Development of User-Friendly System Identification Techniques

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DEVELOPMENT OF USER-FRIENDLY SYSTEM IDENTIFICATION TECHNIQUES

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Acknowledgments

We're all going to die, all of us, what a circus! That alone should make us love each other but it doesn't. We are terrorized and flattened by trivialities, we are eaten up by nothing.

Charles Bukowski
‘The Captain is Out to Lunch and the Sailors have taken over the Ship’

While a PhD only has a single author, it is by no means a solitary experience. Research without collaboration is unlikely to yield new scientific and technical insights since personal interactions allow new ideas and applications to germinate.

In the first place, I would like to thank my colleagues at the ELEC department for being an outgoing group that has been combining a fond interest for engineering, science, and leisure. In particular, I am grateful for the great atmosphere in the office and for my office mates over the last few years: Adam, Anna, David, Diane, Ebrahim, Erlyiang, Francesco, Gustavo, Hannes, Maarten, Mark, Matthias, and Maulik (though not all at the same time). I have fond memories of the time Alexandra and I spend together with Anna and Maarten on our road trip in South Africa and going on a safari to see the local wildlife and smell the wild lavender. I have also enjoyed the no-frills company of Adam, his overall good taste as an engineer and dry sense of humor (‘nobody expects the Cauchy distribution’ was a sparkle of laughter after barren days). I really appreciate Matthias’s deep knowledge and his frankness, especially when the both of us were having rougher times. Also, the interaction I had with Cedric, often concerning good microcontroller code, have always enlightened my day. Apart from the actual research, I had a very good time on social activities with Péter who guided us through Vienna, David who was one of the main organizers of many social activities, and many of the other people at ELEC.

Next to research, quite a considerable amount of my time was spent on teaching assignments. In particular, the many hours we toiled in the ‘mines of Moria’ to aid bachelor students to control pingpong balls went like a breeze thanks to the great company of able-minded colleagues such as Matthias, Evi, Dave, Hans, Sven and Jens. The great technical and administrative support of Gerd, Johan and Luc have also certainly helped in creating a productive setting for those labs. While exhausting, I have immensely enjoyed and benefited from those labs that force you to revise basic concepts that I have come to take for granted over the years.

For the circuits and filters course, I have been really lucky to have had a great predecessor in John whom was always available to help out during the first year when I took over the course project together with Laurent in the first year of my PhD. I have really enjoyed this collaboration since both of us brought
very complementary skills to the project. I am also very grateful for Alexander and Dries for being great colleagues and taking over a larger chunk of this workload when I started writing my dissertation.

During the few summers that I spent at ELEC, I have been fortunate enough to have collaborated with Mikaya and John, on what has become a considerable part of this dissertation. I really admire Mikaya’s joie de vivre, which really made his visits to Brussels a time to look forward to each year. Also, I really love the well-balanced way in which John oversaw this work; I am convinced that in academia, more people like John are needed.

In general, I would like to thank the professors of ELEC: Rik, Yves, Gerd, Ivan, and Leo. Over the last years, I have come to learn Rik as one of the most intelligent, correct and friendly people of ELEC. Gerd is a scientific jack-of-all trades with whom I share a strong love for beer, barbecues and finding practical solutions in the lab. And Yves is one of the people without whom I would not have even started working towards a PhD. I have dearly enjoyed our discussions on software design and philosophy, your very critical—yet constructive—view on some of my papers, and your fascination in the color of my beverages during lunch. Too often, we tend to forget that people in supporting roles are invisible most of the time. So I would like to take this space to also thank our secretaries (Ann, Bea and Annick), our technician Johan and system administrator Sven for running a tight ship most of the time. I also would like to thank Leo, for his political courage and taking on responsibilities when the times call for it. I genuinely appreciate his efforts to provide a stable environment in the department such that research can be done.

During my first real international conference, I was happy to meet with István Kollár, who has unfortunately passed away recently. I have really enjoyed our scientific discussions, grabbing some beers, and the view he offered into the foregone era of the Iron Curtain.

During these last few years, I have been working under the close supervision of my promotor Johan. Although our personalities differ wildly, I truly appreciate Johan for his often refreshing views on the technical content, for his broad overview of engineering and the interesting discussion we have had over the years. I am really grateful, since many of the encounters I had over these last few years, were either enabled and/or strongly encouraged by him.

In the second year of my PhD, I have been a visiting researcher in the Control Systems Theory group of prof. Maarten Steinbuch. I remember my first encounter with Maarten as a friendly, but rather critical, meeting which took me aback at first. I have great admiration for Maarten’s very enjoyable mix of scientific accuracy, critical business acumen and lightheartedness. During these months at the CST group, I much enjoyed the company of Matthijs, Frank, Annemiek and Rick. I was very happy to have had the chance to interact with Rick who had an excellent understanding of the set-up that I was working on and who was always available to discuss or joke around. Also, I took great pleasure in interacting with Matthijs: during his time at ELEC, my time at CST, and all of the times we hung out as colleagues or as friends. Overall, I would like to thank all the people (PhD students, professors, secretaries, …) at CST for their open and warm atmosphere.

However, my main scientific interactions in Eindhoven have been with Tom, who eventually became my second promotor. I have always enjoyed our discussions and the broad and deep scientific knowledge he masters and yet is able to explain with clarity. Thanks to his very open attitude, I was able to learn a whole lot in quite a short time.
I would also like to express my sincere gratitude towards my jury member for their flexibility and great effort in aiding me to ameliorate this dissertation, and my understanding of connections of my work to other fields of science and engineering. In retrospect, I have absolutely enjoyed discussing with them during my private defense and the few hours afterwards. In particular, I have been astounded by the huge amount of work that Koen put into being a great secretary. Also, I appreciate Tomas for taking the time to go through the draft version of this dissertation with me in painstaking detail in the hours after my private defense.

This book also would not have been possible without my fellow members of the matlab2tikz team: Nico, Oleg, Peter, Michael and a handful of incidental other helpers. Most of the graphs in this booklet have passed through the matlab2tikz codebase. I have been really enjoying our collaboration, getting-things-done attitude, and discussions regarding good practices for making figures and coding in MATLAB®, and the future directions of the project.

During my time as a bachelor and master student, I have always enjoyed working together with Jeroen. During both our PhD's at different locations, I have greatly valued how we kept in touch and tried to boost each other's spirits during the darker days of our academic activities and sometimes personal struggles.

At home, I would like to thank my family: my mom and dad for providing a stable yet warm environment, even though not everything has been clear sailing. I really appreciate how the both of you have always found a way to make the best out of it and how I could always count on you. Also my sister Lara and her husband Youri, for being nice people on whom you can always count. Although we are completely different in our professions and thinking, I place great value in getting along very well, despite those differences. I would also like to thank my grandparents (Jo and Ria) for on the one hand bringing me into contact with mathematics from a young age, and for their supportiveness. Also, I really appreciate the great lengths my other grandmother goes through to physically unite the greater Geerardyn family on numerous occasions.

Next to my "original" family, I am very grateful to my family 'in-law' (despite not being married): Linda and Dirk and their kin. As with my own parents, I was able to count on the both of you when times got tough (or when barbecues were involved).

The last person I would like to thank is my girlfriend Alexandra. Without her encouragement, I am convinced I would never have finished my PhD. I am grateful that you pushed me to not quit and that you did everything possible to bring this chapter in our lives to a good end, even though you yourself have not been void of concerns about your PhD.

And last, but not least, I would like to thank our cat Pixie for being an enjoyably crazy critter and for his sounding enthusiasm when bringing home little presents and leaving them either on the floor or on my pillow while I'm fast asleep. His intent has been duly noted and appreciated (the nightly hours and contents of those presents, however, were not always equally well-received).
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<th>Description</th>
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<tbody>
<tr>
<td>ADC</td>
<td>analog-to-digital converter.</td>
</tr>
<tr>
<td>AIC</td>
<td>Akaike's information criterion.</td>
</tr>
<tr>
<td>AICc</td>
<td>Akaike's information criterion corrected for small sample sizes.</td>
</tr>
<tr>
<td>AVIS</td>
<td>active vibration isolation system.</td>
</tr>
<tr>
<td>B TLS</td>
<td>bootstrapped total least squares.</td>
</tr>
<tr>
<td>CDF</td>
<td>cumulative density function.</td>
</tr>
<tr>
<td>CPU</td>
<td>central processing unit.</td>
</tr>
<tr>
<td>CRLB</td>
<td>Cramér-Rao lower bound.</td>
</tr>
<tr>
<td>CZT</td>
<td>chirp Z-transform.</td>
</tr>
<tr>
<td>DAC</td>
<td>digital-to-analog converter.</td>
</tr>
<tr>
<td>DAQ</td>
<td>data acquisition.</td>
</tr>
<tr>
<td>DC</td>
<td>static.</td>
</tr>
<tr>
<td>DC kernel</td>
<td>diagonal correlated kernel.</td>
</tr>
<tr>
<td>DFT</td>
<td>discrete Fourier transform.</td>
</tr>
<tr>
<td>DSA</td>
<td>dynamic signal analyzer.</td>
</tr>
<tr>
<td>DTFT</td>
<td>discrete-time Fourier transform.</td>
</tr>
<tr>
<td>ERNSI</td>
<td>European Research Network on System Identification.</td>
</tr>
<tr>
<td>ETFE</td>
<td>empirical transfer function estimate.</td>
</tr>
<tr>
<td>FIR</td>
<td>finite impulse response.</td>
</tr>
<tr>
<td>FRF</td>
<td>frequency response function.</td>
</tr>
<tr>
<td>GTLS</td>
<td>generalized total least squares.</td>
</tr>
<tr>
<td>IDFT</td>
<td>inverse discrete Fourier transform.</td>
</tr>
<tr>
<td>IEEE</td>
<td>Institute of Electrical and Electronics Engineers.</td>
</tr>
<tr>
<td>IFAC</td>
<td>International Federation of Automatic Control.</td>
</tr>
<tr>
<td>IG</td>
<td>inter-grid.</td>
</tr>
<tr>
<td>i.i.d.</td>
<td>independent and identically distributed.</td>
</tr>
<tr>
<td>IQR</td>
<td>inter-quartile range.</td>
</tr>
<tr>
<td>IRF</td>
<td>impulse response function.</td>
</tr>
<tr>
<td>LML</td>
<td>log marginal likelihood.</td>
</tr>
<tr>
<td>LOOCV</td>
<td>leave-one-out cross-validation.</td>
</tr>
<tr>
<td>Notation</td>
<td>Description</td>
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<tr>
<td>----------</td>
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</tr>
<tr>
<td>LOWESS</td>
<td>locally weighted scatterplot smoothing.</td>
</tr>
<tr>
<td>LPM</td>
<td>Local Polynomial Method.</td>
</tr>
<tr>
<td>LPV</td>
<td>linear parameter-varying.</td>
</tr>
<tr>
<td>LRIC</td>
<td>Local Rational method with Iterative Cost function.</td>
</tr>
<tr>
<td>LRM</td>
<td>Local Rational Method.</td>
</tr>
<tr>
<td>LTI</td>
<td>linear time-invariant.</td>
</tr>
<tr>
<td>MIMO</td>
<td>multiple-input multiple-output.</td>
</tr>
<tr>
<td>ML</td>
<td>maximum-likelihood.</td>
</tr>
<tr>
<td>MLE</td>
<td>maximum-likelihood estimator.</td>
</tr>
<tr>
<td>MSE</td>
<td>mean squared error.</td>
</tr>
<tr>
<td>NI</td>
<td>National Instruments.</td>
</tr>
<tr>
<td>OE</td>
<td>output-error.</td>
</tr>
<tr>
<td>PI</td>
<td>proportional-integral.</td>
</tr>
<tr>
<td>PRESS</td>
<td>prediction residual error sum of squares.</td>
</tr>
<tr>
<td>PSD</td>
<td>power spectral density.</td>
</tr>
<tr>
<td>RAM</td>
<td>random-access memory.</td>
</tr>
<tr>
<td>RARX</td>
<td>regularized auto-regressive exogenous.</td>
</tr>
<tr>
<td>RCF</td>
<td>right co-prime factorization.</td>
</tr>
<tr>
<td>RFIR</td>
<td>regularized finite impulse response.</td>
</tr>
<tr>
<td>RMS</td>
<td>root-mean-square.</td>
</tr>
<tr>
<td>RMSE</td>
<td>root mean squared error.</td>
</tr>
<tr>
<td>SA</td>
<td>spectral analysis.</td>
</tr>
<tr>
<td>SISO</td>
<td>single-input single-output.</td>
</tr>
<tr>
<td>SNR</td>
<td>signal to noise ratio.</td>
</tr>
<tr>
<td>SPICE</td>
<td>Simulation Program with Integrated Circuit Emphasis.</td>
</tr>
<tr>
<td>STFT</td>
<td>short-time Fourier transform.</td>
</tr>
<tr>
<td>TF</td>
<td>transfer function.</td>
</tr>
<tr>
<td>TRIMM</td>
<td>Transient Impulse Modeling.</td>
</tr>
<tr>
<td>VXI</td>
<td>VME extensions for instrumentation.</td>
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# List of Symbols

