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Optimal Estimation of Rational Feedforward Controllers: An Instrumental Variable Approach

Frank Boeren, Lennart Blanken, Dennis Bruijnen, and Tom Oomen

***Abstract*— Iterative control enables a significant control performance enhancement by learning feedforward command signals from previous tasks in a batch-to-batch fashion. The aim of this paper is to develop an approach to estimate the parameters of rational feedforward controllers that provide high performance and extrapolation capabilities towards varying tasks. An instrumental variable-based algorithm is developed that leads to unbiased parameter estimates and optimal accuracy in terms of variance. The approach also enables optimal estimation of a feedforward controller using a polynomial basis. Simulation results confirm that optimal accuracy is obtained with the proposed approach.**

I. INTRODUCTION

Feedforward control is widely used to improve the performance of systems, since feedforward can effectively compensate for the error induced by known, repeating exogenous signals. In fact, typical performance requirements necessitate feedforward in many servo systems, see, e.g., atomic force microscopes [1], [2] and wafer scanners [3].

Iterative Learning Control (ILC) algorithms update the feedforward signal by learning from previous tasks under the assumption that the task is repeating [4], [5]. However, changes in the reference signal typically result in significant performance deterioration, see, e.g., [6], [7]. The observation of these poor extrapolation capabilities of ILC algorithms has led to a significant research effort to combine superior performance with enhanced extrapolation capabilities towards varying tasks. Indeed, in e.g., [8], [9], [10], ILC is extended with basis functions to enhance the extrapolation capabilities. The main drawback of these approaches is that an approximate model of the system is still required, as is the case in standard ILC algorithms.

In [3], the need for an approximate model of the system is eliminated by exploiting results from iterative feedback tuning [11]. In [12], this approach is further investigated, revealing that it suffers from a closed-loop identification problem [13], [14]. To solve this deficiency, an approach based on instrumental variables is presented in [12] for feedforward controllers with a polynomial basis.

Although important steps have been made in the development of data-driven algorithms for feedforward control with a polynomial basis, these approaches do not straightforwardly generalize to generic feedforward controllers with a rational basis. The benefits of a rational basis are well-known in system identification, see, e.g., [15], since the variance of the transfer function estimate is typically proportional to the number of estimated parameters. This paper aims to develop a new algorithm that results in unbiased parameter estimates with optimal accuracy in terms of variance for feedforward controllers with a rational basis.

The main contribution of this paper is the estimation of the parameters of a feedforward controller with a rational basis with optimal accuracy in terms of variance. Feedforward control with a rational basis can enable high performance and enhanced extrapolation capabilities towards varying tasks for all systems described by a rational model. As a special case, the approach for polynomial, i.e., linearly parametrized, feedforward controllers is recovered, as in [12], [16]. However, such polynomial feedforward is only optimal when the system is given by the reciprocal of a polynomial, while rational feedforward is applicable to a large class of systems.

Notation. A discrete-time linear system is denoted as $y(t) = M(q)u(t)$ with input signal $u(t)$, output signal $y(t)$ and system $M(q)$. The variable q denotes the forward shift operator $qu(t) = u(t+1)$. The impulse response representation is given by

$$y(t) = \sum_{k=-\infty}^{\infty} m_k u(t-k), \quad M(q) = \sum_{k=-\infty}^{\infty} m_k q^{-k},$$

with m_k the impulse response parameters. For a vector x, $||x||_W^2 = x^T W x$. A positive-definite matrix A is denoted as $A \succ 0$, while a positive-semidefinite matrix A is denoted as $A \succeq 0$. Furthermore, $\mathbb{E}(x) = \int_{-\infty}^{\infty} x f(x) dx$ with probability density function $f(x)$, and $\bar{\mathbb{E}}(x) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \mathbb{E}(x)$ with N the number of samples. The correlation function based on N samples is denoted as $\hat{R}_{xy} = \frac{1}{N}\sum_{t=1}^{N} x(t)y^T(t)$, with $x(t), y(t) \in \mathbb{R}^n$.

II. PROBLEM FORMULATION

A. Preliminaries

The considered two degree-of-freedom control configuration is shown in Figure 1. The true unknown system $P(q)$ is assumed to be discrete-time, single-input single-output, and linear time-invariant, with rational representation

$$P(q) = \frac{B_0(q)}{A_0(q)}, \tag{1}$$

where $B_0(q), A_0(q)$ are polynomials in q. The control configuration consists of a given stabilizing feedback controller $C_{fb}(q)$, and a feedforward controller $C_{ff}^j(q)$. The index j

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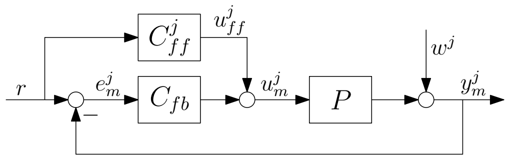

Fig. 1: Two degree-of-freedom control configuration.

denotes the j^{th} task in a sequence of finite time tasks of length N samples, where $j = 1, 2, ..., M$.

