to conclude that the approximate closed-loop discrete-time system, with controller (26), satisfies Assumption 1 with $p = 1$ for $0 < h^* \leq \frac{1}{4}$.

For all three controllers, the second statement of Theorem 1 can now be exploited to conclude that there always exists a sufficiently small nominal sampling interval and a sufficiently small level of uncertainty on the sampling interval such that the exact closed-loop sampled-data networked control system can be guaranteed to be globally exponentially stable. This example also shows that the results proposed in this paper can be used to study both emulation-based controllers as well as discrete-time controllers parametrised by the nominal sampling interval.

Next, let us use condition (14) in Theorem 1 to compute (estimates of the) uncertainty bounds $\underline{h}$, $\overline{h}$ on the sampling interval, depending on $h^*$. Here, we fix $\varepsilon = \frac{1}{2}$ in (8), i.e. $h^*$ represents the average sampling interval. This choice will typically yield symmetric uncertainty intervals for $h$ around $h^*$. Other choices for $\varepsilon$ will, of course, yield different uncertainty intervals. Bounds for $\underline{h}$, $\overline{h}$ are depicted, depending on the choice for $h^*$, in Figure 4. A different perspective on these results is given in Figure 5 in which bounds on the percentage of allowable jitter on the sampling interval are depicted. Figures 4 and 5 indicate that, for this particular example, the controllers that explicitly take into account

![Figure 4](image-url)

**Figure 4.** Bounds $\underline{h}$, $\overline{h}$ on the uncertainty of the sampling interval for controllers (24), (25) and (26) for $L_{fx} = 0.45$.

the nominal sampling interval may allow for a larger uncertainty in the sampling interval than the emulation-based controllers. However, we stress here that this is by no means a generic fact and we note that, firstly, the bounds given here only represent sufficient conditions, which may exhibit a certain level of conservatism and, secondly, that these bounds on the allowable jitter depend on many factors such as the particular controller designed, the particular integration scheme used to obtain the approximate discrete-time plant model, the particular Lyapunov function used to study stability etc. To assess the possible conservatism of these results to some extent, we consider the case in which $f(x) = L_{fx}x$, with $L_{fx} = 0.45$, and the sampling interval is constant. In this case we can straightforwardly compute an upperbound on the sampling interval (because the discrete-time closed-loop system for fixed $h$ is linear), which is $h \approx 1.426$ for controller (24) and $h \approx 0.872$ for controller (25). Considering the fact that we consider an entire class of nonlinear systems and time-varying sampling times, the bounds depicted in Figure 4 are not extremely conservative.

The main purpose of this example is to illustrate the fact that the results presented in this paper allow us to study robustness aspects, e.g. with respect to uncertainty in the sampling intervals, for a class of nonlinear discrete-time controllers, which may either be emulation-based controllers or may be designed have a (non-zero) nominal sampling interval in mind. One of the motivations to study discrete-time controllers based on approximate discrete-time plant models (next to emulation-based controllers) is the fact (see [21, 33] for the case of nonlinear sampled-data systems with fixed sampling times (and no delays)) that controllers based on approximate discrete-time models...
may provide superior performance in terms of convergence speed. The latter fact is illustrated in Figure 6, which shows the continuous-time trajectories of the sampled-data networked control systems resulting from the application of controllers (24), (25) and (26) for \( f(x) = L_{fx} \arctan x, \ L_f = 0.45, \ h = 0.18, \ \overline{h} = 0.22 \) and \( h^* = 0.2 \) (note that Theorem 1 guarantees that all three controllers render the closed-loop system GES for these parameters). The related sequence of uncertain sampling intervals \( h_k \) is depicted in Figure 7.

In the following example we aim to illustrate the applicability of Theorem 2. Hereeto, we consider once more the class of systems as in (23), where now we relax the assumption on the Lipschitz property of the right-hand side in the sense that we only require it to be Lipschitz on compact sets (now e.g. also vectorfields such \( f(x) = x^3 \) can be studied), i.e. Assumption 9 is satisfied. Let us now consider the case in which the sampling interval \( h \) is constant and the delays satisfy \( \tau_k \in [0, \overline{\tau}] \) with \( \overline{\tau} \leq h \). Here we choose \( \tau^* = 0 \) and \( h^* = h \) and use the Euler-type discretisation scheme (see Proposition 1) to construct the following family of approximate discrete-time plant models in terms of the extended state \( \xi_k = \begin{bmatrix} \xi_k^1 \\ \xi_k^2 \end{bmatrix} = \begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix} \):

\[
\xi_{k+1} = \begin{bmatrix} \xi_k^1 \\ \xi_k^2 \end{bmatrix} + h^*(f(\xi_k^1) + u_k), \ u_k \end{bmatrix}^T =: \hat{F}_{h^*}(\xi_k, u_k).
\] (30)

We note that this family of models satisfies Assumption 8. Next, we will show that the controller as in (25) is robust with respect to (small) time-varying delays for sufficiently small sampling periods. Clearly, when the continuous-time plant satisfies Assumption 9, then the controller satisfies Assumption 10. Moreover, we consider the following family
of Lyapunov functions: \( V = |\xi^1| + \alpha |u(\xi^1) - \xi^2| + h^*\alpha |\xi^2|, \) with \( \alpha > 0. \) This family of Lyapunov functions satisfies Assumption 7 with \( L_v = \sqrt{2} \max(1 + \alpha L_u, \alpha(1 + h^*)) \), which is bounded for bounded \( L_u, \alpha \) and \( h^* \). Moreover, the evolution of this family of Lyapunov functions along solutions of the family of closed-loop approximate discrete-time plant models (30), (25) can be shown to satisfy \( \frac{\dot{V}(F^{\star}_{\kappa}, (\xi_k, u_{\theta^*}(\xi_k))) - V(\xi_k)}{h^*} \leq -\alpha |\xi_k| \) with \( \alpha = (1 + h^*)/(1 + 2L_u + h^*L_u) \) for \( 0 < h^* \leq \frac{1}{2}(\sqrt{5}) - 1 \) and \( \frac{1}{2}(\sqrt{5}) - 1 - h^*/(1 + 2L_u + h^*L_u) \) for \( \frac{1}{2}(\sqrt{5}) - 1 \leq h^* \leq 1. \) Consequently, we can conclude that Assumption 6 is satisfied with \( \alpha_1(|\xi|) = \alpha |\xi|, \alpha_2(|\xi|) = L_v |\xi| \) and \( \alpha_3(|\xi|) = \alpha |\xi|, \) where we use that \( u(x) \) is Lipschitz on compact sets since \( f(x) \) is Lipschitz on compact sets. This implies that the approximate closed-loop discrete-time system is globally exponentially stable for sampling intervals \( h^* < 1. \) Now, Theorem 2 can be used to show that the exact closed-loop discrete-time NCS is semi-globally practically asymptotically stable. In other words, by making the nominal sampling period and the uncertainty on the delay sufficiently small, one can attain arbitrarily large domains of attraction and arbitrarily small steady-state errors. We note that a similar analysis can be performed to show that the exact closed-loop discrete-time NCS induced by controller (24) is semi-globally practically asymptotically stable.

6 Conclusions

This paper presents results on the stability analysis of nonlinear Networked Control Systems (NCS) with time-varying sampling intervals, time-varying delays (that may be larger than the sampling interval) and packet dropouts. We have developed a prescriptive framework for the controller design based on approximate discrete-time plant models. As opposed to emulation-based approaches where the effects of sampling-and-hold and delays are ignored in the phase of controller design, we propose an approach in which the controller design is based on approximate discrete-time models constructed for a nominal (non-zero) sampling interval and a nominal delay. Subsequently, sufficient conditions for the global exponential stability and semi-global practical asymptotic stability of the closed-loop NCS with time-varying sampling intervals and delays are provided.