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
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<tbody>
<tr>
<td>( \mathbb{N} )</td>
<td>Set of the natural numbers.</td>
</tr>
<tr>
<td>( \mathbb{Z} )</td>
<td>Set of integer numbers.</td>
</tr>
<tr>
<td>( \mathbb{Q} )</td>
<td>Set of the rational numbers.</td>
</tr>
<tr>
<td>( \mathbb{R} )</td>
<td>Set of the real numbers.</td>
</tr>
<tr>
<td>( \mathbb{C} )</td>
<td>Set of the complex numbers.</td>
</tr>
<tr>
<td>( \mathbb{N}_0 )</td>
<td>Set of the natural numbers without the number 0.</td>
</tr>
<tr>
<td>( \mathbb{R}_0 )</td>
<td>Set of the real numbers without the number 0, i.e. ( \mathbb{R} \setminus {0} ).</td>
</tr>
<tr>
<td>( \mathbb{R}^+ )</td>
<td>Set of the positive real numbers.</td>
</tr>
<tr>
<td>( \mathbb{R}^- )</td>
<td>Set of negative real numbers.</td>
</tr>
<tr>
<td>( j )</td>
<td>The imaginary unit, ( j = \sqrt{-1} ).</td>
</tr>
<tr>
<td>( \xi )</td>
<td>Relative damping of a pole or system.</td>
</tr>
<tr>
<td>( s )</td>
<td>Laplace variable.</td>
</tr>
<tr>
<td>( \omega_n )</td>
<td>Natural (resonance) frequency of a pole or system.</td>
</tr>
<tr>
<td>( q )</td>
<td>Forward shift operator ( (qu(t) = u(t+1)) ).</td>
</tr>
<tr>
<td>( \tau )</td>
<td>Time constant of a pole or system ( (\tau \triangleq (\xi \omega_n)^{-1}) ).</td>
</tr>
<tr>
<td>( \omega )</td>
<td>Frequency (Hz) or angular frequency (rad/s).</td>
</tr>
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</table>

\( n_{\text{BW}} \) Number of excited lines in the 3 dB bandwidth of a resonance. 

\( n_{\text{MC}} \) Number of Monte Carlo runs. 

\( N_A, N_B, N_C \) Order of local models \( \tilde{A}, \tilde{B}, \tilde{C} \). 

\( n_\theta \) Number of parameters. 

\( n_W \) Bandwidth of local window. 

\( \approx \) is approximately equal to .... 

\( \triangleq \) is defined as .... 

\( \sim \) is distributed as .... 

\( \equiv \) is equivalent with .... 

\( \gg \) is much greater than .... 

\( \ll \) is much smaller than .... 

\( \propto \) is proportional to .... 

\( \text{lcm} \) Least common multiple.
<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
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<tbody>
<tr>
<td>ln</td>
<td>Natural logarithm.</td>
</tr>
<tr>
<td>$\log_B$</td>
<td>Logarithm in base $B$ (when $B$ is omitted, $B = 10$ is implied).</td>
</tr>
<tr>
<td>Bias [$\bullet$]</td>
<td>Bias of $\bullet$.</td>
</tr>
<tr>
<td>$E[\bullet]$</td>
<td>Expected value of $\bullet$.</td>
</tr>
<tr>
<td>$\text{Var}[\bullet]$</td>
<td>Variance of $\bullet$.</td>
</tr>
<tr>
<td>$I_n$</td>
<td>$n \times n$ identity matrix.</td>
</tr>
<tr>
<td>$I_{n \times m}$</td>
<td>$n \times m$ identity matrix.</td>
</tr>
<tr>
<td>$0_{n \times m}$</td>
<td>$n \times m$ matrix containing only zeros.</td>
</tr>
<tr>
<td>$|\bullet|_\infty$</td>
<td>$\mathcal{H}_\infty$ norm of the system $\bullet$.</td>
</tr>
<tr>
<td>$|\bullet|_p$</td>
<td>$p$-norm of a signal or system $\bullet$.</td>
</tr>
<tr>
<td>$|\bullet|_2$</td>
<td>$\mathcal{H}_2$ norm of the system $\bullet$.</td>
</tr>
<tr>
<td>$\bar{\bullet}$</td>
<td>Complex conjugate of $\bullet$.</td>
</tr>
<tr>
<td>$\bullet^H$</td>
<td>Hermitian (complex conjugated) transpose of the matrix $\bullet$.</td>
</tr>
<tr>
<td>$\bullet^+$</td>
<td>Moore-Penrose pseudo-inverse of the matrix $\bullet$.</td>
</tr>
<tr>
<td>$\bullet^T$</td>
<td>Transpose of the matrix $\bullet$.</td>
</tr>
<tr>
<td>$O(\bullet)$</td>
<td>Bachmann-Landau “big oh” notation for the order of a function.</td>
</tr>
<tr>
<td>$#\bullet$</td>
<td>Number of elements (cardinality) in the set $\bullet$.</td>
</tr>
<tr>
<td>$\lceil\bullet\rceil$</td>
<td>$\bullet$ rounded upwards (ceil).</td>
</tr>
<tr>
<td>$\lfloor\bullet\rfloor$</td>
<td>$\bullet$ rounded downwards (floor).</td>
</tr>
<tr>
<td>$\lfloor\bullet\rceil$</td>
<td>$\bullet$ rounded towards the nearest integer, as in IEEE 754-2008 (round).</td>
</tr>
<tr>
<td>*</td>
<td>Convolution.</td>
</tr>
<tr>
<td>$\circ$</td>
<td>Hadamard (element-wise) product of matrices ($\cdot \ast$).</td>
</tr>
<tr>
<td>$\otimes$</td>
<td>Kronecker product of matrices (kron).</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

It's a dangerous business, Frodo, going out your door. You step onto the road, and if you don't keep your feet, there's no knowing where you might be swept off to.

John Ronald Reuel Tolkien
'The Fellowship of the Ring'

1.1 Modeling

Models play an important role in today's world, even though we often are not always aware of them. In particular, models enable us to predict the future (within a certain bound of uncertainty):

- supermarkets try to predict how much fruit and vegetables their regular customers buy such that they can order a large enough supply,
- traders on the stock market aim to predict whether companies such as Tesla or Apple will continue to prosper over the next quarter such that they can decide if it's time to buy more stock or rather sell,
- weather forecasters hope to predict what weather it might become tomorrow,
- how much electricity will the whole country use next week (do we need to start up a new power plant or not),
- how much noise the bodywork of a car will produce when it is driving over a bumpy road,
• how many nitrogen oxides will a car exhaust for a certain operating point of its diesel engine,

• the vibrations in a robot are small enough for it to carry out very precise tasks such as positioning electronic components with sub-millimeter precision,

and so on.

Models help us to keep an oversight of complicated matters. Once we understand the behavior of a few systems, this offers great opportunities to understand how those systems work together or even how we can control the world around us. For engineers, models enable us to design larger and more complicated devices and constructions than ever before. E.g. consider a central processing unit (CPU) in a computer: with a few billion transistors, it is no longer possible for a sole human being to understand what every single transistor does in the grand scheme. Instead, models of larger parts of the circuit allow us to retain insight (‘this part of the CPU caches some variables’, ‘these transistors multiply two numbers’, and so on).

### 1.1.1 How to Obtain a Model?

To successfully model a system, one has to delimit what part of the world should be captured in the model, i.e. the system. Since a system is mostly useful if it interacts with the world around it, one often also has to define inputs that influence the system, and outputs which are influenced by the system.

Modeling can be divided into two diametrical groups depending on the starting point that is assumed.

**White box** On the one hand, there is so-called ‘white box’ modeling, or first-principles modeling. The model is constructed by writing down the equations that govern the system, e.g. Newton’s laws of motion for mechanical systems, Kirchhoff’s laws for electrical circuits, etc. This is the kind of modeling that is practiced in school, which explains why it is also popular in industry. Unfortunately, white box modeling quickly becomes unwieldy once more complicated systems are to be studied or many parts of the system are uncertain. In general, many man-hours are required to build a white box model, such that this is often prohibitively expensive for engineering applications.
1.2 Elements of System Identification

**Black box** On the other hand, the diametrically opposed approach is called ‘black box’ modeling. Here, the model structure is chosen such that a wide variety of phenomena can be described. Regardless of the nature of the underlying system, be it mechanical, chemical, electrical, structural, ...; most dynamic phenomena can be described using differential equations (or difference equations). In particular, for black-box models, the structure of the model is chosen such that the data is well-approximated. The downside of this is that interpreting the model physically is not always possible. Fortunately, the goal of a model is often not to interpret the physical reality but to approximate it well enough to allow engineering.

**Gray box** Obviously, in practice nothing is ever as black and white, and neither is modeling. In many cases, one can or even has to incorporate knowledge about the physics of the system in the modeling process. In particular black box models where some parameters are related to physical phenomena, the resulting model is called a ‘gray box’ model.