Let $r(t)$ denote the reference signal. Furthermore, $w^j(t) = H(q)\varepsilon^j(t)$ denotes an unknown disturbance, where $H(q)$ is monic, and $\{\varepsilon^j(t)\}$ is normally distributed white noise with zero mean and variance λ_ε^2. Here, $w^j(t)$ and $r(t)$ are uncorrelated. The known feedback controller $C_{fb}(q)$ is assumed to be designed such that $S(q)H(q) = 1$, with sensitivity function $S(q) = (1 + P(q)C_{fb}(q))^{-1}$, which corresponds to a typical controller design, including LQG control [17, Section 6.2].

The feedforward signal is given by $u_{ff}^j(t)$, while the measured $e_m^j(t)$, $y_m^j(t)$, and $u_m^j(t)$ in task j are given by

$$\begin{aligned} e_m^j(t) &= e_r^j(t) - e_w^j(t), \quad y_m^j(t) = y_r^j(t) + y_w^j(t), \\ u_m^j(t) &= u_r^j(t) - u_w^j(t), \end{aligned}$$

with

$$\begin{aligned} &e_r^j(t) = S(q)(1 - P(q)C_{ff}^j(q))r(t), \quad && e_w^j = S(q)w^j(t), \\ &y_r^j(t) = S(q)P(q)C(q)r(t), && y_w^j = S(q)w^j(t), \\ &u_r^j(t) = S(q)C(q)r(t), \\ &u_w^j(t) = S(q)C_{fb}(q)w^j(t), \end{aligned}$$

and $C(q) = C_{fb}(q) + C_{ff}^j(q)$. Since w^j is an unknown disturbance and $P(q)$ is assumed to be unknown, e_r^j and e_w^j (resp. y_r^j and y_w^j, u_r^j and u_w^j) can not be determined based on the measured signal e_m^j (resp. y_m^j, u_m^j).

B. Iterative Feedforward Control

The goal in iterative feedforward control is to improve the control performance by learning $u_{ff}^j(t)$ based on data from previous tasks in a batch-to-batch fashion [3]. The measured signals $e_m^j(t)$, $y_m^j(t)$, and $u_m^j(t)$, for $t = 1, ..., N$, in task j are stored in a memory buffer. This batch of measured data is used to determine $C_{ff}^{j+1}(q)$ before starting task $j+1$. Next, a parametrization is defined for $C_{ff}^{j+1}(q)$.

Definition 1. *The feedforward controller $C_{ff}(q, \theta)$ parametrized in terms of a rational basis is given by*

$$\mathcal{C}_{rat} = \left\{ C_{ff}(q,\theta) \middle| C_{ff}(q,\theta) = \frac{A(q,\theta)}{B(q,\theta)}, \quad \theta \in \mathbb{R}^{n_a+n_b} \right\}$$

where

$$A(q,\theta) = \sum_{i=1}^{n_a} \psi_i(q^{-1})\theta_i = \Psi_A(q)\theta_A,$$

$$B(q,\theta) = 1 + \sum_{i=n_a+1}^{n_a+n_b} \psi_i(q^{-1})\theta_i = 1 + \Psi_B(q)\theta_B,$$

with parameters $\theta = [\theta_A^T \ \ \theta_B^T]^T$, and polynomial basis functions $\Psi(q) = [\Psi_A(q) \ \ \Psi_B(q)]$.

Typical approaches in iterative feedforward control, including [18], [3], employ $C_{ff}(q, \theta)$ with a polynomial basis, i.e., $B(q, \theta) = 1$. The parametrization in Def. 1 in terms of a rational basis can significantly enhance both control performance and extrapolation properties towards varying tasks. To illustrate this, consider the reference-induced error $e_r^{j+1}(t, \theta^{j+1})$ in task $j + 1$ given by

$$e_r^{j+1}(t,\theta^{j+1}) = S(q)\left(1 - P(q)C_{ff}^{j+1}(q,\theta^{j+1})\right)r(t), \quad (2)$$

which shows that $e_r^{j+1}(t, \theta^{j+1}) = 0$ for all t if $C_{ff}^{j+1}(q, \theta^{j+1}) = P^{-1}(q)$. Then, the performance i) is optimal since $e_r^{j+1}(t, \theta^{j+1}) = 0$ and ii) is invariant with respect to $r(t)$. However, if $C_{ff}^{j+1}(q, \theta^{j+1})$ is parametrized in terms of a polynomial basis, $e_r^{j+1}(t, \theta^{j+1}) = 0$ for all t is in many cases only possible if $P = 1/A_0(q)$. In contrast, optimal control performance and extrapolation properties are achievable for $C_{ff}^{j+1}(q, \theta^{j+1})$ with a rational basis for any $P(q)$ given by (1). This is the key motivation to introduce a rational basis in iterative feedforward control.

C. An Instrumental Variable Approach to Iterative Feedforward Control with a Rational Basis

In iterative feedforward control, $C_{ff}^{j+1}(q, \theta^{j+1})$ in Def. 1 is determined according to the optimization problem

$$\hat{\theta}^{j+1} = \arg\min_{\theta^{j+1}} V(\theta^{j+1}), \quad (3)$$

where the criterion $V(\theta^{j+1})$ depends on the stored signals $e_m^j(t)$, $y_m^j(t)$ and $u_m^j(t)$ as measured in task j, and the known reference $r(t)$, for $t = 1, ..., N$.