The results in this paper represent extensions to the existing literature in several ways. Firstly, the results presented in this paper extend the results in [30, 33] on the controller design for nonlinear sampled-data systems (with constant sampling intervals and no delays) based on approximate discrete-time models to the case of nonlinear sampled-data systems with delays (this represents an extension even for the case with constant delays). Moreover, the results in this paper further extend these works in the sense that we allow for time-varying uncertain sampling intervals and delays. From a different perspective, the results in this paper extend the results on discrete-time modelling and stability analysis for linear NCS with time-varying sampling intervals, delays and packet dropouts to the realm of nonlinear systems.

A Lyapunov characterisations of GES and SGAS

A.1 Lyapunov characterisation of global uniform asymptotic stability

Here, we formulate a Lyapunov-based characterisation of global uniform asymptotic stability (and global exponential stability) for a parametrised family of uncertain discrete-time nonlinear closed-loop systems \( \forall k \in \mathbb{N}, \) with \( F(0, 0) = 0 \) and \( u(0) = 0, \) based on a Lyapunov function \( V_{\theta^*}(\xi_k) \) that is parametrised by a nominal parameter vector \( \theta^*. \)

Theorem 3

Consider a parametrised family of uncertain discrete-time systems (parametrised by \( \theta^* \))

\[
\xi_{k+1} = F_{\theta^*}(\xi_k, u_{\theta^*}(\xi_k)), \quad \theta^* \in \Theta^*, \quad \forall k \in \mathbb{N},
\]

with \( \theta^* \in \Theta^*, \) \( \Theta^* \) as in (12) and \( \Theta \) defined in (5), where \( h, \bar{h}, \tau \) and \( \tau \) may depend on \( \theta^* \) (i.e. we denote \( \Theta = \Theta(\theta^*) \)). If there exist a family of Lyapunov functions \( V_{\theta^*}(\xi), \alpha_1, \alpha_2 \in \mathcal{K}_{\infty}, \) and \( \alpha_3 \in \mathcal{K} \) such that the following conditions hold:

\[
\alpha_1(|\xi|) \leq V_{\theta^*}(\xi) \leq \alpha_2(|\xi|) \quad \text{and} \quad \frac{V_{\theta^*}(F_{\theta^*}(\xi_k, u_{\theta^*}(\xi_k))) - V_{\theta^*}(\xi)}{h} \leq -\alpha_3(|\xi|), \quad \forall \xi \in \mathbb{R}^{n + dm}, \theta \in \Theta(\theta^*), \theta^* \in \Theta^*,
\]

(A.2)
then there exists a KL-function \( \beta \) such that the solutions of the family of systems (A.1) satisfy

\[
|\xi_k| \leq \beta(|\xi_0|,kh_k), \quad \forall k \in \mathbb{N} \text{ and } \forall \xi_0 \in \mathbb{R}^{n+n_m}, \tag{A.3}
\]

and for all \( \theta^* \in \Theta^* \). In other words, the family of systems (A.1) is globally uniformly asymptotically stable for all \( \theta^* \in \Theta^* \) and \( \theta_k \in \Theta(\theta^*) \), \( \forall k \in \mathbb{N} \).

**Proof** The proof is a slight adaptation of the proof of Proposition 1.2 in [21]. \( \square \)

**Remark 8**

If the conditions of Theorem 3 are satisfied for functions \( \alpha_i(s) = a_is^p, \quad i = 1,2,3 \), then there exists an \( \exp - KL \) function \( \beta \) (i.e. there exist \( c, \lambda > 0 \) such that \( \beta(s, t) = cse^{-\lambda t} \)) such that (A.3) is satisfied. In other words, the fixed point \( \xi = 0 \) is a GES fixed point of (A.1).

A.2 Lyapunov characterisation of SGPAS

Here, we formulate a Lyapunov-based characterisation of semi-global practical asymptotic stability (SGPAS) for a parametrised family of uncertain discrete-time nonlinear closed-loop systems \( \xi_{k+1} = F_{\theta_k}(\xi_k, u_{\theta^*}(\xi_k)), \quad \theta_k \in \Theta, \forall k \in \mathbb{N}, \theta^* \in \Theta^* \), with \( F(0,0) = 0 \) and \( u(0) = 0 \), based on a Lyapunov function \( V_{\theta^*}(\xi_k) \) that is parametrised by a nominal parameter vector \( \theta^* \).

**Theorem 4**

Consider a parametrised family of uncertain discrete-time systems (parametrised by \( \theta^* \))

\[
\xi_{k+1} = F_{\theta_k}(\xi_k, u_{\theta^*}(\xi_k)), \quad \theta_k \in \Theta, \forall k \in \mathbb{N}, \tag{A.4}
\]

with \( \theta^* \in \Theta^*_0, \Theta^*_0 \) as in (12) and \( \Theta \) defined in (5). Suppose there exist functions \( \alpha_1, \alpha_2 \in \mathcal{K}_\infty \) and \( \alpha_3 \in \mathcal{K} \) such that for each pair \( d, D > 0 \), with \( d \in (0, \alpha_1(D)] \), there exist \( 0 < \underline{h} \leq \overline{h}, 0 \leq \underline{\lambda} \leq \overline{\lambda}, \tau^* \) according to (8) and a function \( V_{\theta^*}(\xi) \), with \( \theta^* \) defined in (8), such that the following conditions hold:

\[
\begin{align*}
\alpha_1(|\xi|) \leq V_{\theta^*}(\xi) & \leq \alpha_2(|\xi|), \\
\max\{V_{\theta^*}(F_{\theta_k}(\xi_k, u_{\theta^*}(\xi_k))), V_{\theta^*}(\xi)\} \geq d & \Rightarrow \frac{V_{\theta^*}(F_{\theta_k}(\xi_k, u_{\theta^*}(\xi_k))) - V_{\theta^*}(\xi)}{h_k} \leq -\alpha_3(|\xi|), \quad \forall \theta_k \in \Theta
\end{align*}
\tag{A.5}
\]

with \( \Theta \) defined in (5). Then, there exist a KL-function \( \beta \) and a \( \delta = \alpha_1^{-1}(d) > 0 \) such that the solutions of (A.4) satisfy

\[
|\xi_k| \leq \beta(|\xi_0|,kh_k) + \delta, \quad \forall \xi_0 \leq \alpha_2^{-1} \circ \alpha_1(D) \quad \text{and} \quad \forall k \in \mathbb{N}. \tag{A.6}
\]

Since \( d, D > 0 \), with \( d \in (0, \alpha_1(D)] \), can be chosen arbitrarily the system (A.4) is semi-globally practically asymptotically stable (SGPAS).

**Proof** The proof is a slight adaptation of the first part of the proof of Theorem 2 in [33]. \( \square \)

B Proofs

B.1 Proof of Theorem 1

Let us first prove the first statement of the theorem and consider fixed \( \theta^* \in \Theta^*_0 \). Let us study the evolution of the candidate Lyapunov function \( V_{\theta^*}(\xi) \) along solutions of the closed-loop uncertain exact discrete-time system (7), (11):

\[
\Delta V_k := \frac{V_{\theta^*}(F_{\theta_k}(\xi_k, u_{\theta^*}(\xi_k))) - V_{\theta^*}(\xi_k)}{h_k} \tag{B.1}
\]
Below, we exploit the mean value theorem [5] to obtain $V_\theta^*(x) - V_\theta^*(y) \in \partial V_\theta^T(z)(x - y)$ for some $z = \sigma x + (1 - \sigma)y$, $\sigma \in [0, 1]$. Hence, $V_\theta^*(x) - V_\theta^*(y) \leq \sup_{z \in \partial V_\theta^T(z)} \| \xi \| (x - y)$. Using Assumption 2, we obtain $V_\theta^*(x) - V_\theta^*(y) \leq L_u \| z \|^{p-1} \| x - y \|$. Using Assumption 1 and the latter inequality in (B.1) gives

$$\Delta V_k = \frac{V_\theta^* (\bar{F}_\theta^e (\xi_k, u_\theta^* (\xi_k))) - V_\theta^* (\xi) + V_\theta^* (\bar{F}_\theta^e (\xi_k, u_\theta^* (\xi_k))) - V_\theta^* (\bar{F}_\theta^e (\xi_k, u_\theta^* (\xi_k)))}{h}$$

$$\leq \frac{h^*}{h} \frac{V_\theta^* (\bar{F}_\theta^e (\xi_k, u_\theta^* (\xi_k))) - V_\theta^* (\bar{F}_\theta^e (\xi_k, u_\theta^* (\xi_k)))}{h^*} + \frac{V_\theta^* (\bar{F}_\theta^e (\xi_k, u_\theta^* (\xi_k))) - V_\theta^* (\bar{F}_\theta^e (\xi_k, u_\theta^* (\xi_k)))}{h} \frac{h^*}{h}$$