### 1.2 Elements of System Identification

System identification is the science (and often art) that deals with converting observations of a system into mathematical models to describe the behavior of the system under test. As such, system identification rather fits in the category of black box modeling (or at least dark gray). Successfully identifying a system hinges on a plethora of choices that are to be made by the user. To make these choices well, a practitioner needs a considerable amount of experience. In Figure 1.1, a high-level flow diagram containing some key steps that are performed when identifying a system. For the technical details, we refer to the many textbooks on system identification (Keesman, 2011; Ljung, 1999b; Norton, 1986; Pintelon and Schoukens, 2012; Söderström and Stoica, 1989). Here, we give the high-level overview of the different steps. For a more complete overview, we also refer to Ljung, (2010a).

**Experiment design** In the first step, an experiment is constructed to gain insight into the behavior of a system. For system identification, experiment design is a question of designing input signals that will be applied to the system under test. Typically, the design consists of constructing a signal such that the obtained model has a small uncertainty in some sense (Goodwin and Payne, 1977; Goodwin, Rojas, et al., 2006; Levadi, 1966; Mehra, 1974). On the one hand, if one has very precise prior information, one could even construct optimal signals (Gagliardi, 1967; Gevers, Bombois, et al., 2011; Karlin and Studden, 1966; Zarrop, 1979). Typically, the design is very sensitive to the prior knowledge of the system. This is necessary to focus all the energy in the signal towards exciting
the system dynamics to the end of reducing the model uncertainty. However, when the prior knowledge is incomplete, uncertain or even incorrect, optimal signals can lead to very poor models as they might not excite other dynamics. An alternative approach is to incorporate the uncertainty of the prior knowledge into the signal design. This leads to so-called robust optimal experiment design (Goodwin, Welsh, et al., 2006; Rojas, Aguero, et al., 2012) that construct signals that optimize the worst-case model uncertainty.

The design of good experiments hence already forces the user to make a choice that can have far-reaching consequences for the model quality.

**Select a model structure** Once an experiment has been performed, the obtained input-output data are to be processed. In many cases, the eventual goal of identification is to obtain a parametric model, so the user has to select a model structure and complexity based on both his prior knowledge and on the data. A useful tool to help choose the model structure and/or model orders, is to inspect a nonparametric model of the system. One such tool is the frequency response function. For instance, by inspecting a frequency response function, a control engineer can have a rough idea of the system behavior in terms of approximate resonance frequencies, asymptotic slopes, etc. This gives a qualitative view of the minimally required complexity of the model and is hence a useful diagnostic tool. However, since most nonparametric estimators enforce very little prior knowledge about the system, it can sometimes be problematic to clearly distinguish noise in the measurements from the actual systematic behavior.
1.2 Elements of System Identification

**Estimate the model parameters** Once the model structure has been chosen, its parameters are to be estimated. This is often done by formulating an optimization problem which serves to minimize a particular cost function. The cost function is in essence a measure of the differences between the measured data and the model evaluated for particular parameter values. By minimizing the cost function, the optimization procedure yields parameter values that minimize the misfit between the data and the model. Unfortunately, for many identification settings, these optimization problems are non-convex (Boyd and Vandenberghe, 2004). This means that initial values need to be available to start the optimization process. Moreover, for poor starting values, optimization algorithms can get stuck in local minima of the cost function. Consequently, without reasonable starting values, the ‘optimized’ model can also be of poor quality.

**Validate the model** Finally, the estimated model is validated. Essentially, one checks whether the obtained model meets the predefined requirements, e.g. with respect to allowable uncertainty, complexity, ... This can happen in a few different ways. On the one hand, in-sample measures such as statistical tests on the value of the cost function, correlation tests on the residuals, etc. allow to verify that the model performs well on the measured dataset. Such approaches help to detect underfitting, where the model is not flexible enough to describe the measurements. On the other hand, the ultimate test is to validate the model against new measurements that have not been used during the estimation. This prevents overfitting of the model, where the model not only captures the behavior of the system but also describes the particular realization of the noise in the measurements. The latter leads to models that are overly tuned to the measurements such that their results cannot be replicated for new measurements. In case the model is not validated, this provides the user with new knowledge: that particular model is not good enough and at least one of the previous steps (or the prior knowledge) has to be altered.
1.3 User-Friendly System Identification

In the previous section, it has been seen that some of the system identification steps require a significant amount of prior knowledge and craftsmanship of the user to make good choices.

In this dissertation, we focus on developing system identification techniques that are ‘user-friendly’. The availability of easy to use and fast methods has been underlined by Gevers, (2011). In this context, ‘user-friendly’ should be understood as being easy to use for users ranging from novices to well-seasoned identification specialists.

Concretely, for novice users it is important to have straightforward techniques that require little interaction. In practice, this boils down to methods that work well for a wide class of systems. As such, only very generic assumptions of the system under test should be made and these should be easy to interpret.

For seasoned identification practitioners, methods that require little interaction are an opportunity for automation. This is especially important for complicated systems: systems that have high-order dynamics, multiple-input multiple-output (MIMO) systems with a high dimensionality, and networked systems amongst others. For such complex systems, building a model can be time-consuming and laborious if one has to supervise every step of the process. For more advanced users, user-friendly methods can hence allow to deal with more complex systems in a shorter amount of time such that the economical cost of building a model is reduced.

In this dissertation, the focus lies on linear time-invariant (LTI) systems. While this might seem as a very restrictive choice, this is one of the fundamental settings for system identification. In particular, LTI systems are a first step to build a system model. In many cases, such a linear model is accurate enough (Schoukens, Swevers, et al., 2004), e.g. to build a nominal controller, to design electrical filters and obtain a reasonable intuition of the system under test. Also, most engineers and scientists have a good understanding of linear systems (Kailath, 1980; Mandal and Asif, 2007; Oppenheim, Willsky, et al., 1996) such that LTI models align well with their prior knowledge and experience. Nevertheless, when a linear model is not adequate, a more flexible model needs to be constructed. Such a model could be more flexible, e.g. by relinquishing the time-invariance and/or its linearity. However, many of those advanced modeling approaches employ LTI models as starting values or intermediate result in building the actual non-linear (Giri and Bai, 2010) or time-varying model (Lataire et al., 2012; Louarroudi et al., 2014). As such, improvements in estimating LTI models also indirectly improve more advanced methods.
1.3 User-Friendly System Identification

1.3.1 Contributions

In this dissertation, we look into a few aspects of the identification workflow with the goal to make the whole process more user-friendly. Each of these aspects will be dealt with in its own chapter.

Research Question 1: How should a good experiment be designed?
Can we design a ‘good experiment’ to identify a LTI model? Particularly, this means that a robust input signal needs to be constructed without relying on extensive prior knowledge of the system. Such a signal should cover a wide frequency band to excite all dynamics of interest in the system during the experiment. However, the signal should ensure that systems in different frequency bands can be identified with a specified level of accuracy.

Research Question 2: Which non-parametric method should be used for a frequency response function (FRF)?
Developing a full parametric model for complicated systems requires considerable effort. Instead, could we leverage non-parametric or locally parametric approaches to obtain insight in the behavior of the system? Methods such as the Local Polynomial Method (LPM) and Local Rational Method (LRM) can offer good results. However, it is not clear how these should be tuned for different circumstances (near sharp resonance peaks, in low noise circumstances, ...). Can we devise simple rules that allow one to obtain good FRFs?

Research Question 3: Can we estimate the peak gain of a system in a non-parametric way?
To robustly control complicated systems, it can be overly cumbersome and even unwanted to construct high-order parametric models. As a result, some lightly damped resonances can dominate the model uncertainty. For robust control, this poses serious problems as such very sharp resonances are hard to observe in an FRF measurement. Instead, can we capture the peak gain of resonances using non-parametric models (i.e. without resorting to a global parametric model)?

Research Question 4: Can we avoid local minima by using smoothers?
When a parametric model is required, this often involves solving a nonlinear optimization problem. Such optimization problems require good initial parameter estimates to allow iterative methods to converge to a good local optimum, or, preferably even the global optimum. Can non-parametric smoothers be used to help avoid such local optima?
1.3.2 Outline and Publications

The lion’s share of this thesis has been published in either peer-reviewed journals or conferences. This section links my different publications to the different sections in this thesis. For an overview of publications grouped by type, please refer to page 185.

In Chapter 2, quasi-logarithmic multisines are proposed as a good (robust) excitation signal. That chapter is based on a journal article published in Institute of Electrical and Electronics Engineers (IEEE) Transactions on Instrumentation & Measurement:


preliminary results were also presented at the 2012 International Federation of Automatic Control (IFAC) symposium on System Identification (SYSID) and the IEEE International Instrumentation and Measurement Conference (I2MTC):


This work has also been presented at the following local (non-refereed) conferences:

- Egon Geerardyn, Yves Rolain, Johan Schoukens. “How to obtain a broad band FRF with constant uncertainty?” In: 31th Benelux Meeting on Systems and Control, March 27-29. CenterParcs Heijderbos, Heijden, Mar. 2012, p. 121, and


The non-parametric FRF estimation methods in Chapter 3 are based on a yet-unpublished manuscript. The so-called time-truncated LPM that is presented in the same chapter
is based on a journal article published in IEEE Transactions on Instrumentation & Measurement:


In particular, this smoother enables one to reduce the effect of noise on an estimated FRF using an automated approach. Relatedly, a preliminary study of the LPM in the context of lightly-damped MIMO systems has been presented at the (non-refereed) Benelux Meeting on Systems and Control:


The use of the non-parametric FRF estimation methods for $H_\infty$ gain estimation (and in general the FRF interpolation from Chapter 4) has been presented at the 2014 IFAC World Congress in South Africa and the 2014 Leuven Conference on Noise and Vibration Engineering (ISMA) in Leuven:

- **Egon Geerardyn**, Tom Oomen, Johan Schoukens. “Enhancing $H_\infty$ Norm Estimation using Local LPM/LRM Modeling: Applied to an AVIS”. In: *19th IFAC World Congress*. Cape Town, South Africa, Aug. 2014. DOI: 10.3182/20140824-6-ZA-1003.01186,


Preliminary results have been presented at local (non-refereed) conferences:

- **Egon Geerardyn**, Tom Oomen, Johan Schoukens. “The Local Polynomial Method (LPM) for $H_\infty$ Robust Control”. In: *32nd Benelux Meeting on Systems and Control, March 26-28*. Ol Fosse D’Outh, Houffalize, Belgium, Mar. 2013,


Experimental work related to Chapter 4 has also been presented at the 2015 IFAC Symposium on System Identification (SYSID) and at the 2015 workshop of the European Research Network on System Identification (ERNSI):


The study of different initialization strategies as depicted in Chapter 5 has been published in the IEEE Transactions on Instrumentation & Measurement:


and also at the local (non-refereed) 2015 Benelux Meeting on Systems and Control:


In collaboration with fellow researchers, I have written a few other publications. However, these publications are not covered in this dissertation.
• Matthijs van Berkel, Gerd Vandersteen, **Egon Geerardyn**, Rik Pintelon, Hans Zwart, Marco de Baar. “Frequency domain sample maximum likelihood estimation for spatially dependent parameter estimation in PDEs”. In: *Automatica* 50.8 (Aug. 2014), pp. 2113–2119. ISSN: 0005-1098. DOI: 10.1016/j.automatica.2014.05.027,


• Adam Cooman, **Egon Geerardyn**, Gerd Vandersteen, Yves Rolain. “Where are the Dominant Nonlinear Contributions in a Feedback Op-amp?” In: IUAP/PAI DYSCO Workshop. Château-Ferme de Profondval, Louvain-la-neuve, 2012,

Design of Excitation Signals
This chapter is based on Geerardyn et al., (2013, 2012) and Larsson et al., (2012).