In this paper, iterative feedforward control based on instrumental variables is pursued, as in, e.g., [12] and [19]. The corresponding criterion $V(\theta^{j+1})$ is defined next.

Definition 2. *The criterion $V(\theta^{j+1})$ in iterative feedforward control based on instrumental variables is given by*

$$V(\theta^{j+1}) = \left\| \frac{1}{N}\sum_{t=1}^{N} z(t)L(q)\hat{e}^{j+1}(t,\theta^{j+1}) \right\|_W^2, \quad (4)$$

where $z(t) \in \mathbb{R}^{n_z}$ are instrumental variables that are a function of (derivatives of) $r(t)$, W is a positive-definite weighting matrix, $n_z \geq n_\theta$, $L(q)$ is a prefilter, and the predicted error $\hat{e}^{j+1}(t, \theta^{j+1})$ in task $j + 1$ is given by

$$\hat{e}^{j+1}(t,\theta^{j+1}) = \tilde{e}_m^j(t) - S(q)P(q)C_{ff}^{j+1}(q,\theta^{j+1})r(t), (5)$$

with $\tilde{e}_m^j(t) = e_m^j(t) + S(q)P(q)C_{ff}^j(q, \theta^j)r(t)$.

The goal of this paper is to develop an approach that results in unbiased estimates θ^{j+1} with optimal accuracy for C_{ff}^{j+1} with a rational basis. The following steps are presented:

S1. Show that the optimization problem has in general no analytic solution θ^{j+1} for $C_{ff}(q, \theta)$ as in Def. 1;
S2. Derive expressions for the design variables $z(t)$, $L(q)$ and W such that optimal accuracy is obtained;
S3. Propose an iterative scheme to determine $\hat{\theta}^{j+1}$ with optimal accuracy;
S4. Confirm that the proposed approach results in optimal accuracy in a simulation study.

III. ANALYSIS OF THE OPTIMIZATION PROBLEM

In this section, an analysis is provided for the optimization problem in (3) with criterion $V(\theta^{j+1})$ in (4). It is shown that this optimization problem has in general no analytic solution θ^{j+1} for a rational parametrization of $C_{ff}(q,\theta)$.

Recall from Sect. II that $\hat{\theta}^{j+1}$ is determined based on measured data from task j, without estimating a model of $P(q)$. To achieve this, the predicted error $\hat{e}^{j+1}(t,\theta^{j+1})$ as defined in (5) is expressed as

$$\hat{e}^{j+1}(t,\hat{\theta}^{j+1}) = \frac{1}{B(q,\hat{\theta}^{j+1})}\tilde{e}_m^j(t) - \varphi^T(t,\hat{\theta}^{j+1})\hat{\theta}^{j+1}, \quad (6)$$

where

$$\varphi(t,\hat{\theta}^{j+1}) = \frac{1}{B(q,\hat{\theta}^{j+1})}\begin{bmatrix}\Psi_A(q)C^{-1}(q)y_m^j(t)\\ -\Psi_B(q)\tilde{e}_m^j(t)\end{bmatrix}. \quad (7)$$

The derivation of (6) is omitted due to space restrictions. Any solution $\hat{\theta}^{j+1}$ to the optimization problem posed in (3) has to satisfy the necessary condition for optimality $\partial V(\theta^{j+1})/\partial\theta^{j+1} = 0$, which is for $V(\theta^{j+1})$ in (4) equal to

$$\hat{R}_{z\partial}^T W\left(\frac{1}{N}\sum_{t=1}^{N} z(t)L(q)\hat{e}^{j+1}(t,\hat{\theta}^{j+1})\right) = 0, \quad (8)$$

where $\hat{R}_{z\partial} = \frac{1}{N}\sum_{t=1}^{N} z(t)L(q)\frac{\partial\hat{e}^{j+1}(t,\hat{\theta}^{j+1})}{\partial\hat{\theta}^{j+1}}$, and

$$\frac{\partial\hat{e}^{j+1}(t,\hat{\theta}^{j+1})}{\partial\hat{\theta}^{j+1}} = \frac{1}{B(q,\hat{\theta}^{j+1})}\begin{bmatrix}-\Psi_A(q)C^{-1}(q)y_m^j(t)\\ \Psi_B(q)\frac{A(q,\hat{\theta}^{j+1})}{B(q,\hat{\theta}^{j+1})}C^{-1}(q)y_m^j(t)\end{bmatrix}^T.$$