(B.2)

For notational convenience we will drop the arguments of $\bar{F}_\theta^e$ and $\bar{F}_\theta^a$ from now on. Let us first investigate the term $(|\bar{F}_\theta^e| + |\bar{F}_\theta^a|)^{p-1}$ in (B.2). By the definitions of $\bar{F}_\theta^e$ and $\bar{F}_\theta^a$ in (7) and (10), respectively, and Assumption 5 we have that

$$|\bar{F}_\theta^e| \leq |\bar{F}_\theta^e| + |\xi_k| + |u_k| \leq |\bar{F}_\theta^e| + (1 + L_u)|\xi_k|,$$

(B.3)

and the fact that $e^{L_I t_{j+1}^k} \leq e^{L_I t_{j+1}^k - 1}$, $\forall j \in \{1, \ldots, d - 1\}$ to obtain $|\bar{F}_\theta^e| \leq |\xi_k| + \sum_{j=0}^{d-2} \left( e^{L_I t_{j+1}^k - 1} \right) |\epsilon_k| + (d - 1) |u_k|$. Using Assumption 5, we obtain that

$$|\bar{F}_\theta^e| \leq \left( 1 + L_u (d - 1) \left( e^{L_I t_{j+1}^k - 1} \right) \right) |\xi_k|. \quad (B.6)$$

Combining (B.3) and (B.6) and using the definition of $L_c$ in the theorem yields

$$|\bar{F}_\theta^e| \leq (2 + L_u) (d - 1) (e^{L_I t_{j+1}^k - 1}) |\xi_k| = L_c |\xi_k|. \quad (B.7)$$

Next, $|\bar{F}_\theta^a|$ can be upperbounded using Assumptions 3 and 5:

$$|\bar{F}_\theta^a| \leq h^* \hat{\rho}(h^*) (1 + L_u) |\xi_k| \Rightarrow |\bar{F}_\theta^a| \leq |\bar{F}_\theta^a| + h^* \hat{\rho}(h^*) (1 + L_u) |\xi_k|$$

$\Rightarrow |\bar{F}_\theta^a| \leq (L_c + h^* \hat{\rho}(h^*) (1 + L_u)) |\xi_k| = L_a |\xi_k|$, \quad (B.8)
where in the last inequality we used (B.7), the fact that \( h^* \leq \overline{h} \) and the definition of \( L_a \) in the theorem. Combining (B.7) and (B.8), the term \((|\bar{F}_{\theta_k}^c| + |\bar{F}_{\theta_0}^a|)^{p-1}\) in (B.2) can be upperbounded as follows:

\[
(|\bar{F}_{\theta_k}^c| + |\bar{F}_{\theta_0}^a|)^{p-1} \leq (L_a + L_e)^{p-1} |\xi_k|^{p-1}.
\]

(B.9)

Next, we investigate the term \(|\bar{F}_{\theta_k}^c - \bar{F}_{\theta_0}^a|\) in (B.2) in more detail:

\[
|\bar{F}_{\theta_k}^c - \bar{F}_{\theta_0}^a| = |\bar{F}_{\theta_k}^c - \bar{F}_{\theta_k}^c + \bar{F}_{\theta_k}^c - \bar{F}_{\theta_0}^a| \leq |\bar{F}_{\theta_k}^c - \bar{F}_{\theta_k}^c| + |\bar{F}_{\theta_k}^c - \bar{F}_{\theta_0}^a|,
\]

(B.10)

where the first term in the right-hand side of (B.10) relates to the sensitivity of the exact discrete-time model to time-varying parameters \( \theta_k \) as a consequence of time-varying sampling intervals \( h_k \) and time-varying delays \( \tau_k \) and the second term relates to the consistency of the exact and approximate discretisations of the continuous-time plant model for the fixed parameter vector \( \theta^* \). Using Assumptions 3 and 5, the second term in the right-hand side of (B.10) can be upperbounded as follows

\[
|\bar{F}_{\theta_k}^c - \bar{F}_{\theta_0}^a| \leq h^* \rho(h^*) (1 + L_a) |\xi_k|, \forall \xi_k \in \mathbb{R}^{n+\overline{d}m}.
\]

(B.11)

Next, let us study the first term in the right-hand side of (B.10). Clearly, the term \(|\bar{F}_{\theta_k}^c - \bar{F}_{\theta_k}^c|\) reflects the difference in the exact discrete-time plant induced by the difference between \( \theta^* \) and \( \theta_k \). Let us define \( \Delta t^k_j := t^*_j - t^*_j \), \( j \in \{0, 1, 2, \ldots, \overline{d} - d\} \). Since \( t^*_j \in [t_j, 
\overline{t}_j] \forall j \in \{1, 2, \ldots, \overline{d} - d\} \), and \( t^*_j \in [t_j, 
\overline{t}_j] \forall k \) and \( j \in \{1, 2, \ldots, \overline{d} - d\} \), we have that \( \Delta t^k_j \in [-\overline{\Delta t}_j, \Delta t^k_j] \), with \( \overline{\Delta t}_j = \overline{t}_j - t_j, j \in \{1, 2, \ldots, \overline{d} - d\} \). These bounds can be used to evaluate the term \(|\bar{F}_{\theta_k}^c - \bar{F}_{\theta_0}^c|\) in more detail using the definition in (7):

\[
|\bar{F}_{\theta_k}^c - \bar{F}_{\theta_0}^c| \leq \sum_{j=0}^{\overline{d}-d} \left( \int_{s_k}^{s_k+|\Delta t^k_j|} f(x(s), u_{k+j-\overline{d}}) \|x\| \, ds + \int_{s_k}^{s_k+|\Delta t^k_j|} f(x(s), u_{k+j-\overline{d}}) \|u\| \, ds \right),
\]

where in the last inequality we exploited Assumption 4. Exploiting Assumption 4 and the Gronwall-Bellman inequality again we can obtain:

\[
|\bar{F}_{\theta_k}^c - \bar{F}_{\theta_0}^c| \leq \sum_{j=0}^{\overline{d}-d} \left( e^{L_J t^*_j} \left( e^{L_J t^*_j} - 1 \right) \int_{s_k}^{s_k+|\Delta t^k_j|} f(x(s), u_{k+j-\overline{d}}) \|x\| \, ds \right) + \left( e^{L_J t^*_j} - 1 \right) \int_{s_k}^{s_k+|\Delta t^k_j|} f(x(s), u_{k+j-\overline{d}}) \|u\| \, ds.
\]