A common mistake that people make when trying to design something completely foolproof is to underestimate the ingenuity of complete fools.

---

Douglas Adams
‘Mostly Harmless’

2.1 Introduction

Input design deals with the task of choosing an excitation signal in a way so that the information obtained from the experiment is maximized. This is a problem which has a long history in statistics. In system identification, the focus has mostly been on input design for dynamic systems. A common approach is to design the input spectrum and thereby maximizing the information matrix of the estimates, see e.g. (Fedorov, 1972; Goodwin and Payne, 1977).

A complicating issue in identification of dynamic system is the fact that the information matrix often depends on the true parameter values. Consequently, an optimal input design, based on the information matrix, will depend on quantities that are not known at the time of the design. There are two main solutions to this in the engineering literature (Goodwin, Rojas, et al., 2006):

1. Iterative input design where the design is updated as more information about the system is available, see e.g. (Gevers, 2005; Hjalmarsson, 2005). The disadvantage of this approach is that multiple experiments are required to converge to the optimal
design. Moreover, since the signal is optimized for a specific system model, it is likely that changes in the dynamics are hard to observe using such a signal.

2. Robust (optimal) input design makes the input design robust to uncertainty in the prior knowledge of the parameter values, see e.g. (Goodwin, Rojas, et al., 2006; Rojas, Aguero, et al., 2012; Rojas, Welsh, et al., 2007) and references therein. Although the signal is more robust towards uncertainty in the prior knowledge, it requires more forethought than (nominal) optimal designs. E.g. the approach followed by Rojas, Welsh, et al., (2007) relies on both nominal parameter values and knowledge of the distribution function of these parameters to design a robust (optimal) excitation signal. As such, this requires more advanced prior knowledge of the system to cope with the uncertainty of the prior knowledge.

2.1.1 General Purpose Robust Excitation Signals

In this chapter, the goal is to construct excitation signals that bridge the properties of general purpose and robust excitation signals to alleviate the chicken-and-the-egg problem that optimal strategies typically incur. In particular, a generic excitation signal will be used, but its properties are tuned in such a way that it attains some properties of robust excitation signals without requiring extensive prior information. We will consider input design when it is known that the system has resonances, possibly spanning a frequency band that may cover several decades. It is, however, not necessary to know how many resonances are in the system and at what precise frequencies those may occur. In short, we consider an input signal to be robust, if such signal allows one to measure the frequency response function (FRF) of a wide variety of systems well, without requiring a lot of specific prior information regarding the system. Moreover, a robust input signal should not be very sensitive towards assumptions regarding the system under test (or should be easily adaptable if the assumptions are only approximately valid).

Such problem settings occur in many diverse fields of experimentation, such as the measurement of biochemical impedances (Bragós et al., 2001; Sanchez et al., 2011), electrochemical impedance spectroscopy (Breugelmans et al., 2010; Niedostatkiewicz and Zielonko, 2009; Van Gheem et al., 2004), electronics (Munir et al., 2011), mechatronics (Oomen and Steinbuch, 2016; Steinbuch and Norg, 1998), vibration analysis (Karnopp, 1995; Voorhoeve et al., 2015) and acoustic testing (D’Amico and Desmet, 2014; Garcia Lopez et al., 2014) amongst others.

Any linear time-invariant (LTI) system of finite order can be expanded into a finite sum of sub-systems of first and second order by means of a partial fraction expansion (Oppenheim, Willsky, et al., 1996). Therefore the performance of the proposed method is
assessed on a prototype system of second order. This does not limit the generality of the approach as we can add several of such systems to construct a general LTI system.

To ensure that any second order system with its resonance frequency in the specified frequency range is measured with an equal accuracy, the system output should contain a comparable power level irrespective of its resonance frequency. For a system with a sufficiently low damping, one can intuitively understand that most of the information is contained within the 3 dB bandwidth of the system. Remember that an LTI system is naturally described in a logarithmic frequency axis (the slopes of such a system are specified in dB/decade). Hence, an excitation with a flat power spectrum and a linearly spaced frequency grid will over-excite the high end of the band at the cost of the low frequencies. To restore equal excitation signal levels over the complete range of frequencies, we will need to distribute the power in $1/\omega$ over the frequency band (Goodwin, Rojas, et al., 2006; Goodwin, Welsh, et al., 2006). In a noise excitation context, this results in pink noise. In a periodic setting, this results in a multisine whose frequency components are spaced equidistantly on a logarithmic axis.

The main problem is that network and dynamic signal analyzers typically rely on an equidistant frequency grid that comes with the discrete Fourier transform (DFT). A logarithmically spaced signal, on the other hand, does not match well to an equidistant frequency grid. More specifically, the lack of frequency resolution at the lowest frequencies hampers the quality of the power distribution at the low edge of the frequency band. To circumvent this disadvantage, we will construct quasi-logarithmically spaced signals. Their frequency spacing is approximately logarithmic but nevertheless conforms to the linear frequency grid of the DFT.

We will also present a simple method to determine a suitable density of a logarithmically spaced signal for a predefined minimal damping of the system to be tested.

**Contents** In Section 2.2, we first consider (optimal) input design for an example second-order system. Section 2.3 expounds the use of a signal with $1/f$ power spectrum on some basic sub-systems. Next, in Section 2.4 LTI systems are parametrized to facilitate a robust input design based on the observations of the preceding section. In Section 2.5 a comparison of different discrete frequency grids is provided, this is related to the problem of generating a multisine signal with a $1/f$ power spectral density. Next, Section 2.6 describes the selection of a suitable frequency ratio $\alpha$ for a logarithmically spaced grid such that constant model quality is obtained. The usefulness of the proposed signals is illustrated on simulations in Section 2.7 and validated on the measurement of a set of band-pass filters in Section 2.8. Conclusions are formulated in Section 2.9.
2.2 Input Design

For the theoretical analysis, we consider single-input single-output (SISO) output-error (OE) systems, i.e.,

\[ y(t) = G(s, \theta)u(t) + H(s, \vartheta)e(t), \]  

(2.1)

where \( u(t) \) is a known input sequence, \( y(t) \) the output sequence, and \( e(t) \) is zero mean Gaussian white noise with variance \( \sigma^2_e \). \( G \) and \( H \) are rational, time-invariant transfer function in continuous time. For notational simplicity, \( s \) is used both as the complex Laplace variable and the differentiation operator; it should be clear from the context which is used. The transfer function is parameterized by the vector \( \theta \). These parameters are unknown and hence we are interested in estimating their values.

**Assumption 2.1.** The noise filter \( H(s, \vartheta) \) is assumed to be independently parametrized from the plant \( G(s, \theta) \).

**Assumption 2.2.** For the derivations, \( H(s, \vartheta) = 1 \) is used as a working assumption such that the disturbing noise on the output is zero mean Gaussian white noise. When this assumption is violated, one has to first measure the noise coloring and use this knowledge to predistort the input to account for the noise coloring.

The goal of input design is to find an input that under the given experimental conditions, gives as much information as possible about the system. This can be done by optimizing the input power spectrum such that a scalar criterion involving the Fisher information matrix is minimized.

In open loop experiments, the scaled average information matrix is given by

\[ \text{Fi} (\theta) = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \left( \frac{\partial \hat{y}(t)}{\partial \theta} \right) \left( \frac{\partial \hat{y}(t)}{\partial \theta} \right)^T, \]  

(2.2)

\[ \frac{\partial \hat{y}(t)}{\partial \theta} = \frac{\partial G(s, \theta)}{\partial \theta} u(t). \]  

(2.3)

By Parseval’s theorem, \( \text{Fi} (\theta) \) can be expressed as

\[ \text{Fi} (\theta) = \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{\partial G(j\omega)}{\partial \theta} \left( \frac{\partial G(j\omega)}{\partial \theta} \right)^H \right] \phi_u(j\omega) d\omega. \]  

(2.4)

The information matrix is a function of the input power spectrum \( \phi_u \), which is the only entity that can be used to improve the quality of the estimates. A common optimality criterion is D-optimality where \( \text{det} \text{Fi} (\theta) \) is maximized (Goodwin and Payne, 1977).
2.3 Logarithmic Power Distributions

Remark 2.1. D-optimal input designs are quite popular since this criterion minimizes the volume of the uncertainty ellipsoid of the parameters. Moreover, the D-optimal input design is independent of the parametrization of the system (as long as the different parametrizations can be related by a bijective function) (Goodwin and Payne, 1977, Chapter 6).

The information matrix tells us how good the individual parameter estimates are. However, if we are interested in estimating a transfer function $G(s, \theta)$ the quality of this estimate is of more importance. It is possible to find an expression for the variance of $\hat{G}(j\omega, \theta)$ using the Gauss approximation formula (Ljung, 1999b; Pintelon and Schoukens, 2012). This gives

$$
\text{Var} \left[ \hat{G}(j\omega) \right] \approx \left( \frac{\partial G(j\omega)}{\partial \theta} \right)^H \text{Fi}^{-1}(\theta) \frac{\partial G(j\omega)}{\partial \theta}.
$$

Typically, the input design formulation depends on the true system parameters. This means that the optimal input depends on the unknown system that one wants to identify. Furthermore, classical input designs, such as the D-optimal design, typically give signals with sparse spectra. This means that only a few frequencies are excited. A sparse spectrum is bad from a robustness point of view; there is very limited information about what happens at the unexcited frequencies. Therefore, a signal design to be optimal for one system may result in very bad estimates if applied to another system. To increase the robustness of the design, one could take a small fraction $\delta$, from the optimal power and redistribute this power over the whole frequency band of interest. From (2.4) and (2.5) we see that scaling the optimal input power by $1 - \delta$, has the following effect on the variance of the estimate:

$$
\text{Var} \left[ \hat{G}(j\omega) \right]_{(1-\delta)\text{optimal}+\delta\text{robust}} \approx \frac{1}{1-\delta} \text{Var} \left[ \hat{G}(j\omega) \right]_{\text{optimal}}.
$$

This is a small loss while the increase in robustness can be significant if the added signal is robust. Such a robust signal is studied in the following parts of the chapter.

2.3 Logarithmic Power Distributions

When no information about the system is known, it is desirable that the excitation signal is such that regardless of the true system characteristics, the resulting estimates have equally good statistical properties. This can be ensured by a suitable choice of the input power spectrum.
2.3.1 A Second Order System

Consider the second order system

\[ y(t) = G(s, \theta)u(t) = \frac{1}{s^2 + 2\xi\omega_n s + 1}u(t), \]  

(2.7)

with parameters \( \theta = [\omega_n \, \xi]^T \). The resonance frequency of the system is \( \omega_R \equiv \sqrt{1 - \xi^2}\omega_n \).