The derivation of $\frac{\partial\hat{e}^{j+1}(t,\hat{\theta}^{j+1})}{\partial\hat{\theta}^{j+1}}$ is omitted due to space restrictions. By substituting (6) in (8) and rearranging terms, it follows that

$$\hat{R}_{z\partial}^T W\left(\hat{R}_{z\tilde{e}} - \hat{R}_{z\varphi}\hat{\theta}^{j+1}\right) = 0, \quad (9)$$

with

$$\begin{aligned}\hat{R}_{z\tilde{e}} &= \frac{1}{N}\sum_{t=1}^{N} z(t)L(q)\frac{1}{B(q,\hat{\theta}^{j+1})}\tilde{e}_m^j(t),\\ \hat{R}_{z\varphi} &= \frac{1}{N}\sum_{t=1}^{N} z(t)L(q)\varphi^T(t,\hat{\theta}^{j+1}),\end{aligned} \quad (10)$$

and $\varphi(t,\hat{\theta}^{j+1})$ as defined in (7). The optimality condition (9) has in general no analytic solution since $\hat{R}_{z\partial}$, $\hat{R}_{z\tilde{e}}$, and $\hat{R}_{z\varphi}$ all depend on $\hat{\theta}^{j+1}$. For the general case with a rational parameterization for $C_{ff}^{j+1}(q,\theta^{j+1})$, an iterative procedure can be used to determine $\hat{\theta}^{j+1}$. Before developing such a procedure in Sect. V, the accuracy properties of the estimate $\hat{\theta}^{j+1}$ are analyzed in Sect. IV.

IV. OPTIMAL ACCURACY FOR ITERATIVE FEEDFORWARD WITH A RATIONAL BASIS

In the previous section, an optimization problem is described for iterative feedforward control with a rational basis. Suppose that $\hat{\theta}^{j+1}$ can be determined based on (9) by means of an iterative procedure. Then, it turns out that the statistical properties of $\hat{\theta}^{j+1}$ depend on $z(t)$, $L(q)$ and W as defined in Def. 2, as is well-known in IV-based identification approaches, see, e.g., [20]. This raises the following question: How to select $z(t)$, $L(q)$ and W such that unbiased estimates $\hat{\theta}^{j+1}$ with optimal accuracy are obtained for feedforward control with a rational basis? This question will be investigated in this section.

A. Accuracy for General Instruments

The asymptotic covariance matrix P_{eIV} is typically used as a performance measure for the optimality of $\hat{\theta}^{j+1}$ in terms of accuracy, see, e.g., [21], [20]. Consider the asymptotic distribution of $\hat{\theta}^{j+1}$ in (9) given by

$$\sqrt{N}\left(\hat{\theta}^{j+1} - \theta_0\right) \xrightarrow{\text{dist}} \mathcal{N}(0, P_{eIV}),$$

where θ_0 is the asymptotic parameter estimate, defined as the parameters such that $e_r^{j+1}(t,\theta_0) = 0$ for all t. Based on (2), this implies that

$$e_r^{j+1}(t,\theta_0) = S(q)\left(1 - P(q)C_{ff}^{j+1}(q,\theta_0)\right)r(t) = 0,$$

i.e., θ_0 are the parameters such that $C_{ff}^{j+1}(q,\theta_0) = P^{-1}(q)$. This corresponds to the optimal performance achievable for feedforward control in the considered control configuration. Under mild technical conditions as in, e.g., [21, Chapter 7], it can be shown that the asymptotic normal distribution of $\hat{\theta}^{j+1}$ exists and the covariance matrix P_{eIV} is given by

$$P_{eIV} = \left[R_{z\partial}^T W R_{z\varphi}\right]^{-1} R_{z\partial}^T W J W^T R_{z\partial}\left[R_{z\partial}^T W R_{z\varphi}\right]^{-T}, \quad (11)$$

with $R_{z\partial} = \bar{\mathbb{E}}z(t)L(q)\frac{\partial\hat{e}^{j+1}(t,\theta_0)}{\partial\theta_0}$, and

$$R_{z\varphi} = \bar{\mathbb{E}}z(t)L(q)\varphi^T(t,\theta_0),$$

$$J = \lambda_\varepsilon^2\bar{\mathbb{E}}\left[L(q)\frac{S(q)H(q)}{B(q,\theta_0)}z(t)\right]\left[L(q)\frac{S(q)H(q)}{B(q,\theta_0)}z(t)\right]^T,$$

with $\varphi(t,\theta_0)$ as in (7) for θ_0. Note that P_{eIV} in (11) holds for any $z(t)$, $L(q)$, and W. Next, a lower bound is derived for P_{eIV} as a function of $z(t)$, $L(q)$, and W. By reaching this lower bound on P_{eIV}, optimal accuracy is obtained for iterative feedforward control based on instrumental variables.