(B.13)
Let us use the fact that \( t^*_j \leq h^* \) for all \( j \in \{0, 1, \ldots, \overline{d} - d + 1 \} \), the fact that \( \Delta \theta_j^{k+1} = \theta_j^k - \overline{\theta}_k \) and the fact that, since \( t_0^k = 0, \forall k \), we have \( t_0^* = 0 \), which yields \( \Delta \theta^k = t_0^k - t_0^* = 0 \), in (B.13) to obtain

\[
|F_{0}^e - F_{0}^{\overline{e}}| \leq e^{L_f h^*} \left( \sum_{j=1}^{\overline{d} - d + 1} \left( e^{L_f h^* - 1} \left( |x_k| + |u_{k-d}| + \sum_{j=1}^{\overline{d} - d} \left( e^{L_f |\Delta \theta_j^k| - 1} - 1 \right) \left( 2|x_k| + |u_{k+j-\overline{d}}| + |u_{k+j-\overline{d}-1}| \right) \right) \right) .
\]

(B.14)

Next, using Assumption 5, the definition of \( \overline{u}_k \) and the following facts:

- \( 2|x_k| + |u_{k+j-\overline{d}}| + |u_{k+j-\overline{d}-1}| \leq 2|x_k| + \max(|u_k|, |\overline{u}_k|) + |\overline{u}_k| \leq 3|\xi_k| + \max(|\xi_k|, |u_k|), \forall j \in \{1, \ldots, \overline{d} - d \} \);
- \( |x_k| + |u_{k-d}| \leq |\xi_k| + \max(|\overline{u}_k|, |u_k|) \leq |\xi_k| + \max(|\xi_k|, |u_k|) \);

in (B.14) we obtain

\[
|F_{0}^e - F_{0}^{\overline{e}}| \leq e^{L_f h^*} \left( \sum_{j=1}^{\overline{d} - d + 1} \left( e^{L_f h^* - 1} \left( |\xi_k| + \max(|\xi_k|, |u_k|) \right) + \sum_{j=1}^{\overline{d} - d} \left( e^{L_f |\Delta \theta_j^k| - 1} - 1 \right) \left( 3|\xi_k| + \max(|\xi_k|, |u_k|) \right) \right) \right)
\]

(B.15)

Then, the inequality in (B.15) can be rewritten as follows:

\[
|F_{0}^e - F_{0}^{\overline{e}}| \leq \rho_\theta \left( h^*, M_h, M_{t_1}, \ldots, M_{t_{\overline{d} - d}} \right) |\xi_k|
\]

(B.16)

with \( \rho_\theta \left( h^*, M_h, M_{t_1}, \ldots, M_{t_{\overline{d} - d}} \right) \) defined in (15). Next, we return to the evaluation of the increment \( \Delta V_k \) of the candidate Lyapunov function given in (B.2) by using (B.9), (B.10), (B.11) and (B.16):

\[
\Delta V_k \leq \left( -a_3 \frac{h^*}{h} + \frac{L_v (L_a + L_e)^{p-1}}{h} \left( h^* \dot{\rho}(h^*) \left( 1 + L_u + \rho_\theta \left( h^*, M_h, M_{t_1}, \ldots, M_{t_{\overline{d} - d}} \right) \right) \right) |\xi_k|^p.\]

(B.17)

Let us now exploit the fact that \( h^* = \varepsilon \frac{h}{h} + (1 - \varepsilon) \overline{h} \) given the definition in (8) and the fact that \( \overline{h} > 0 \). Hence, we can conclude that \( h^*/\overline{h} \geq 1 - \varepsilon \) and using this fact in (B.17) gives

\[
\Delta V_k \leq \left( -(1 - \varepsilon)a_3 + \frac{L_v (L_a + L_e)^{p-1}}{\overline{h}} \left( h^* \dot{\rho}(h^*) \left( 1 + L_u + \rho_\theta \left( h^*, M_h, M_{t_1}, \ldots, M_{t_{\overline{d} - d}} \right) \right) \right) \right) |\xi_k|^p.
\]

(B.18)

So, if the condition in (14) is indeed satisfied for some \( 0 < \beta < 1 - \varepsilon \), then

\[
\Delta V_k \leq -a_3 \beta |\xi_k|^p.
\]

(B.19)

Note that (B.19) with the definition of \( V_k \) in (B.1) implies that

\[
\frac{\overline{V}_\theta \left( F_{0}^e (\xi_k, \theta^*(\xi_k)) - V_{\theta}^e (\xi_k) \right)}{h_k} \leq -a_3 \beta |\xi_k|^p, \forall \theta_k \in \Theta,
\]

(B.20)

since \( h_k \in \left[ \frac{\overline{h}}{\overline{h}}, \overline{h} \right), \forall k \in \mathbb{N} \). Given the fact that the function \( V_{\theta}^e \) satisfies the conditions in (A.2) of Theorem 3 (see Assumption 1 and (B.20)) we can conclude that the closed-loop uncertain exact discrete-time system (7), (11) is globally exponentially stable (see also Remark 8). This proves the first statement in the theorem.
Let us next prove the second statement of the theorem. Note that the term \( \frac{Lc(Lc+La)^{p-1}}{\mathcal{H}} \rho_\theta (h^*, M_h, M_\ell_1, \ldots, M_\ell_{\mathcal{L}-2}) \) in (14) can always be made arbitrarily small by an appropriate choice of \( \mathcal{H} - h^* \) and \( \mathcal{L} - \ell \) (i.e., by making these uncertainty intervals sufficiently small). Moreover, using the fact that \( \hat{\rho} \) is a \( \mathcal{K}_\infty \) function, the fact that \( \frac{h^*}{\mathcal{H}} \leq 1 \) and the fact that Assumptions 1-3 and 5 hold for all \( \theta^* \in \Theta_0^\mathcal{H} \), where the definition of \( \Theta_0^\mathcal{H} \) in (12) allows \( h^* \) to be taken arbitrarily close to zero, the term \( \frac{Lc(Lc+La)^{p-1}}{\mathcal{H}} \rho(h^*) (1 + L_a) \) in (14) can always be made arbitrarily small by making the nominal sampling interval \( h^* \) small enough since \( \hat{\rho}(h^*) \in \mathcal{K}_\infty \). Consequently, there exists an \( h^*_{max} \leq h^* \) such that for all \( h^* \in (0, h^*_{max}) \), there exist \( \mathcal{H}, \mathcal{L}, \mathcal{H}, \mathcal{T} \) and \( 0 < \beta < 1 - \varepsilon \) satisfying (14). In turn, this implies, using (B.18) and (B.19) and the definition of \( \bar{V}_k \) in (B.1), that there exists \( 0 < \beta < 1 - \varepsilon \) such that

\[ \frac{V_{\theta^*}(F_{\theta^*}^e(\xi_k, u_{\theta^*}(\xi_k))) - V_{\theta^*}(\xi_k)}{h_k} \leq -a_3 \beta |\xi_k|^p, \]

(B.21)

for all \( \theta^* \in \Theta_0^\mathcal{H}(h^*_{max}, \underline{d}, \bar{d}) \) and for all \( \theta^* \in \Theta(\theta^*) \forall k \in \mathbb{N} \), where \( \Theta \) may depend on \( \theta^* \) since \( \mathcal{L}, \mathcal{T}, \mathcal{H}, \mathcal{T} \) and \( \mathcal{T} \) may depend on \( h^* \) when guaranteeing the satisfaction of condition (14). Using Theorem 3, and Remark 8, we can now conclude that the closed-loop exact uncertain discrete-time model is GES for all \( \theta^* \in \Theta_0^\mathcal{H}(h^*_{max}, \underline{d}, \bar{d}) \) and \( \theta^* \in \Theta(\theta^*) \forall k \in \mathbb{N} \). This proves the second statement in the theorem and completes the proof.