When \( \xi < \frac{1}{\sqrt{2}} \), such a second order system exhibits a peak in the amplitude of the transfer function (Oppenheim, Willsky, et al., 1996):

\[ |G(\omega_M)| \equiv \max_{\omega} |G(\omega)| = \frac{1}{2\xi\sqrt{1 - \xi^2}} \]  

(2.8)

\[ \omega_M = \sqrt{1 - 2\xi^2}\omega_n. \]  

(2.9)

The 3 dB bandwidth around this resonance peak is given by \( BW_{3\, dB} = 2\xi\omega_n \) for systems that have a sufficiently low damping.

Since information is local in the frequency domain, the input signal power inside \( BW_{3\, dB} \), contributes significantly to the information matrix while input power outside \( BW_{3\, dB} \) has a smaller contribution. This can be seen by considering the optimal design for a second order system. From (Goodwin and Payne, 1977, Example 6.4.5) we have that the D-optimal design for (2.7) is a single sinusoid at frequency

\[ \omega^* = \frac{\omega_n\sqrt{(1 - 2\xi^2) + \sqrt{(1 - 2\xi^2)^2 + 15}}}{\sqrt{5}}. \]  

(2.10)

For lightly-damped systems, i.e. \( \xi \ll 1 \), \( \omega^* \approx \omega_n \), which is inside the bandwidth of the system. From (2.10) it is clear that the optimal design depends on the resonance of the system to be identified. Input design can be based on a guess of \( \omega_n \), however if this guess is wrong, the resulting estimates may be unacceptable (Rojas, Welsh, et al., 2007).

When no information about the system is available, we propose using an input power spectrum with a logarithmic power distribution. This can be achieved either with a continuous band-limited \( 1/f \) spectrum or a quasi-logarithmic multisine spectrum (Pintelon and Schoukens, 2012). We choose to work with an equidistant frequency grid with resolution \( f_0 \) and design the input in the frequency band \( [\omega_nk_{\text{min}} \omega_nk_{\text{max}}] \), where the resolution is \( k_0 = 1/T_{\text{meas}} \) and \( T_{\text{meas}} \) is the measurement time. Hence, a finite measurement time limits the frequency resolution and therefore we have to put a lower limit on the frequency band of interest. Moreover, we can only impose the logarithmic distribution sufficiently well if \( k_{\text{min}} \) is large enough.
First, we look at the band-limited $1/f$ power spectrum given by

$$\phi_u(\omega) = \begin{cases} \frac{1}{|\omega|} & \text{if } |\omega| \in [\omega_n k_{\text{min}}, \omega_n k_{\text{max}}], \\ 0 & \text{otherwise}. \end{cases}$$

With this input spectrum, the power inside $BW_{3\text{dB}} \approx [(1 - \xi)\omega_n, (1 + \xi)\omega_n]$ is

$$\frac{1}{2\pi} \int_{(1-\xi)\omega_n}^{(1+\xi)\omega_n} \frac{1}{\omega} \, d\omega = \frac{1}{2\pi} \ln \frac{1 + \xi}{1 - \xi}, \quad (2.11)$$

which is independent of $\omega_n$. The robustness of the $1/f$ input has been noted previously by e.g. Goodwin, Rojas, et al. (2006), Rojas, Aguero, et al. (2012), and Rojas, Welsh, et al. (2007).

Alternatively, one could also construct a signal with power spectrum concentrated at discrete frequency lines. Such signals are elaborated in Section 2.5.

The result of these two choices of input power spectrum is that for a given damping, $\text{Var} \left[ \hat{G} \right]$ is approximately the same independent of $\omega_n$. This is illustrated in Section 2.7.

### 2.3.2 Sums of Second Order Systems

This section extends the previous section to more general systems than second order. Consider a system

$$G(s, \theta) = \sum_{i=1}^{K} G_i(s, \theta_i) = \sum_{i=1}^{K} \frac{1}{s^2 + \frac{\omega_n^2}{\omega_n^2} + 2 \xi_i s + 1}, \quad (2.12)$$

$$\theta_i = [\omega_n, \xi_i]^T. \quad (2.13)$$

The systems $G_i$ will be called the sub-systems of $G$. Without loss of generality, we assume that $\omega_{n1} < \omega_{n2} < \cdots < \omega_{nK}$. To properly identify the system $G$, excitation is required such that all sub-systems are properly excited, regardless of their individual resonance frequencies.
The information matrix for systems of the type (2.13) will have a block structure commensurate with the dimension of \( \theta \) given by

\[
F_i (\theta) = \frac{1}{\pi} \begin{bmatrix}
\partial G_{1,1} & \partial G_{1,2} & \cdots & \partial G_{1,K} \\
\partial G_{2,1} & \partial G_{2,2} & \cdots & \partial G_{2,K} \\
\vdots & \vdots & \ddots & \vdots \\
\partial G_{K,1} & \partial G_{K,2} & \cdots & \partial G_{K,K}
\end{bmatrix},
\]

(2.14)

\[
\partial G_{i,j} = \int_0^\infty \Re \left\{ \frac{\partial G_i}{\partial \theta_i} \frac{\partial G_j^H}{\partial \theta_j} \right\} \phi_u(\omega) \, d\omega.
\]

(2.15)

Furthermore, when the damping of the systems is low or the resonances of the sub-systems are well separated, the estimates decouple in the sense that the elements of the off-diagonal blocks become small. The effect is that the variance of the estimated transfer function approximately becomes the sum of the variances of the sub-systems. To illustrate this we look at a system with \( K = 2 \), i.e. the sum of two second order systems. The conclusions from the example are easily extended to systems consisting of more than two sub-systems.

### 2.3.2.1 Example: Sum of Two Second-Order Systems

Consider the system

\[
G(s, \theta) = G_1(s, \theta_1) + G_2(s, \theta_2)
\]

(2.16)

\[
= \frac{1}{s^2/\omega_n^2 + 2 \xi_1 \omega_n s + 1} + \frac{1}{s^2/\omega_n^2 + 2 \xi_2 \omega_n s + 1},
\]

(2.17)

with \( \theta_1 = [\omega_{n1} \, \xi_1]^T \) and \( \theta_2 = [\omega_{n2} \, \xi_2]^T \). The information matrix for this system is

\[
F_i (\theta) = \begin{bmatrix}
\partial G_{1,1} & \partial G_{1,2} \\
\partial G_{2,1} & \partial G_{2,2}
\end{bmatrix}.
\]

(2.18)

To investigate the decoupling, we study the (1, 1) elements of blocks \( \partial G_{1,1} \) and \( \partial G_{1,2} \) of the information matrix (2.18), when the input spectrum (2.11) is used; similar arguments can be made for the other elements of the two matrix blocks.

The (1, 1) element of the \( \partial G_{1,1} \) entry of the information matrix is

\[
\int_0^\infty \Re \left\{ \frac{\partial G_1}{\partial \omega_{n1}} \frac{\partial G_1^H}{\partial \omega_{n1}} \right\} \frac{1}{\omega} \, d\omega.
\]

(2.19)
which evaluates to

\[ \frac{\pi}{16\sqrt{1 - \xi^2\omega_n^2\xi^2}} + \frac{2 \arctan\left(\frac{2\xi^2 - 1}{\sqrt{1 - \xi^2}}\right)}{16\sqrt{1 - \xi^2\omega_n^2\xi^2}} + \frac{(2\xi^2 + 1)}{4\sqrt{1 - \xi^2\omega_n^2\xi^2}} \]  

(2.20)
after lengthy calculations. For \( \xi \ll 1 \) this can be simplified to

\[ \int_0^\infty \Re \left[ \frac{\partial G_1}{\partial \omega_n} \cdot \frac{\partial G_1}{\partial \omega_n} \right] \frac{1}{\omega} \, d\omega = \frac{\pi}{8\omega_n^2} \xi^{-3} + O(\xi^{-2}) \]  

(2.21)
since the \( \arctan \) in (2.20) tends to \( \frac{\pi}{2} \) for \( \xi \to 0 \). For the corresponding element in \( \partial G_{1,2} \), we start by bounding the absolute value of the integral by

\[ \left| \int_0^\infty \Re \left[ \frac{\partial G_1}{\partial \omega_n} \cdot \frac{\partial G_2}{\partial \omega_n} \right] \frac{1}{\omega} \, d\omega \right| \leq \int_0^\infty \left| \Re \left[ \frac{\partial G_1}{\partial \omega_n} \cdot \frac{\partial G_2}{\partial \omega_n} \right] \right| \frac{1}{\omega} \, d\omega \leq \int_0^\infty \left| \frac{\partial G_1}{\partial \omega_n} \cdot \frac{\partial G_2}{\partial \omega_n} \right| \frac{1}{\omega} \, d\omega. \]  

(2.22)

To proceed, consider

\[ \frac{\partial G_1}{\partial \omega_n} = \frac{2\xi j\omega}{\omega_n^2 \left[ 1 + 2\xi j\omega/\omega_n + (j\omega/\omega_n)^2 \right]^2}, \]  

(2.25)

which has zeros in \( j\omega = \{ 0, \xi\omega_n \} \). Hence \( \left| \frac{\partial G_1}{\partial \omega_n} \right| \) has asymptotic slope 20 dB/dec. for low frequencies, at \( j\omega = \xi\omega_n \) the slope increases to 40 dB/dec.. There are four poles which give a \(-40\) dB/dec. roll-off after the peak. The shape of \( \frac{\partial G_1}{\partial \omega_n} \) is illustrated in Figure 2.2.

A crude, albeit useful, bound on the integrand in (2.24) is

\[ f(\omega, \theta_1, \theta_2) = \begin{cases} \frac{\omega^2}{\omega_n^2} \frac{1}{\omega} \frac{\partial G_1}{\partial \omega_n} & \text{if } \omega \leq \omega_n, \\ \frac{1}{\omega} \frac{\partial G_1}{\partial \omega_n} & \text{if } \omega_n < \omega < \omega_n, \\ \frac{\omega^2}{\omega_n^2} \frac{1}{\omega} \frac{\partial G_1}{\partial \omega_n} & \text{if } \omega \geq \omega_n, \end{cases} \]  

(2.26)

where \( \frac{\partial G_1}{\partial \omega_n} \) = \( \left\| \frac{\partial G_1}{\partial \omega_n} \right\|_\infty \). Again, direct calculations give

\[ \int_0^\infty f(\omega, \theta_1, \theta_2) \, d\omega = \frac{\omega_n}{\omega_n^2} \left[ 2 - \ln(\frac{\omega_n^2}{\omega_n}) \right] \xi^{-2} + O(1). \]  

(2.27)
Hence (2.22) cannot grow faster than $O \left( \xi^{-2} \right)$ as $\xi \to 0$. From this analysis we conclude that:

- the integral (2.22) decreases as $\omega_{n2} - \omega_{n1}$ increases, and
- lower damping further decreases the integral (2.22) in comparison to (2.21).

To further illustrate the decoupling of the two systems, we calculate (2.18) for different values of the damping $\xi$ and separations of the two resonances $\omega_{n1}$ and $\omega_{n2}$. If the separation is taken in terms of $BW_{3 \text{ dB}}$-units for the lower resonance, the result only depends on the difference $\omega_{n1} - \omega_{n2}$ and not on the actual frequencies. The resulting information matrix is then normalized so that the largest element is 1.