B. Lower Bound for the Covariance Matrix P_{eIV}

Optimal accuracy is obtained if $z(t)$, $L(q)$ and W are determined such that P_{eIV} in (11) is as small as possible. Consider the following lower bound for P_{eIV}, i.e., $P_{eIV} \succeq P_{eIV}^{opt}$, with optimal covariance matrix P_{eIV}^{opt} given by

$$P_{eIV}^{opt} = \lambda_\varepsilon^2\left[\bar{\mathbb{E}}\left[\frac{B(q,\theta_0)}{S(q)H(q)}\varphi_r(t,\theta_0)\right]\left[\frac{B(q,\theta_0)}{S(q)H(q)}\varphi_r(t,\theta_0)\right]^T\right]^{-1}$$

with $\varphi_r(t,\theta_0)$ the reference-induced contribution of $\varphi(t,\theta_0)$ in (7), given by

$$\varphi_r(t,\theta_0) = \frac{1}{B(q,\theta_0)}\begin{bmatrix}\Psi_A(q)C^{-1}(q)y_r^j(t)\\ -\Psi_B(q)\tilde{e}_r^j(t)\end{bmatrix},$$

and $\tilde{e}_r^j(t) = S(q)r(t)$. The optimal covariance matrix P_{eIV}^{opt} is derived as in [21, Section 8.2] for open-loop identification and [20] for closed-loop identification.

Equivalence between P_{eIV} and P_{eIV}^{opt} is obtained if $z(t)$, $L(q)$ and W are designed as:

- $z_{\text{e,opt}}(t) = \frac{B(q,\theta_0)}{S(q)H(q)}\varphi_r(t,\theta_0)$;
- $L_{\text{e,opt}}(q) = \frac{1}{S(q)H(q)}$;
- $W_{\text{e,opt}} = I$ and $n_z = n_\theta$.

This result follows by substituting $z_{\text{e,opt}}(t)$, $L_{\text{e,opt}}(q)$ and $W_{\text{e,opt}}$ in (11). Recall from Sect. II-A that $C_{fb}(q)$ is assumed to be designed such that $S(q)H(q) = 1$. Based on this assumption, the optimal prefilter simplifies to $L_{\text{opt}}(q) = 1$. Furthermore, the optimal instruments $z_{\text{opt}}(t)$ become

$$z_{\text{opt}}(t) = C^{-1}(q)\begin{bmatrix} \Psi_A(q)y_r^j(t) \\ -\Psi_B(q)u_r^j(t) \end{bmatrix}. \tag{12}$$

A derivation of (12) is omitted due to space restrictions.

C. Concluding Remarks

In this section, it is shown that unbiased estimates $\hat{\theta}^{j+1}$ with optimal accuracy are obtained by selecting $z(t)$ as in (12), $L(q) = 1$, and $W = I$. Recall that the statistical properties of $\hat{\theta}^{j+1}$ are analyzed under the assumption that $\hat{\theta}^{j+1}$ can be determined from the optimization problem in (3). The computation of $\hat{\theta}^{j+1}$ is a non-trivial problem for iterative feedforward control with a rational basis, see Sect. III. Therefore, as foreshadowed in Sect. III, an iterative procedure is proposed in Sect. V to determine $\hat{\theta}^{j+1}$.

V. TOWARDS OPTIMAL ACCURACY FOR RATIONAL FEEDFORWARD

In this section, an iterative scheme is proposed for the computation of $\hat{\theta}^{j+1}$ for rational feedforward that attains optimal accuracy in terms of variance. The proposed iterative scheme is closely related to pseudo-linear regression and bootstrap methods in system identification, see, e.g., [22].

By using $z_{\text{opt}}(t)$ as given in (12), $L_{\text{opt}}(q) = 1$ and $W_{\text{e,opt}} = I$, the optimality condition (9) becomes

$$\hat{R}_{z\tilde{e}}^{\text{opt}} - \hat{R}_{z\varphi}^{\text{opt}}\hat{\theta}^{j+1} = 0, \tag{13}$$

where $\hat{R}_{z\tilde{e}}^{\text{opt}}$ and $\hat{R}_{z\tilde{e}}^{\text{opt}}$ follow from substituting $z_{\text{opt}}(t)$ and $L_{\text{opt}}(q)$ in (10) as

$$\begin{aligned} \hat{R}_{z\tilde{e}}^{\text{opt}} &= \frac{1}{N}\sum_{t=1}^{N} z_{\text{opt}}(t)\frac{1}{B(q,\hat{\theta}^{j+1})}\tilde{e}_m^j(t), \\ \hat{R}_{z\varphi}^{\text{opt}} &= \frac{1}{N}\sum_{t=1}^{N} z_{\text{opt}}(t)\varphi^T(t,\hat{\theta}^{j+1}), \end{aligned} \tag{14}$$

with $\varphi(t,\hat{\theta}^{j+1})$ in (7) and $\tilde{e}_m^j(t) = e_m^j(t) + S(q)P(q)C_{ff}^j(q,\theta^j)r(t)$. Similar to (9), the optimality condition (13) has no analytic solution since $\hat{R}_{z\tilde{e}}^{\text{opt}}, \hat{R}_{z\tilde{e}}^{\text{opt}}$ in (14) depends on $\hat{\theta}^{j+1}$ through $\frac{1}{B(q,\hat{\theta}^{j+1})}$.