B.2 Proof of Proposition 1

Let us first prove the first statement of the proposition. Hereto, we first study the one-step consistency of (17) with the exact discrete-time model \( F_{\theta}^e(\xi, u) \):

\[ |F_{\theta}^e(\xi_k, u_k) - F_{\theta}^{Euler}(\xi_k, u_k)| = |F_{\theta}^e(\xi_k, u_k) - F_{\theta}^{Euler}(\xi_k, u_k)| \]

\[ = \sum_{j=0}^{\bar{d} - d} \int_{s_k + t_j}^{s_k + t_j + 1} f(x(s), u_{k+j-\bar{d}}) ds - \sum_{j=0}^{\bar{d} - d} (t_{j+1} - t_j) f(x_k, u_{k+j-\bar{d}}) \]

\[ = \sum_{j=0}^{\bar{d} - d} \left( \int_{s_k}^{s_k + (t_{j+1} - t_j)} f(x(s) + t_j, u_{k+j-\bar{d}}) ds - (t_{j+1} - t_j) f(x_k, u_{k+j-\bar{d}}) \right). \]

(B.22)

Hence, \( |F_{\theta}^e(\xi_k, u_k) - F_{\theta}^{Euler}(\xi_k, u_k)| \) can be upperbounded as follows:

\[ |F_{\theta}^e(\xi_k, u_k) - F_{\theta}^{Euler}(\xi_k, u_k)| \leq \sum_{j=0}^{\bar{d} - d} (T_{1,j} + T_{2,j}) \]

(B.23)

with

\[ T_{1,j} = \left| \int_{s_k}^{s_k + (t_{j+1} - t_j)} f(x(s), u_{k+j-\bar{d}}) ds - (t_{j+1} - t_j) f(x_k, u_{k+j-\bar{d}}) \right| \]

\[ T_{2,j} = \int_{s_k}^{s_k + (t_{j+1} - t_j)} \left| f(x(s) + t_j, u_{k+j-\bar{d}}) - f(x(s), u_{k+j-\bar{d}}) \right| ds, \quad j \in \{1, 2, \ldots, \bar{d} - d\}. \]

(B.24)

In order to upperbound the terms \( T_{1,j}, j \in \{0, 1, 2, \ldots, \bar{d} - d\} \), we use the fact that \( F_{\theta_0}^e(\xi_k, u_k) \), with \( \theta_0 := \left[ h \ 0 \ 0 \ \ldots \ 0 \right]^T \), i.e., the exact discretisation in the case of constant sampling intervals and no delays, is one-step consistent with \( F_{\theta_0}^{Euler}(\xi_k, u_k) := x_k + hf(x_k, u_k) \), see e.g. [21]. Consequently, there exists a function \( \rho \in \mathcal{K}_\infty \) such that

\[ T_{1,j} \leq (t_{j+1} - t_j) \rho(t_{j+1} - t_j) \left( |x_k| + |u_{k+j-\bar{d}}| \right), \quad j \in \{0, 1, 2, \ldots, \bar{d} - d\}. \]

(B.25)
Next, let us upperbound the terms $T_{2,j}$ using Assumption 4:

$$T_{2,j} \leq L_f \int_{s_k}^{s_{k+j+t_j-t_j}} |x(s + t_j) - x(s)| \, ds, \ j \in \{0, 1, 2, \ldots, \overline{d} - d\},$$

where we used that in (B.26) both $x(s + t_j)$ and $x(s)$ are solutions corresponding to the input $u_{k+j-\overline{d}}$. Using the latter fact and Assumption 4 once more and by exploiting the Gronwall-Bellman inequality we obtain:

$$T_{2,j} \leq L_f (e^{L_f t_j} - 1) \int_{s_k}^{s_{k+j+t_j-t_j}} (|x(s)| + |u_{k+j-\overline{d}}|) \, ds$$

$$\leq L_f (e^{L_f t_j} - 1) \int_{s_k}^{s_{k+j+t_j-t_j}} e^{L_f (s-s_k)} (|x_k| + |u_{k+j-\overline{d}}|) \, ds$$

$$= (e^{L_f t_j} - 1) (e^{L_f (t_j+1-t_j)} - 1) (|x_k| + |u_{k+j-\overline{d}}|), \ j \in \{0, 1, 2, \ldots, \overline{d} - d\}.$$  

Combining (B.23), (B.25) and (B.27) gives

$$|\hat{F}_{\theta}^e(\xi_k, u_k) - \hat{F}_{\theta}^{\mathrm{Euler}}(\xi_k, u_k)| \leq \sum_{j=0}^{\overline{d} - d} ((t_j + 1 - t_j) \rho(t_j + 1 - t_j) + (e^{L_f t_j} - 1) (e^{L_f (t_j+1-t_j)} - 1)) (|x_k| + |u_{k+j-\overline{d}}|).$$

Let us now use the fact that $t_j \leq h$ for all $j \in \{0, 1, \ldots, \overline{d} - d + 1\}$ and the fact that $|x_k| + |u_{k+j-\overline{d}}| \leq |x_k| + |\bar{u}_k| + |u_k| \leq 2|\xi_k| + |u_k|$, $\forall j \in \{0, 1, \ldots, \overline{d} - d\}$ to construct a (more conservative) upperbound for $|\hat{F}_{\theta}^e(\xi_k, u_k) - \hat{F}_{\theta}^{\mathrm{Euler}}(\xi_k, u_k)|$ in terms of the sampling interval $h$: $|\hat{F}_{\theta}^e(\xi_k, u_k) - \hat{F}_{\theta}^{\mathrm{Euler}}(\xi_k, u_k)| \leq (\overline{d} - d + 1) (h \rho(h) + (e^{L_f h} - 1)^2) (2|\xi_k| + |u_k|)$. Next, we will exploit the following inequality:

$$\frac{(e^{L_f h} - 1)^2}{h} = \frac{1}{h} \sum_{i=1}^{\infty} \frac{(L_f h)^i}{i!} (e^{L_f h} - 1) = L_f \sum_{i=1}^{\infty} \frac{(L_f h)^{i-1}}{(i-1)!} (e^{L_f h} - 1)$$

$$\leq L_f \sum_{i=1}^{\infty} \frac{(L_f h)^{i-1}}{(i-1)!} (e^{L_f h} - 1) = L_f e^{L_f h} (e^{L_f h} - 1).$$

Hence, we can conclude that there exists a function $\rho^* \in \mathcal{K}_{\infty}$ defined by $\rho^*(h) = 2(\overline{d} - d + 1) (\rho(h) + L_f e^{L_f h} (e^{L_f h} - 1))$ such that

$$|\hat{F}_{\theta}^e(\xi_k, u_k) - \hat{F}_{\theta}^{\mathrm{Euler}}(\xi_k, u_k)| \leq h \rho^*(h) (|\xi_k| + |u_k|).$$

Note that $\rho^*$ is a $\mathcal{K}_{\infty}$ function since both $\rho$ and $L_f e^{L_f h} (e^{L_f h} - 1)$ are $\mathcal{K}_{\infty}$ functions. The inequality in (B.30) expresses that fact that $\hat{F}_{\theta}^{\mathrm{Euler}}(\xi_k, u_k)$ is indeed one-step consistent with $\hat{F}_{\theta}^e(\xi_k, u_k)$. This concludes the proof of the first statement in the proposition.