In Figure 2.1 the largest element in $\partial G_{1,2}$ is plotted for different dampings and separations. It is clear that highly resonant systems have a large degree of decoupling and that resonances that are well-separated increases the decoupling. The diagonal blocks of the normalized $Fi(\theta)$ in all cases have some elements close to 1. In this example, the $\xi$ is set to be the same in both sub-systems. Using different dampings in the sub-systems would, however, not change the argument, as the sub-systems with the lowest damping will be the limiting factor.

Based on the above arguments, we approximate the information matrix for lightly-damped systems as a block-diagonal matrix:

$$ Fi(\theta) \approx \begin{bmatrix} \partial G_{1,1} & 0 \\ 0 & \partial G_{2,2} \end{bmatrix}. $$

(2.28)

Hence the variance of the estimate $\hat{G}(s, \theta)$ becomes

$$ \text{Var} \left[ \hat{G}(\omega) \right] \approx \frac{\partial G^H(e^{j\omega})}{\partial \theta} \begin{bmatrix} \partial G_{1,1} & 0 \\ 0 & \partial G_{2,2} \end{bmatrix}^{-1} \frac{\partial G(e^{j\omega})}{\partial \theta} $$

(2.29)

$$ = \text{Var} \left[ \hat{G}_1(\omega) \right] + \text{Var} \left[ \hat{G}_2(\omega) \right]. $$

(2.30)

This results extends *mutatis mutandis* to systems consisting of more sub-systems.
2.4 System Parametrization

We consider the transfer function models of proper LTI systems of arbitrary but finite orders \( N_B/N_A \):

\[
G(\Omega, \theta) = \frac{B(\Omega, \theta)}{A(\Omega, \theta)} = \frac{\sum_{i=0}^{N_B} b_i \Omega^i}{\sum_{i=0}^{N_A} a_i \Omega^i}.
\] (2.31)

The model is parametrized by the vector \( \theta = [b_0 \cdots b_{n_B-1} a_0 \cdots a_{n_A-1}]^T \) and is evaluated at different frequencies \( \Omega \) in the complex plane. For a continuous time model \( \Omega = s \), while for the discrete time case \( \Omega = z \). For the sake of notational simplicity, \( \Omega = s \) will be used further on, but the results are similar for \( \Omega = z \). Remember that the rational transfer function of any strictly proper LTI system \( G \) can be decomposed in its...
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partial fraction form (Oppenheim, Willsky, et al., 1996, Appendix):

\[
G(s) = \frac{\sum_{i=0}^{n_B} b_i s^i}{\sum_{i=0}^{n_A} a_i s^i} = \sum_{i=1}^{l_1} \prod_{i=1}^{M_{1,i}} \left( s - p_i \right) M_{1,i} \prod_{i=1}^{l_2} \left( s^2 + 2\xi_\text{ni} s + \omega_\text{ni}^2 \right)^{M_{2,i}}
\]

\[
= \sum_{i=1}^{I_1} \sum_{\nu=1}^{M_{1,i}} \frac{K_i}{(s - p_i)^\nu} + \sum_{i=1}^{I_2} \sum_{\nu=1}^{M_{2,i}} \frac{K_{i,\nu} \omega_{\text{ni}}^2 (s + \omega_\text{zi})}{\omega_{\text{zi}} (s^2 + 2\xi_\text{ni} s + \omega_\text{ni}^2)^\nu}
\]

where \( I_1 \) denotes the number of real-valued poles, \( I_2 \) denotes the number of complex conjugated pole pairs. \( M_{1,i} \) and \( M_{2,i} \) denotes the multiplicity of the corresponding poles, such that \( n_A = \sum_{i=1}^{I_1} M_{1,i} + 2 \sum_{i}^{I_2} M_{2,i} \). Each term in (2.33) represents a sub-system of the system \( G(s) \). Within these sub-systems, we can distinguish first-order sub-systems and second-order sub-systems, the latter of which describe resonant phenomena. Since such resonances are often a limiting factor, we first focus on these resonant sub-systems.

**Remark 2.2.** For non-proper systems, one can write the transfer function \( G_{\text{non-proper}}(s) = P(s) + G_{\text{proper}} \) where \( P(s) \) is a polynomial and \( G_{\text{proper}}(s) \) a proper rational function. As such, the polynomial \( P(s) \) should be added to equation (2.33) for non-proper systems.

Consider the second-order sub-system with multiplicity \( \nu \):

\[
G_2(s, \theta) = \frac{K \omega_n^{2\nu} (s + \omega_\text{zi})}{\omega_\text{zi} (s^2 + 2\xi_\text{ni} s + \omega_\text{ni}^2)^\nu}
\]

parametrized in \( \theta = [K, \omega_n, \omega_z, \xi]^T \). To determine the Fisher Information matrix, the derivative \( \frac{\partial G(s, \theta)}{\partial \theta} \) is required; its components are given by:

\[
\frac{\partial G_2}{\partial K} = \frac{G_2}{K} = \frac{\omega_n^{2\nu} (s + \omega_\text{zi})}{\omega_\text{zi} (s^2 + 2\xi_\text{ni} s + \omega_\text{ni}^2)^\nu}
\]

\[
\frac{\partial G_2}{\partial \omega_\text{zi}} = \frac{-K \omega_n^{2\nu}}{\omega_\text{zi}} \frac{s}{(s^2 + 2\xi_\text{ni} s + \omega_\text{ni}^2)^\nu}
\]

\[
\frac{\partial G_2}{\partial \omega_n} = \frac{2K \omega_n^{2\nu-1} \nu}{\omega_\text{zi}} \frac{s (s + \omega_\text{zi})}{(s^2 + 2\xi_\text{ni} s + \omega_\text{ni}^2)^{\nu+1}}
\]

\[
\frac{\partial G_2}{\partial \xi} = \frac{-2K \omega_n^{2\nu+1} \nu}{\omega_\text{zi}} \frac{s (s + \omega_\text{zi})}{(s^2 + 2\xi_\text{ni} s + \omega_\text{ni}^2)^{\nu+1}}.
\]

The poles of these derivatives coincide with the system poles (possibly with increased multiplicity). Consequently, this means that for lightly-damped systems \( (\xi \ll \sqrt{2}) \),
these derivatives exhibit a sharp resonance peak at the same frequency as the actual sub-system \( G_2 \), see also Figure 2.2. The roll-off near the resonance is determined mainly by the damping \( \xi \) and the multiplicity \( \nu \) of the sub-system. The relationship between the frequency of the zero \( \omega_z \) and the resonance frequency \( \omega_n \) alters the shape of these derivatives, but has little impact in the vicinity of the resonance peak. As such, the Fisher information matrix is pre-dominantly determined by the input power spectrum near \( \omega_n \). Note that \( \frac{\partial G_2}{\partial \omega_n} \) and \( \frac{\partial G_2}{\partial \xi} \) have the steepest roll-off near the resonance. As such, those parameters are the most important for the experiment design.

Moreover, the results from Section 2.3.2.1 for the sum of simplified second order systems, i.e. equation (2.13), extend directly to the generic LTI system in (2.33) for the second-order sub-systems under the assumptions given below.

**Assumption 2.3.** The sub-systems in (2.33) all are either lightly damped \( (\xi \ll \sqrt{1/2}) \) or are not of interest.

**Assumption 2.4.** The resonance frequencies \( \omega_{ni} \) of all the sub-systems in (2.33) are well-separated in the frequency domain:

\[
\forall i, j \in \mathbb{N}, i \neq j : |\omega_{ni} - \omega_{nj}| > \frac{BW_i + BW_j}{2},
\]

where \( BW_i \approx 2\xi_i\omega_{ni} \) denotes the 3 dB bandwidth of the \( i \)th resonant sub-system.

**Remark 2.3.** The aforementioned assumption practically implies that there is no overlap between different resonances and hence there is little so-called ‘modal overlap’. The modal overlap \( \mu \) is given (Ege et al., 2009) as the ratio of the average 3 dB bandwidth to the average difference in resonance frequencies for all resonances in the frequency band of interest. Often, the modal overlap is linked to the model density \( n_m (\omega) = BW_{3dB}n_m (\omega) \).

In acoustic and structural engineering, expressions exist for the theoretical modal density \( n_m (f) \) in specific geometries such as beams, plates, square volumes (Bies and C. H. Hansen, 2009; Hart and C., 1971). Specifically, beams exhibit a modal density \( n_m (\omega) = \mathcal{O} (\omega^{-1/2}) \), for plates this becomes \( n_m (\omega) = \mathcal{O} (1) \), and for volumes \( n_m (\omega) = \mathcal{O} (\omega) \). As such, the previous assumption practically limits the use of the excitation signals to beams or the low-frequency region of plates and volumes (Ege et al., 2009).

**Assumption 2.5.** It is assumed that the excitation signal contains many more frequency lines than parameters required to describe the system under test well. Hence, we can neglect the bias of the estimate and focus on its variance error by assuming that a proper model order can be chosen and fitted from the data.

Under these assumptions, the Fisher information matrix will be approximately a diagonal block matrix, where the \( i \)th block corresponds to the parameters \( [\omega_{ni}, \omega_{zi}, \xi_i, K_{i,1}, \ldots, K_{i,M_2,i}] \) of a particular sub-system. Using the same logic as in Section 2.3.2.1, this means that the variance of the complete system is approximately equal to the sum of the variances obtained for each sub-system, due to the decoupling of the Fisher information matrix.
Figure 2.2: Bode plots of the elements of \( \frac{\partial G_2}{\partial \omega} \) for small \( \xi \). The roll-off near \( \omega_n \) is significant and depends strongly on \( \xi \). It can be seen that the derivatives are dominated by the resonance peak, an effect that is even more prominent when \( \nu \) is large. This means that for a good input signal, it is critical to have power near the resonance peak. Note also that \( \frac{\partial G_2}{\partial \omega_n} \) and \( \frac{\partial G_2}{\partial \xi} \) exhibit a sharper peak and hence \( \xi \) and \( \omega_n \) are the most important parameters of the model. In this example, \( \xi = 0.025, K = 0.5, \frac{\omega_n}{\omega_n} \in \{0.1, 1, 10\}, \nu = 1 \) and \( \omega_n = 1 \text{ rad/s} \) were used.
2.5 Multisine Excitations

In contrast to signals with a continuous $1/f$ power spectral density (PSD) presented in Section 2.3, similar results can be obtained by using signals that have a discrete frequency grid. Such signals are presented in this section.

To excite the system under test we consider a generalization of multisine excitations (Pintelon and Schoukens, 2012):

**Definition 2.1** (Generalized multisine). A generalized multisine $u(t)$ is a signal consisting of $F$ sine waves with a different frequencies $f_k$, amplitudes $A_k$ and phase shifts $\phi_k$:

$$u(t) = \frac{1}{\sqrt{F}} \sum_{k=1}^{F} A_k \sin(2\pi f_k t + \phi_k).$$  (2.40)

Without loss of generality, one can stipulate that the frequency grid is sorted, i.e. $f_1 < f_2 < \ldots < f_F$.

For LTI systems, it is well-known that the information matrix only depends upon the amplitude spectrum of the input and not upon its phase spectrum as indicated in (2.4). As such, the results presented in this chapter also do not depend on the particular choice of the phase spectrum. This means a user is free to make an appropriate choice for their application.