The use of the optimal instruments $z_{\text{opt}}(t)$ in (13) introduces additional complexity in the computation of $\hat{\theta}^{j+1}$. To see this, recall from Sect. II-A that $y_r^j(t)$ and $u_r^j(t)$ can not be determined based on the measured $e_m^j(t)$, $y_m^j(t)$, $u_m^j(t)$ and known $r(t)$ without estimating a model of $P(q)$. Hence, $z_{\text{opt}}(t)$ in (9) needs to be approximated in the pursued data-driven approach. Next, an iterative scheme is presented that exploits approximate implementations of $\varphi(t,\hat{\theta}^{j+1})$ and $z_{\text{opt}}(t)$ to determine $\hat{\theta}^{j+1}$ with optimal accuracy.

A. Approximate Implementation of $z_{opt}(t)$ and $\varphi(t,\hat{\theta}^{j+1})$

The key idea behind the proposed iterative scheme is to solve (13) by means of a sequence of convex optimization problems. To this end, introduce an auxiliary index i, i.e., $\hat{\theta}_{<i>}^{j+1}$ and $\hat{\theta}_{<i-1>}^{j+1}$, and define the approximate implementation of $\varphi(t,\hat{\theta}^{j+1})$, denoted as $\varphi_{<i>}(t,\hat{\theta}_{<i-1>}^{j+1})$, as

$$\varphi_{<i>}(t) = \frac{1}{B(q,\hat{\theta}_{<i-1>}^{j+1})}\begin{bmatrix} \Psi_A(q)C^{-1}(q)y_m^j(t) \\ -\Psi_B(q)\tilde{e}_m^j(t) \end{bmatrix}^T, \tag{15}$$

while an approximate implementation of the optimal instruments $z_{\text{opt}}(t)$, denoted as $z_{\text{p},<i>}(t,\hat{\theta}_{<i-1>}^{j+1})$, yields

$$z_{\text{p},<i>}(t) = C_{<i>}^{-1}(q)\begin{bmatrix} \Psi_A(q)r(t) \\ -\Psi_B(q)C_{ff,<i>}^{j+1}(q,\hat{\theta}_{<i-1>}^{j+1})r(t) \end{bmatrix}, \tag{16}$$

where $C_{<i>}^{-1}(q) = (C_{fb}(q) + C_{ff,<i>}^{j+1}(q,\hat{\theta}_{<i-1>}^{j+1}))^{-1}$. By replacing $\varphi(t,\hat{\theta}^{j+1})$ and $z_{\text{opt}}(t)$ in the optimality condition (14) by (15) and (16), respectively, it follows that $\hat{\theta}_{<i>}^{j+1}$ is the solution to

$$\hat{\theta}_{<i>}^{j+1} = (\hat{R}_{z\varphi,<i>}^{\text{opt}})^{-1}\hat{R}_{z\tilde{e},<i>}^{\text{opt}}, \tag{17}$$

in iteration i of the iterative scheme, with

$$\begin{aligned} \hat{R}_{z\tilde{e},<i>}^{\text{opt}} &= \frac{1}{N}\sum_{t=1}^{N} z_{\text{p},<i>}(t,\hat{\theta}_{<i-1>}^{j+1})\frac{1}{B(q,\hat{\theta}_{<i-1>}^{j+1})}\tilde{e}_m^j(t), \\ \hat{R}_{z\varphi,<i>}^{\text{opt}} &= \frac{1}{N}\sum_{t=1}^{N} z_{\text{p},<i>}(t,\hat{\theta}_{<i-1>}^{j+1})\varphi_{<i>}^T(t,\hat{\theta}_{<i-1>}^{j+1}). \end{aligned}$$

Since $\theta_{<i-1>}^{j+1}$ is known in iteration i of the iterative scheme, (17) has an analytic solution $\hat{\theta}_{<i>}^{j+1}$. Upon convergence of the iterative scheme, $\varphi_{<i>}(t,\hat{\theta}_{<i-1>}^{j+1})$ and $z_{\text{p},<i>}(t,\hat{\theta}_{<i-1>}^{j+1})$ approximate (7) and (12), respectively. An analysis of this result is beyond the scope of this paper.

B. Proposed Algorithm

The proposed algorithm to determine $\hat{\theta}^{j+1}$ is presented in Algorithm 1.

Algorithm 1. Determine $\hat{\theta}^{j+1}$ with optimal accuracy

a) Initialize $\hat{\theta}_{<i-1>}^{j+1} = \theta^j$.
b) Construct $C_{ff,<i>}^{j+1}(q,\hat{\theta}_{<i-1>}^{j+1}) = \dfrac{A(q,\hat{\theta}_{A,<i-1>}^{j+1})}{B(q,\hat{\theta}_{B,<i-1>}^{j+1})}$.
c) Construct $\varphi_{<i>}(t,\hat{\theta}_{<i-1>}^{j+1})$ as in (15) and instruments $z_{\text{p},<i>}(t,\hat{\theta}_{<i-1>}^{j+1})$ as in (16).
d) Solve $\hat{\theta}_{<i>}^{j+1}$ according to (17).
e) Set $i \to i+1$ and repeat from Step b) until a stopping criterion is met.