Let us now prove the second statement of the proposition, i.e. we address the question of the one-step consistency of $\hat{F}_{\theta}^e(\xi_k, u_k)$ with $\hat{F}_{\theta}^e(\xi_k, u_k)$. Using the assumptions in the proposition and (B.30), it holds that

$$|\hat{F}_{\theta}^e(\xi_k, u_k) - \hat{F}_{\theta}^a(\xi_k, u_k)| = |\hat{F}_{\theta}^e(\xi_k, u_k) - \hat{F}_{\theta}^{Euler}(\xi_k, u_k) + \hat{F}_{\theta}^{Euler}(\xi_k, u_k) - \hat{F}_{\theta}^a(\xi_k, u_k)|$$

$$\leq |\hat{F}_{\theta}^e(\xi_k, u_k) - \hat{F}_{\theta}^{Euler}(\xi_k, u_k)| + |\hat{F}_{\theta}^{Euler}(\xi_k, u_k) - \hat{F}_{\theta}^a(\xi_k, u_k)|$$

$$\leq (h \rho^*(h) + h \rho(h)) (|\xi_k| + |u_k|).$$

By defining $\hat{\rho}(h) := \rho^*(h) + \rho(h)$, implying that $\hat{\rho} \in \mathcal{K}_{\infty}$, we obtain that $|\hat{F}_{\theta}^e(\xi_k, u_k) - \hat{F}_{\theta}^a(\xi_k, u_k)| \leq h \hat{\rho}(h) (|\xi_k| + |u_k|)$, which shows that indeed the approximate discrete-time plant model $\hat{F}_{\theta}^e(\xi, u)$ is one-step consistent with the exact discrete-time model $\hat{F}_{\theta}^e(\xi, u)$. This concludes the proof of the second statement on the proposition and completes the proof.
B.3 Proof of Theorem 2

Let us study the evolution of the candidate Lyapunov function $V_{q^*}$ along solutions of the closed-loop uncertain exact discrete-time system (7), (11): $\Delta V := (V_{q^*}(F^o_{q^*}(\xi, u_{q^*}(\xi))) - V_{q^*}(\xi)) / h$. Note that we aim to show that $\Delta V$ is strictly negative under the conditions in (A.5) for any $\theta \in \Theta$, with $\Theta$ as in (5). Using Assumption 1 and the definitions in (7) and (10) we obtain

$$
\Delta V = \frac{V_{q^*}(F^o_{q^*}(\xi, u_{q^*}(\xi))) - V_{q^*}(\xi) + \Delta_{vb}(\xi)}{h} + \frac{V_{q^*}(F^o_{q^*}(\xi, u_{q^*}(\xi))) - V_{q^*}(\xi)}{h} \leq - \frac{h^*}{h} \alpha_3(\|\xi\|) + \frac{1}{h} (\Delta V_1 + \Delta V_2)
$$

with

$$
\Delta V_1 := V_{q^*}(F^o_{q^*}(\xi, u_{q^*}(\xi))) - V_{q^*}(F^o_{q^*}(\xi, u_{q^*}(\xi))), \quad \Delta V_2 := V_{q^*}(F^o_{q^*}(\xi, u_{q^*}(\xi))) - V_{q^*}(F^o_{q^*}(\xi, u_{q^*}(\xi))).
$$

Herein, $\Delta V_1$ relates to the sensitivity of the exact discrete-time model to time-varying parameters $\theta_k$ as a consequence of time-varying sampling intervals $h_k$ and time-varying delays $\tau_k$ and $\Delta V_2$ relates to the consistency of the exact and approximate discretisations of the plant model for the nominal parameter vector $\theta^*$. Let us first upperbound the term $\Delta V_2$. We aim to show that

$$
\Delta V_2 \leq \frac{h^*}{2} \alpha_3(\|\xi\|),
$$

so that the effect of the approximate discretisation only ‘consumes’ half of the contraction that the stability of the approximate model for $\theta^*$ provides, see (B.32). If $F^o_{q^*}(\xi, u_{q^*}(\xi)) \leq \Delta$ and if $F^o_{q^*}(\xi, u_{q^*}(\xi)) \leq \Delta$, then Assumption 7 gives $\Delta V_2 \leq L_u F^o_{q^*}(\xi, u_{q^*}(\xi)) - F^o_{q^*}(\xi, u_{q^*}(\xi))$. Additionally, if $|\xi| \leq \Delta$ and $|u_{q^*}(\xi)| \leq \Delta_u$, then Assumption 8 in combination with the latter inequality gives $\Delta V_2 \leq L_u h^* \tilde{\rho}(h^*)$. So, the requirement in (B.34) translates to the requirement $L_u h^* \tilde{\rho}(h^*) \leq \frac{h^*}{2} \alpha_3(\|\xi\|)$ or $\tilde{\rho}(h^*) \leq \frac{\alpha_3(\|\xi\|)}{2L_u}$. Resuming, we can say that (B.34) is validated if

\begin{enumerate}
\item $|\xi| \leq \Delta$;
\item $|u_{q^*}(\xi)| \leq \Delta_u$;
\item $|F^o_{q^*}(\xi, u_{q^*}(\xi))| \leq \Delta$;
\item $|F^o_{q^*}(\xi, u_{q^*}(\xi))| \leq \Delta$;
\item $\tilde{\rho}(h^*) \leq \frac{\alpha_3(\|\xi\|)}{2L_u}$.
\end{enumerate}

Let us check these requirements and, where necessary, pose appropriate conditions under which these are fulfilled:

**AD 1.** $|\xi| \leq \Delta$. Since we aim to show that (A.5) holds, we assume that $|\xi| \leq D$. Hence the requirement that $|\xi| \leq \Delta$ is satisfied if we choose $\Delta \geq \delta_2 \geq D$, which is guaranteed by the choice for $\Delta \geq q\delta_2$ with $q > 1$ in the theorem.

**AD 2.** $|u_{q^*}(\xi)| \leq \Delta_u$. Under Assumption 10, we have that $|u(\xi)| \leq L_u|\xi|$ for $|\xi| \leq \Delta$. So using the fact that $|\xi| \leq D \leq \Delta$, we have that $|u(\xi)| \leq L_uD$. Since under the assumptions in the theorem $\Delta_u \geq (1 + L_u)D > L_uD$, the requirement $|u_{q^*}(\xi)| \leq \Delta_u$ under point 2 is validated.

**AD 3.** $|F^o_{q^*}(\xi, u_{q^*}(\xi))| \leq \Delta$. The fact that $|\xi| \leq D$, the definition of $\delta_2$ in (19), Assumption 6 and our choice for $\Delta \geq \delta_2 \geq D$ in the theorem imply that

$$
|F^o_{q^*}(\xi, u_{q^*}(\xi))| \leq \alpha_1^{-1} (V_{q^*}(F^o_{q^*}(\xi, u_{q^*}(\xi)))) \leq \alpha_1^{-1} (V_{q^*}(\xi)) \leq \alpha_1^{-1} \circ \alpha_2(|\xi|) \leq \alpha_1^{-1} \circ \alpha_2(D) \leq \delta_2 \leq \Delta.
$$

**AD 4.** $|F^o_{q^*}(\xi, u_{q^*}(\xi))| \leq \Delta$. From Assumption 8, it follows that $|F^o_{q^*}(\xi, u_{q^*}(\xi)) - F^o_{q^*}(\xi, u_{q^*}(\xi))| \leq h^* \tilde{\rho}(h^*)$ and $|F^o_{q^*}(\xi, u_{q^*}(\xi))| \leq |F^o_{q^*}(\xi, u_{q^*}(\xi))| + h^* \tilde{\rho}(h^*)$, since by the choice of $\Delta$ and $\Delta_u$ it holds that $|\xi| \leq \Delta$ and $|u_{q^*}(\xi)| \leq D$. 