In this chapter, random-phase multisines are used since this is a reasonable choice for systems that are dominantly linear (Schoukens, Swevers, et al., 2004). Hence, $\phi_k$ is the outcome of a uniform random process over $[0, 2\pi[$. This leaves two sets of free parameters to create a signal that suits our needs:

- the amplitude of the spectral lines $A_k$, and
- the frequency grid $\{f_1, \ldots, f_F\}$.

Note that the $k^{th}$ term in (2.40) is periodic with period $T_k \triangleq f_k^{-1}$.

Consequently, the complete multisine has a period $T_{u(t)}$ that is the (least) common multiple of the period of its components:

$$T_{u(t)} = \text{lcm} \{ T_1, \ldots, T_F \} = \text{lcm} \{ f_1^{-1}, \ldots, f_F^{-1} \}$$  (2.41)

if a common multiple exists, i.e. $T_{u(t)} = n_1 T_1 = \ldots = n_F T_F$ with $n_i \in \mathbb{N}_0$ for $i \in \{ 1, \ldots, F \}$. 

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Example 2.1. Consider a generalized multisine where \( f_1 = 1 \) Hz and \( f_2 = \frac{2}{3} \) Hz. The corresponding periods are \( T_1 = f_1^{-1} = 1 \) s and \( T_2 = f_2^{-1} = 1.5 \) s. The overall period of the multisine is \( T_u(t) = n_1 T_1 = n_2 T_2 = 3 \) s, i.e. \( n_1 = 3 \) and \( n_2 = 2 \).

Example 2.2. Consider the generalized multisine with \( f_1 = 1 \) Hz, \( f_2 = \pi \) Hz, and \( f_3 = \pi^2 \) Hz. The corresponding periods are then \( T_1 = 1 \) s, \( T_2 = \pi^{-1} \) s, and \( T_3 = \pi^{-2} \) s. However, a common multiple of these periods cannot be found since there exist no positive integers \( n_1 \), \( n_2 \), and \( n_3 \) that ensure that \( T_u(t) = n_1 T_1 = n_2 T_2 = n_3 T_3 \). As such, this generalized ‘multisine’ is not periodic.

This last example illustrates that a least common multiple of real numbers is not guaranteed to exist. Consequently, the generalized multisine of Definition 2.1 is not guaranteed to be periodic.

2.5.1 Equidistant (Linear) Grid Multisine

In many applications, an equidistant frequency grid is used for \( u(t) \) because this fits best with the classical DFT analysis (Mandal and Asif, 2007; Oppenheim, Schafer, et al., 1999). It allows to analyze periodic signals on an equidistant frequency grid very efficiently. For such a frequency grid, it is guaranteed that the signal is periodic in the time domain, unlike for the generalized multisines of the previous section. Moreover, such a signal can be created easily using a waveform generator that has a constant sampling frequency \( f_s \).

An equidistant frequency grid \( \{ f_1, \ldots, f_F \} \) with spacing \( \Delta f \) consisting of \( F \) lines conforms to the following relation:

\[
f_k = k \Delta f + k_0 \Delta f \quad \forall k \in \{ 1, \ldots, F \}, \quad k_0 \in \mathbb{N}.
\]  

(2.42)

Using such linear grid, we can easily construct a representation of a multisine in discrete time that is suited for use with the DFT by sampling at a given sampling rate \( f_s \):

\[
u[n] = \frac{1}{\sqrt{F}} \sum_{k=1}^{F} A_k \sin \left( \frac{2\pi n (k + k_0) \Delta f}{f_s} + \phi_k \right)
\]

(2.43)

Such an equidistant grid is a very common choice. Note that it has more frequency lines per decade (or octave) at higher frequencies. Therefore, the excitation power per octave is larger at higher frequencies when a constant amplitude spectrum \( A_k = A_{in} \) is used. This is a rather common choice in multisine excitations when no other constraints are taken into account. For a constant amplitude spectrum, this means that a model of the
low-frequency dynamics is typically plagued by a higher uncertainty than for models of high-frequency dynamics. This is caused by the fact that the lower frequency bands receive less signal power than the high frequency bands.

Therefore a signal with an equidistant frequency grid is undesirable for our situation as it wastes signal power at high frequencies at the cost of an increased variance in the lower side of the band.

### 2.5.2 (Ideal) Logarithmic Grid Multisine

For a logarithmic frequency grid \( \{ f_1, \ldots, f_F \} \) of \( F \) lines, the following relation holds between excited frequencies:

\[
  f_k = \alpha \cdot f_{k-1} \quad \forall k \in \{ 1, \ldots, F \}
\]

for a given lowest frequency \( f_0 \) and frequency ratio \( \alpha > 1 \). Note that for \( \alpha \in \mathbb{N} \), such a frequency grid is a subset of a linear frequency grid and hence the multisine is periodic. However, when \( \alpha \not\in \mathbb{N} \), the signal is not guaranteed to be periodic (see Example 2.2).

When designing an excitation signal, it is common to only specify a (large) frequency band by its boundary frequencies \( f_{\min} \) and \( f_{\max} \). For a given \( \alpha \) it is easy to determine the number of excited lines within the frequency range of interest:

\[
  F = \left\lfloor \frac{\log f_{\max} - \log f_{\min}}{\log \alpha} \right\rfloor
\]

where \( \lfloor \cdot \rfloor \) denotes rounding towards the nearest smaller integer value. In the Section 2.6, the choice of a suitable \( \alpha \) is elaborated.

**Property 2.1.** Each frequency band \( B = [f, \kappa f] \) with \( \kappa \in \mathbb{R}^+ \setminus \{ 0, 1 \} \) that is completely within the range of the excited logarithmic grid of frequencies \( \{ f_1, \ldots, f_F \} \) contains a constant number of frequency lines

\[
  F_B = \left\lfloor \frac{\log(\kappa f) - \log(f)}{\log \alpha} \right\rfloor = \left\lfloor \log_{\alpha} \kappa \right\rfloor.
\]

**Example 2.3.** For the frequency decade band \( \text{dec.}(f) \triangleq [f, 10f] \), a logarithmic frequency grid contains \( F_{\text{dec.}} = \lfloor \log_{\alpha} 10 \rfloor \) excited lines. Note that this is independent of the frequency \( f \).

**Remark 2.4.** In different contexts, e.g. electrical circuit simulators such as Simulation Program with Integrated Circuit Emphasis (SPICE) (Kundert, 1995), measurement equipment such as dynamic signal analyzers (DSAs) and programming languages, it is more common to specify the spacing of a logarithmic grid using the total number of frequencies \( F \) or the number of frequencies per decade \( F_{\text{dec.}} \) instead of the grid ratio \( \alpha \). Obviously, such alternate specifications are equivalent.
Example 2.4. The MATLAB® code $f = \logspace(\log10(f_{\text{min}}), \log10(f_{\text{max}}), F)$ produces a logarithmically spaced frequency vector $f$ with $F$ elements in the range $[f_{\text{min}}, f_{\text{max}}]$. As such, the effective frequency ratio $\alpha = \sqrt[0]{f_{\text{max}}/f_{\text{min}}}$ in this vector.

Example 2.5. The HP 3562A Dynamic Signal Analyzer (1985) has a logarithmic measurement mode that ensures that each decade is covered by $F_{\text{dec.}} = 80$ lines to measure an FRF. This is equivalent to a grid spacing $\alpha = \sqrt[0]{80} \approx 1.029$.

From Property 2.1, it can be seen that for any set of frequency bands with a constant relative bandwidth (e.g. decades ($\kappa = 10$), octaves ($\kappa = 2$),...), a logarithmic frequency grid contains a constant number of lines per frequency band. Consequently, the power in each relative bandwidth is identical when a constant amplitude spectrum ($A_k = A_{\text{in}}$) is used.

To return to the FRF, in Section 2.6 we will show that for a dense logarithmic frequency grid, any second order system with damping $\xi$ and a resonance within the bulk of the frequency grid, will receive the same number of frequency lines in its $3$ dB bandwidth regardless of the actual value of the resonance frequency. As most of the information of such systems is obtained from the measurements at frequencies inside the $3$ dB bandwidth, one expects an equal variance for each system, as will be shown in Section 2.6.5.

For that reason, a logarithmic multisine is a good candidate for our purposes. However, it is more involved both to generate and to analyze such a signal, as the excited frequencies do not lie on a commensurate frequency grid. Especially for periodic measurements, this is a major downside since the period of such signal does not always exist (i.e. this leads to infinite measurement times). It is therefore not practical to use a perfectly logarithmically spaced frequency grid as the DFT or chirp $Z$-transform (CZT) (Rabiner, 2004) explicitly rely on an equidistantly-spaced frequency grid.

### 2.5.3 Quasi-Logarithmic Grid Multisine

Since most measurement equipment and analysis methods rely on signals with an equidistant frequency grid, we create a frequency grid $\{f_1, f_2, f_3, \ldots, f_F\}$ for which (2.44) is approximately valid but with the strict constraint that the frequencies must be a subset of an equidistant frequency grid. This yields the following relation for subsequent frequency lines:

$$f_k \approx \alpha \cdot f_{k-1} \quad \forall k \in \{1, \ldots, F\}$$

(2.47)

under the constraint that

$$f_k = N_k \cdot \Delta f, \quad N_k \in \mathbb{N}.$$  (2.48)
2.5 Multisine Excitations

Such a grid is called a quasi-logarithmic (quasi-log) grid (Pintelon and Schoukens, 2012).

Remark 2.5. When \( \alpha \not\in \mathbb{N} \), it is possible that at the low end of the frequency band, \( \Delta f > (\alpha - 1) f_{k-1} \) for some integer values of \( k \). In such cases, the commensurate frequency grid is not dense enough to realize a frequency ratio \( \alpha \). Consequently, the power spectrum at these low frequencies will be less than the one of an ideal logarithmic frequency grid.

Definition 2.2. We denote the ‘bulk’ of the quasi-logarithmic frequency grid as the frequency region where \( \frac{f_k}{f_{k-1}} \approx \alpha \) can be attained. A necessary condition for \( f_k \) to be part of the bulk is that \( \Delta f \leq (\alpha - 1) f_{k-1} \). Equivalently, the bulk can be seen as the frequency interval \( \left[ \frac{\Delta f}{\alpha-1}, +\infty \right[ \).

Definition 2.3. The effective frequency ratio \( \alpha_k \) of a frequency grid is

\[
\alpha_k \triangleq \frac{f_k}{f_{k-1}}. \tag{2.49}
\]

Definition 2.4. For the implementation of a quasi-logarithmic frequency grid, the approximate equality (2.47) does not define the grid unambiguously. The quasi-logarithmic grids depicted in this chapter are constructed using the relationship

\[
f_k = \alpha_k(n_k) f_{k-1} \quad \forall k \in \{1, \ldots, F\} \tag{2.50}
\]

with

\[
\alpha_k(N_k) \triangleq \left\lceil \frac{\alpha^{n_k} f_{k-1}}{\Delta f} \right\rceil \approx \alpha, \text{ and} \tag{2.51}
\]

\[
n_k \triangleq \min \{ n \mid n \in \mathbb{N}, \alpha_k(n) > 1 \}. \tag{2.52}
\]

In the equation above, \( \lfloor \cdot \rfloor \) denotes rounding towards the nearest integer according to the IEEE 754 standards and as implemented in the MATLAB® function round. This satisfies the approximate definition.