TABLE I: Mean and standard deviation of $\bar{\theta}$ confirm that the standard deviation for the proposed instruments with iterations $z_{p,<3>}(t,\theta^1_{<2>})$ is almost equal to the standard deviation obtained with the optimal instruments $z_{\text{opt}}(t)$, while for the proposed instruments without iterations $z_{p,<1>}(t,\theta^1_{<0>})$ the standard deviation is typically a factor 10^2 larger than for $z_{p,<3>}(t,\theta^1_{<2>})$.

	True Value	Instruments $z_{p,<1>}(t,\theta^1_{<0>})$	Instruments $z_{p,<3>}(t,\theta^1_{<2>})$	Instruments $z_{\text{opt}}(t)$
$\bar{\theta}_1$	6.0×10^{-2}	$6.0\times10^{-2}\pm3.9\times10^{-6}$	$6.0\times10^{-2}\pm2.9\times10^{-8}$	$6.0\times10^{-2}\pm2.8\times10^{-8}$
$\bar{\theta}_2$	1.1×10^{-5}	$1.1\times10^{-5}\pm1.3\times10^{-6}$	$1.1\times10^{-5}\pm1.6\times10^{-8}$	$1.1\times10^{-5}\pm1.5\times10^{-8}$
$\bar{\theta}_3$	3.1×10^{-7}	$3.1\times10^{-7}\pm5.7\times10^{-9}$	$3.1\times10^{-7}\pm3.4\times10^{-11}$	$3.1\times10^{-7}\pm3.3\times10^{-11}$
$\bar{\theta}_4$	1.8×10^{-4}	$1.8\times10^{-4}\pm2.0\times10^{-5}$	$1.8\times10^{-4}\pm2.5\times10^{-7}$	$1.8\times10^{-4}\pm2.3\times10^{-7}$
$\bar{\theta}_5$	2.0×10^{-5}	$2.0\times10^{-5}\pm7.2\times10^{-8}$	$2.0\times10^{-5}\pm3.5\times10^{-10}$	$2.0\times10^{-5}\pm3.4\times10^{-10}$

Even though global convergence of the algorithm is not guaranteed, practical use has showed that convergence is generally good. These observations are in accordance with the convergence properties of similar algorithms in system identification [22]. An approach to deal with possible instability of $C^{-1}(q)$ is given in [12, Appendix A].

VI. SIMULATION EXAMPLE

The system $P(q)$ and feedback controller $C_{fb}(q)$ are given by

$$P(q) = \frac{1.032\times10^{-5}\left(1-1.981q^{-1}+0.9888q^{-2}\right)}{\left(1-q^{-1}\right)\left(1-1.927q^{-1}+0.9565q^{-2}\right)},$$
$$C_{fb}(q) = \frac{305.8q^{-1}-604.4q^{-2}+299.7q^{-3}}{1-2.721q^{-1}+2.461q^{-2}-0.7396q^{-3}}.$$

The disturbance $w^j(t)$ is given by $w^j(t) = H(q)\varepsilon^j(t)$, where $\left\{\varepsilon^j(t)\right\}$ is normally distributed white noise with zero mean and standard deviation $\lambda_\varepsilon = 5.0\times10^{-8}$. Furthermore, $H(q)$ is determined such that $S(q)H(q) = 1$. The closed-loop system is excited by a fourth-order reference trajectory $r(t)$. The feedforward controller $C_{ff}(q,\theta)$ is parametrized as in Definition 1, with basis functions

$$\Psi_A(q) = \left[\begin{array}{ccc}\psi_1 & \psi_2 & \psi_3\end{array}\right],\ \Psi_B(q) = \left[\begin{array}{cc}\psi_4 & \psi_5\end{array}\right],$$

with

$$\psi_1 = \psi_5 = \left(\frac{1-q^{-1}}{T_s}\right)^2,\quad \psi_2 = \left(\frac{1-q^{-1}}{T_s}\right)^3,$$
$$\psi_3 = \left(\frac{1-q^{-1}}{T_s}\right)^4,\quad \psi_4 = \left(\frac{1-q^{-1}}{T_s}\right),$$

parameters $\theta = [\theta_A^T,\ \theta_B^T]^T$, and $T_s = 4.0\times10^{-4}$ s.

Since $P(q)$ is known in this simulation example, it is possible to i) determine the true parameters θ_0 such that $e_r^{j+1})(t) = 0$ for all $t = 1,...,N$, and ii) construct the optimal instruments $z_{\text{opt}}(t)$ as defined in (12). The true parameters θ_0 are given in Table I. The parameter estimates obtained with $z_{\text{opt}}(t)$ are used as a benchmark for the proposed approach in Algorithm 1 based on $z_{p,<i>}(t,\theta^{j+1}_{<i-1>})$. The number of computational iterations for the proposed approach is given by $K = 3$, with corresponding instruments denoted as $z_{p,<3>}(t,\theta^{j+1}_{<2>})$. To demonstrate the advantages of refining $z_{\text{p},<i>}(t,\theta^{j+1}_{<i-1>})$ and $\varphi_{<i>}(t,\theta^{j+1}_{<i-1>})$, additional simulations are performed with instruments $z_{p,<1>}(t,\theta^{j+1}_{<0>})$ where $K = 1$.