It is important to note that the developments in the proof of Theorem 1 leading to the bound in (B.16) exploit a delay 
\[ h^* \dot{\rho}(h^*) \leq \frac{1}{2} \alpha_2^{-1} \left( \frac{d}{2} \right) = \delta_1, \]  
which is guaranteed by the choice of \( L_1 \) and the condition (20) in the theorem. We have that  
\[ |\bar{F}_{\theta^*}(\xi, u_{\theta^*}(\xi))| \leq \alpha_1^{-1} \circ \alpha_2(D) + h^* \dot{\rho}(h^*). \]  
Consequently, under the condition  
\[ |\bar{F}_{\theta^*}(\xi, u_{\theta^*}(\xi))| \leq \alpha_1^{-1} \circ \alpha_2(D) + \frac{1}{2} \alpha_2^{-1} \left( \frac{d}{2} \right) = \delta_2 \leq \Delta. \] 
We will address the satisfaction of this assumption when evaluating the \( \Delta V_1 \)-term later. Equation (B.38), together with Assumption 6 and the definition of \( \delta_1 \) in (19), gives  
\[ |\bar{F}_{\theta^*}(\xi, u_{\theta^*}(\xi))| \geq \frac{\alpha_3}{2L_v} |\xi|. \]  
Using the latter fact again in (B.16) gives  
\[ |\bar{F}_{\theta^*}(\xi, u_{\theta^*}(\xi))| \leq \alpha_2^{-1} \left( \frac{d}{2} \right) = \delta_1. \]  
Using this inequality in combination with the inequalities in (B.35), we obtain  
\[ |\xi| \geq d \alpha_2^{-1} \circ \alpha_1(\delta_1). \]  
Consequently, the condition \( \dot{\rho}(h^*) \leq \frac{\alpha_3}{2L_v} |\xi| \) under point 5 can be written as  
\[ \dot{\rho}(h^*) \leq \frac{\alpha_3 \circ \alpha_2^{-1} \circ \alpha_1(\delta_1)}{2L_v}, \]  
which is guaranteed by the choice of \( L_1 \) and condition (20) in the theorem.

Let us next evaluate the \( \Delta V_1 \)-term in (B.32), as defined in (B.33), which relates to the sensitivity of the exact discrete-time model to time-varying parameters \( \theta_k \) as a consequence of time-varying sampling intervals \( h_k \) and time-varying delays \( \tau_k \):  
\[ \Delta V_1 = V_{\theta^*}(\bar{F}_{\theta^*}(\xi, u_{\theta^*}(\xi)) - V_{\theta^*}(\bar{F}_{\theta^*}(\xi, u_{\theta^*}(\xi))). \]  
Using Assumption 7 we obtain  
\[ \Delta V_1 \leq L_v |\bar{F}_{\theta^*}(\xi, u_{\theta^*}(\xi)) - \bar{F}_{\theta^*}(\xi, u_{\theta^*}(\xi))| \]  
(B.42) for  
\[ |\bar{F}_{\theta^*}(\xi, u_{\theta^*}(\xi))| \leq \Delta \text{ and } |\bar{F}_{\theta^*}(\xi, u_{\theta^*}(\xi))| \leq \Delta. \]  
Note that we have already shown that  
\[ |\bar{F}_{\theta^*}(\xi, u_{\theta^*}(\xi))| \leq \Delta \]  
under point 4 above by the choice for \( \Delta \geq \delta_2 \) in the theorem. Now, we will show under which conditions we can also guarantee that  
\[ |\bar{F}_{\theta^*}(\xi, u_{\theta^*}(\xi))| \leq \Delta \text{ for all } \theta \in \Theta. \]  
Here, we exploit the upperbounding of the term  
\[ |\bar{F}_{\theta^*}(\xi, u_{\theta^*}(\xi)) - \bar{F}_{\theta^*}(\xi, u_{\theta^*}(\xi))| \]  
as previously pursued in the proof of Theorem 1 ultimately leading to the bound in (B.16), where we used Assumption 10 and the fact that  
\[ |\xi| \leq D \leq \Delta. \]  
Using the latter fact again in (B.16) gives  
\[ |\bar{F}_{\theta^*}(\xi, u_{\theta^*}(\xi)) - \bar{F}_{\theta^*}(\xi, u_{\theta^*}(\xi))| \leq \rho_0 \left( h^*, M_h, M_{t_1}, \ldots, M_{t_{\tau-1}} \right) D. \]  
(B.43) It is important to note that the developments in the proof of Theorem 1 leading to the bound in (B.16) exploit a Lipschitz condition on the right-hand side \( f(x, u) \) of the continuous-time plant model, which under Assumption 9
only holds for $|x| \leq \Delta_x$ and $|u| \leq \Delta_u$. To be more specific, in the proof of Theorem 1 we use that

$$|f(x(s + t_j^k), u_{k+j-\bar{d}})| \leq L_f \left(|x(s + t_j^k)| + |u_{k+j-\bar{d}}|\right), \forall j \in \{0, 1, \ldots, \bar{d} - \bar{d}\}$$
$$|f(x(s + t_{j+1}^k), u_{k+j-\bar{d}})| \leq L_f \left(|x(s + t_{j+1}^k)| + |u_{k+j-\bar{d}}|\right), \forall j \in \{0, 1, \ldots, \bar{d} - \bar{d}\}$$

$$|x(s + t_j^k) + |u_{k+j-\bar{d}}| \leq e^{L_f(t_j^k + s - s_k)} \left(|x_k| + |u_{k+j-\bar{d}}|\right), \forall j \in \{0, 1, \ldots, \bar{d} - \bar{d}\} \text{ and } s \in [s_k, s_k + \Delta t_j^k)$$

$$|x(s + t_{j+1}^k) + |u_{k+j-\bar{d}}| \leq e^{L_f(t_{j+1}^k + s - s_k)} \left(|x_k| + |u_{k+j-\bar{d}}|\right), \forall j \in \{0, 1, \ldots, \bar{d} - \bar{d}\} \text{ and } s \in [s_k, s_k + \Delta t_{j+1}^k).$$

(B.44)

Since in Assumption 9 the Lipschitz property on $f(x, u)$ only holds for $|x| \leq \Delta_x$ and $|u| \leq \Delta_u$, we require that

- the following bound on the continuous-time evolution of the state $x$ is satisfied: $|x(s + t_j^k)| \leq \Delta_x$ for $s \in [s_k, s_k + \Delta t_j^k), j \in \{0, 1, \ldots, \bar{d} - \bar{d} + 1\}$. Given the definitions of $t_j^k$, $t_j$ and $\Delta t_j^k$ this requirement can be replaced by the requirement $|x(s)| \leq \Delta_x$ for $s \in [s_k, s_k + 1]$.
- the following bound on the (past) control input is satisfied: $|u_{k+j-\bar{d}}| \leq \Delta_u, \forall j \in \{0, 1, \ldots, \bar{d} - \bar{d}\}$.

Using the definition of $\xi$, we can conclude that $|x_k| \leq |\xi_k|$ and $|u_{k+j-\bar{d}}| \leq |\xi_k| + |u_k|, \forall j \in \{0, 1, \ldots, \bar{d} - \bar{d}\}$. Since $|\xi_k| \leq D$ and $|u_k| \leq L_u D$, this implies that $|x_k| \leq D$ and $|u_{k+j-\bar{d}}| \leq (1 + L_u) D, \forall j \in \{0, 1, \ldots, \bar{d} - \bar{d}\}$. Because $x(s)$ corresponds to a solution related to the input $\bar{u}$ we have that, if $|x(s)| \leq \Delta_x$ for $s \in [s_k, s_k + 1)$, and $|u_{k+j-\bar{d}}| \leq \Delta_u$ (where we now exploit the choice for $\Delta_u \geq (1 + L_u) D$ in the theorem), then

$$|x(s)| \leq e^{L_f(s - s_k)} |x_k| + |u_{k+j-\bar{d}}| \left(e^{L_f(s - s_k)} - 1\right) \leq e^{L_f h_k} D + (1 + L_u) D \left(e^{L_f h_k} - 1\right),$$

(B.45)

$$\leq (2 + L_u) e^{L_f h_k} D - (1 + L_u) D,$$

where we exploited Assumption 9 and the Gronwall-Bellman inequality. Hence, we require that $\Delta_x \geq D \left(2 + L_u\right) e^{L_f \bar{t}} - (1 + L_u) \bar{t}$, since $h_k \leq \bar{t}, \forall k$, which translates into the following condition on the upperbound for the sampling interval:

$$\bar{t} \leq \frac{1}{L_f} \ln \left(\frac{\Delta_x + (1 + L_u)}{(2 + L_u) D}\right),$$

(B.46)

which is guaranteed by the choice of $L_2$ and the condition (21) in the theorem. Now, under condition (B.46), the inequality in (B.43) is valid and if we additionally use that $|\bar{F}_0^\circ(\xi, u_\theta(\xi))| \leq \delta_2$, see (B.37), we can conclude that $|\bar{F}_0^\circ(\xi, u_\theta(\xi))| \leq \delta_2 + \rho_0 \left(h^*, M_h, M_{t_1}, \ldots, M_{\tau - \bar{d}}\right) D$. To ensure that $|\bar{F}_0^\circ(\xi, u_\theta(\xi))| \leq \Delta$, which was needed to validate (B.42), we require that

$$\Delta \geq \delta_2 + \rho_0 \left(h^*, M_h, M_{t_1}, \ldots, M_{\tau - \bar{d}}\right) D,$$

(B.47)

which is guaranteed by the choice of $L_3$ and the condition (22) in the theorem since $\Delta \geq q \delta_2$.

Below, we will address the fact that the formulation of the condition in (B.41) was hinging on the fact that the inequality in (B.38) is satisfied for which (B.39) is a sufficient condition.

Let us aim to satisfy the conditions in (A.5) for any $\theta \in \Theta$, with $\Theta$ as in (5), in the following way

$$\left\{ \begin{array}{l}
|\xi| \leq D \\
\max\{V_\theta(\bar{F}_0^\circ(\xi, u_\theta(\xi)), V_\theta(\xi)) \geq d\} \Rightarrow \frac{\Delta V}{h} \leq -\frac{1}{4\ h} \alpha_3(|\xi|),
\end{array} \right.$$
Hence, we can conclude that

$$L_v \rho_0 \left( h^*, M_h, M_{t_1}, \ldots, M_{\varpi-\delta} \right) D \leq \frac{d}{4}, \quad \text{(B.49)}$$

which is guaranteed by the choice of $L_3$ and the condition (22) in the theorem. Herewith, the derivation of condition (B.41) is valid. Moreover, (B.42) and (B.43) also show that

$$\Delta V_1 \leq L_v \rho_0 \left( h^*, M_h, M_{t_1}, \ldots, M_{\varpi-\delta} \right) D. \quad \text{(B.50)}$$

Next, we return to the evaluation of $\Delta V$ in (B.32)

$$\Delta V \leq - \frac{h^*}{h} \alpha_3(|\xi|) + \frac{1}{h} (\Delta V_1 + \Delta V_2). \quad \text{(B.51)}$$

Given the fact that conditions (B.36), (B.41), (B.46), (B.47) and (B.49) are satisfied by the choice of $L_1, L_2, L_3$ and the conditions (20), (21), and (22) in the theorem and the fact that $V_{\theta^*}(F_{\theta}(\xi, u_{\theta^*}(\xi))) \geq \frac{\delta}{4} d$, we have that $\Delta V_2 \leq \frac{h^*}{h} \alpha_3(|\xi|)$ and, consequently (B.51) gives $\Delta V \leq - \frac{h^*}{h} \alpha_3(|\xi|) + \frac{1}{h} \Delta V_1$. Combining this with (B.50) gives $\Delta V \leq - \frac{h^*}{h} \alpha_3(|\xi|) + \frac{1}{h} \rho_0 \left( h^*, M_h, M_{t_1}, \ldots, M_{\varpi-\delta} \right) D$. Consequently, we have that

$$\Delta V \leq - \frac{h^*}{h} \alpha_3(|\xi|) \quad \text{if} \quad L_v \rho_0 \left( h^*, M_h, M_{t_1}, \ldots, M_{\varpi-\delta} \right) D \leq \frac{1}{4} h^* \alpha_3(|\xi|). \quad \text{(B.52)}$$

Using (B.40), the condition in (B.52) can be formulated as follows:

$$L_v \rho_0 \left( h^*, M_h, M_{t_1}, \ldots, M_{\varpi-\delta} \right) D \leq \frac{1}{4} h^* \alpha_3 \circ \alpha_2^{-1} \circ \alpha_1(\delta_1), \quad \text{(B.53)}$$

which is guaranteed by the choice of $L_3$ and the condition (22) in the theorem. So, we can conclude that under the conditions (B.36), (B.41), (B.46), (B.47), (B.49) and (B.53) and for the case that $V_{\theta^*}(F_{\theta}(\xi, u_{\theta^*}(\xi))) \geq \frac{\delta}{4} d$, it holds that

$$\Delta V \leq - \frac{h^*}{h} \alpha_3(|\xi|). \quad \text{(B.54)}$$

**Case 1** $V_{\theta^*}(F_{\theta}(\xi, u_{\theta^*}(\xi))) \geq \frac{\delta}{4} d$ and $V_{\theta^*}(\xi) \geq d$;

Let us use Assumption 6

$$V_{\theta^*}(F_{\theta}(\xi, u_{\theta^*}(\xi))) - V_{\theta^*}(\xi) \leq - h^* \alpha_3(|\xi|) \Rightarrow V_{\theta^*}(\xi) \geq V_{\theta^*}(F_{\theta}(\xi, u_{\theta^*}(\xi))) + h^* \alpha_3(|\xi|) \Rightarrow V_{\theta^*}(\xi) \geq h^* \alpha_3(|\xi|). \quad \text{(B.55)}$$

Hence, $V_{\theta^*}(F_{\theta}(\xi, u_{\theta^*}(\xi))) - V_{\theta^*}(\xi) \leq \frac{1}{4} d - V_{\theta^*}(\xi) = \frac{1}{4} d - V_{\theta^*}(\xi) - \frac{1}{4} V_{\theta^*}(\xi) \leq - \frac{h^*}{4} \alpha_3(|\xi|)$. Consequently, also in this case we have that

$$\Delta V = \frac{V_{\theta^*}(F_{\theta}(\xi, u_{\theta^*}(\xi))) - V_{\theta^*}(\xi)}{h} \leq - \frac{h^*}{4} \alpha_3(|\xi|). \quad \text{(B.56)}$$

where we used (B.55) and the fact that in this case $V_{\theta^*}(\xi) \geq d$.

Let us now exploit the fact that $h^* = \varepsilon \frac{h}{h} + (1 - \varepsilon) \frac{h}{h} \geq (1 - \varepsilon) \frac{h}{h}$ given the definition in (8) and the fact that $h > 0$. Hence, we can conclude that $h^* / \bar{h} > 1 - \varepsilon$. Consequently, from (B.54) in Case 1 and (B.56) in Case 2, we have that

$$\Delta V = \frac{V_{\theta^*}(F_{\theta}(\xi, u_{\theta^*}(\xi))) - V_{\theta^*}(\xi)}{h} \leq - \frac{1}{4}(1 - \varepsilon) \alpha_3(|\xi|), \quad \text{(B.57)}$$
which implies that \[ \frac{V_\ast((\bar{F}_2(\xi_{\text{aux}}(\xi))) - V_\ast(\xi))}{\dot{\alpha}_3(|\xi|)} \leq -\frac{1}{4}(1 - \varepsilon)\alpha_3(|\xi|) \] since \( h \leq \bar{h}. \) Clearly, the conditions of Theorem 4 are now satisfied for the closed-loop exact discrete-time model (7), (11), implying that there exists a \( KL \)-function \( \beta \) and a \( \delta = \alpha^{-1}(d) > 0 \) such that the solutions of the closed-loop exact discrete-time model (7), (11) satisfy (A.6). Moreover, since conditions (20), (21), (22) can be satisfied for any \( d, D > 0, \) with \( d \in (0, \alpha_1(D)] \), by choosing \( h^*, \bar{h} \) and the uncertainties \( \bar{h} - h^* \) and \( \bar{\tau} - \tau \) small enough and Assumptions 6, 7, 8 and 10 hold for all \( \theta^* \in \Theta_\ast \), the closed-loop exact discrete-time model (7), (11) is semi-globally practically asymptotically stable. This completes the proof.

References