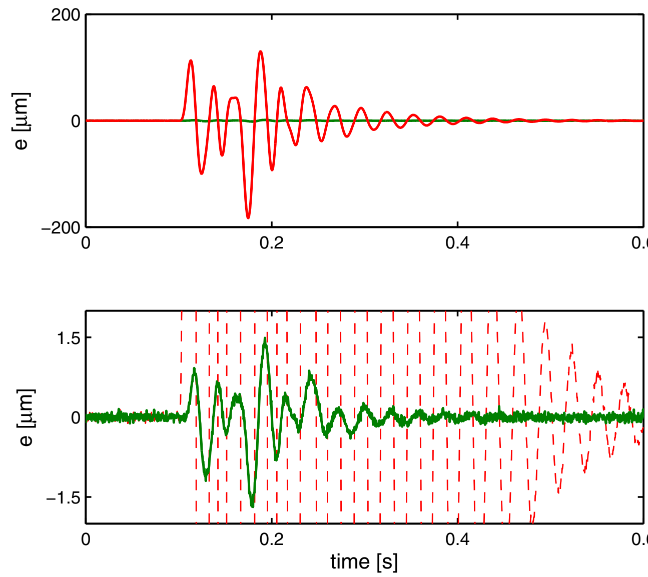

Fig. 2: Simulation results: The worst-case error signal $e^1_{\text{wc}}(t)$ shows that the control performance with instruments $z_{p,<1>}(t,\theta^1_{<0>})$ (red) is significantly worse compared to $z_{p,<3>}(t,\theta^1_{<2>})$ (green). This confirms that the iterative algorithm to refine $z_{\text{p},<i>}(t,\theta^{j+1}_{<i-1>})$ and $\varphi_{<i>}(t,\theta^{j+1}_{<i-1>})$ can significantly improve performance.

For all considered approaches, a Monte Carlo simulation is performed with $m = 200$ realizations. The number of tasks in each realization is equal to $M = 1$, and the initial parameters in each realization are set to $\theta^0 = [5.4\times10^{-2}\,0\,0\,0\,0]^T$. Furthermore, the sample mean in task j is defined as

$$\bar{\theta}^j = \frac{1}{m}\sum_{l=1}^{m}\hat{\theta}^j_l,$$

where $\hat{\theta}^j_l$ are the parameter estimates in the l^{th} realization. The worst-case error over all realizations is denoted $e^1_{\text{wc}}(t,\theta^1_{\text{wc}})$, where θ^1_{wc} is defined as the parameter vector such that $V(\theta^1_{\text{wc}}) \geq V(\theta^1_l)$ for $l = 1,...,m$.

The results of the Monte Carlo simulation study are presented in Table I. The following observations are made:

i) Unbiased estimates of θ_0, i.e. $\bar{\theta} = \theta_0$, are obtained by all approaches. This shows that the considered IV-based approaches lead to unbiased estimates.

ii) The standard deviation of $\bar{\theta}$ is significantly smaller for the optimal IV-based iterative procedure with $z_{p,<3>}(t, \theta^1_{<2>})$ when compared to $z_{p,<1>}(t, \theta^1_{<0>})$. This confirms that the iterative algorithm to refine $z_{\mathrm{p},<i>}(t, \theta^{j+1}_{<i-1>})$ and $\varphi_{<i>}(t, \theta^{j+1}_{<i-1>})$ enhances the accuracy of the parameters in terms of variance.

iii) The standard deviation of $\bar{\theta}$ for the iterative design procedure with $z_{p,<3>}(t, \theta^1_{<2>})$ closely approximates that of the iterative procedure with optimal instruments $z_{\mathrm{opt}}(t)$. This confirms that the iterative design procedure achieves optimal accuracy of the parameters.

Finally, the connection between accuracy of the estimated parameters and control performance is analyzed for the considered simulation example. Figure 2 illustrates that the worst-case error signal $e^1_{\mathrm{wc}}(t, \theta^1_{\mathrm{wc}})$ with instruments $z_{p,<1>}(t, \theta^1_{<0>})$ (top) is significantly deteriorated compared to $z_{p,<3>}(t, \theta^1_{<2>})$ (bottom). This shows that Algorithm 1 can significantly improve the control performance by refining $z_{\mathrm{p},<i>}(t, \theta^{j+1}_{<i-1>})$ and $\varphi_{<i>}(t, \theta^{j+1}_{<i-1>})$.

VII. CONCLUSIONS

In this paper, a new approach is developed that extends iterative feedforward control approaches to feedforward controllers with a rational parametrization. This approach enables high control performance and enhances extrapolation capabilities towards varying tasks for the general class of systems described by a rational model. As a special case, polynomial feedforward is recovered, as in [12], [16]. Unbiased estimates with optimal accuracy are obtained by using an instrumental variable framework. An explicit expression for the asymptotic covariance matrix is derived and subsequently used to determine instruments that result in optimal accuracy. Simulation results confirm that the proposed algorithm leads to optimal accuracy and improved performance. Ongoing research focuses on extensions to optimal input design [23], inferential control [24], and experimental verification on an industrial motion system.

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