Remark 2.6. The use of different rounding strategies — e.g. the ceil function \( \lceil \cdot \rceil \) and floor function \( \lfloor \cdot \rfloor \) — in (2.51), allows for slightly different guarantees on the effective \( \alpha_k \) attained by the grid. By denoting \( \alpha_k^\star \) as the effective frequency ratio for a rounding strategy \( \bullet \) in that equation, one can trivially see that the following relationships hold:

\[
\alpha_k^\lfloor \leq \alpha \leq \alpha_k^\rceil \quad \text{and} \quad \alpha_k^\lceil \leq \alpha_k^\lfloor \leq \alpha_k^\rceil \quad \forall k \in \mathbb{N}_0. \tag{2.53}
\]

This means that rounding using the floor operator favors to strictly attain the specified frequency bin density at the cost of a few more excited lines (and hence less allottable power per line). The ceil function produces a grid that has more power per line at the
cost of a reduced bin density. On the other hand, using round as in (2.51), provides a grid that is somewhere in-between both extremes. For this reason, the grid constructed using $a_k\lfloor \cdot \rceil$, i.e. by using the normal round function, has been used in the remainder of this chapter.

**Example 2.6.** An easy way to produce a quasi-logarithmic grid in MATLAB® is to first produce the logarithmic grid $f_{\text{Log}}$ as in Example 2.4 and reducing it to a quasi-logarithmic grid with resolution $\Delta f$ using $f = \text{unique}(\text{round}(f_{\text{Log}}/\Delta f)) \times \Delta f$. Note that this is a computationally efficient way since both logspace and unique require at most $O(F)$ time, where $F = \text{numel}(f_{\text{Log}})$.

As this signal approximates a logarithmically spaced signal, we expect that it will exhibit comparable properties. On the other hand, by imposing (2.48), it remains possible to use signal techniques which rely on a commensurate frequency grid.

### 2.5.4 Compensated Quasi-Logarithmic Grid Multisine

The power density of the quasi-log multisine at the low frequencies is reduced in comparison to that of the logarithmic grid multisine. This is due to the limited available frequency resolution if one sticks to the DFT frequency grid. To compensate for this loss in PSD, we can increase the amplitude spectrum $A_k$ at these lines. Note that we are not able to restore the frequency resolution, only the power in a certain band is made equal to the power in the logarithmic multisine in the same band.

First of all, we need to determine the factor $A_k$ needed to correct the power spectrum to match that of a log grid multisine. To this end, one first constructs a logarithmic frequency grid $LG = \{ \Omega_1, \ldots, \Omega_F \}$ for a given factor $\alpha$ and the corresponding quasi-log grid $QLG = \{ f_1, \ldots, f_F \}$ for a resolution $\Delta f$. For each frequency $f_k$ in the quasi-log grid, we define $n(f_k)$ as the number of frequencies in the logarithmic grid that are nearer to $f_k$ than to any other grid line in the quasi-log grid:

$$n(f_k) = \#\text{nearestLines}_{f_k}$$  \hspace{1cm} (2.54)

$$\text{nearestLines}_{f_k} = \left\{ \Omega_l \mid \| \Omega_l - f_k \| < \| \Omega_l - f_{k'} \|, \quad \forall k' \neq k : f_k, f_{k'} \in QLG \right\}$$  \hspace{1cm} (2.55)

When approximating a log multisine with constant amplitude spectrum $A_{\text{in}}$, we choose the amplitude spectrum of the compensated quasi-log multisine as follows

$$A_k = A_{\text{in}} \cdot \sqrt{n(f_k)} \quad \forall k \in \{1, \ldots, F\}.$$  \hspace{1cm} (2.56)
This effectively concentrates the total power of the \( n (f_k) \) surrounding lines in the log grid multisine at the frequency \( f_k \) in the compensated quasi-log multisine.

The power spectrum of this compensated signal approximates the one of the log grid multisine better than an uncompensated quasi-log grid. As the power spectrum approximates the power spectrum of the log grid multisine, the uncertainty of the frequency response function will be approximately independent of the frequency within the measured frequency band.

### 2.5.5 Pink Quasi-Logarithmic Multisine

In (Rojas, Welsh, et al., 2007) it is suggested that band-limited \( 1/f \) noise is a reasonable, albeit not optimal, robust excitation signal.

For \( 1/f \) noise (‘pink noise’), the power spectral density \( S(f) \) is proportional to \( 1/f \). We can again approximate such a signal by means of a quasi-logarithmic grid multisine. To do so, we construct a quasi-logarithmic frequency grid \( \{ f_1, \ldots, f_k, \ldots, f_F \} \) and determine the corresponding amplitude spectrum \( A_k \) such that each excited line carries the same power as the corresponding frequency band does for pink noise.

This allows to define the amplitude spectrum \( A_k \) as

\[
A_k = A_{\text{in}} \sqrt{\int_{f_k}^{f_k} \frac{1}{f} \, df} = A_{\text{in}} \cdot \sqrt{\ln \frac{f_k}{f_k}} \quad (2.57)
\]

where \([ f_k, \bar{f}_k ]\) denotes the frequency range over which the PSD of pink noise is to be approximated by the power at frequency line \( f_k \).

To reduce the complexity of the expressions for \( f_k \) and \( \bar{f}_k \), we extend the frequency grid by one component to the left and to the right by means of the grid relation (equations (2.47) and (2.48)) to obtain the grid \( \{ f_0, f_1, \ldots, f_F, f_{F+1} \} \).

Using this extension, the frequency range covered by \( f_k \) can then be calculated as

\[
f_k = \sqrt{f_{k-1} f_k} \quad (2.58)
\]

\[
\bar{f}_k = \sqrt{f_{k+1} f_k} \quad (2.59)
\]

This range can be seen as the power-spectral density counterpart of the discrete spectrum relation (2.55). Consequently, both signals can be expected to behave in approximately the same way.
Note that instead of the geometric mean, the arithmetic mean can be used in (2.59) to calculate almost identical boundaries.

### 2.5.6 Comparison of the Frequency Grids

To gain some insight into these different grids, Figure 2.3 depicts the amplitude spectrum for a linear, log, compensated quasi-log and pink quasi-log grid multisine that all carry the same total power. For this example, a frequency spacing $\Delta f = 1$ Hz and frequency ratio $\alpha = 1.05$ are used.

![Figure 2.3: Comparison of the amplitude spectrum of a linear grid multisine, a logarithmic grid multisine, the compensated quasi-logarithmic multisine and the 'pink' quasi-log multisine covering the frequency band [1 Hz, 1 kHz]. $\omega_{\text{LL}}$ indicates the cross-over between the linear and logarithmic regions in the lower two decades of the excited frequency band.](image)

At the lowest frequencies the quasi-log grid and the linear grid frequencies do coincide and each line in the quasi-log grid is surrounded by many lines of the log grid. The amplitudes of the compensated and pink quasi-log lines show that the compensation only has effect at the lower frequencies since the uncompensated counterpart has an amplitude spectrum close to 1. The properties of these frequency grids are summarized in Table 2.1 and Figure 2.4.

We can expect a very similar behavior from properly designed compensated quasi-log multisines and from the ‘pink’ quasi-log multisine, since their amplitude spectra are almost identical. As such, it is advised to use ‘pink’ quasi-log multisines as those can be
Table 2.1: Summary of properties of the different frequency grids.

<table>
<thead>
<tr>
<th>Properties</th>
<th>Linear</th>
<th>Logarithmic</th>
<th>Quasi-logarithmic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{k+1}$</td>
<td>$f_k + \Delta f$</td>
<td>$\alpha f_k$</td>
<td>$\left\lfloor \frac{\alpha^{N_k} f_k}{\Delta f} \right\rfloor \Delta f, N_k \in \mathbb{N}_0$</td>
</tr>
<tr>
<td>$\Delta f_k$</td>
<td>$f_{k+1} - f_k$</td>
<td>$\Delta f$</td>
<td>$(\alpha - 1) f_k \in \mathbb{N}_0 \Delta f$</td>
</tr>
<tr>
<td>$\alpha_k$</td>
<td>$1 + \frac{\Delta f}{f_k}$</td>
<td>$\alpha$</td>
<td>$\approx \alpha$</td>
</tr>
<tr>
<td>$F_{[a,b]}$</td>
<td>$\left\lfloor \frac{b-a}{\Delta f} \right\rfloor \log_\alpha \frac{b}{a}$</td>
<td>$\approx \left\lfloor \log_\alpha \frac{b}{a} \right\rfloor$</td>
<td></td>
</tr>
<tr>
<td>$F_{[f,\kappa f]}$</td>
<td>$\left\lfloor \frac{(\kappa - 1) f}{\Delta f} \right\rfloor \log_\alpha \kappa$</td>
<td>$\approx \left\lfloor \log_\alpha \kappa \right\rfloor$</td>
<td></td>
</tr>
</tbody>
</table>

Figure 2.4: Actual frequency grid spacing $\Delta f_k$ and ratio $\alpha_k$ as a function of the frequency, for the different grid types with $f_0 = 1$ Hz, $\Delta f = 1$ Hz and $\alpha = 1.05$ as in Figure 2.3. $\omega_{\ell\ell}$ marks the transition between linear and logarithmic regime in the quasi-logarithmic grid.
constructed using more straightforward code (equation (2.57)) than the compensated quasi-log multisines (equations (2.54), (2.55), and (2.56)).

2.6 Optimization of the Frequency Spacing

2.6.1 Calculation of the Model Uncertainty

In the previous section logarithmically spaced multisines were introduced, but up to now their major parameter, the frequency spacing $\alpha$, has not been designed.

In this section we show a possible approach to compare the performance of different excitation signals with respect to the theoretically minimal attainable uncertainty of the transfer function as expressed by the Cramér-Rao lower bound (CRLB). Doing so allows to determine a suitable value for the frequency spacing $\alpha$.

Denote $U(\omega)$ as the excitation signal, $Y_\circ(\omega)$ the exact, unperturbed system output, $E(\omega)$ for the output noise source, $Y(\omega) = Y_\circ(\omega) + E(\omega)$ depicts the measured output signal as shown in Figure 2.5. The output noise is assumed to be zero-mean and white with a standard deviation of $\sigma_e$ in the time domain or $\sigma_E$ in the frequency domain.

![Figure 2.5: Output-error set-up.](image)

As in (Gallager, 2008; Petersen and Pedersen, 2012; Pintelon and Schoukens, 2012), and equivalently to the derivation in Section 2.2, the covariance matrix $C_{G(\omega_{\text{int}})}$ of the parametric transfer function $G(\theta, \omega)$ can be shown to be approximately equal to

$$C_{G(\omega_{\text{int}})} \approx J_{\text{int}}^H F_{\theta}^{-1} J_{\text{int}}$$

(2.60)

with

$$F_{\theta} = 2 \Re \left[ J_{\text{exc}}^H C_X^{-1} J_{\text{exc}} \right]$$

(2.61)

the Fisher information matrix (Pintelon and Schoukens, 2012).

In these expressions, the Jacobian $J_{\text{int}}$ matrix of the transfer function evaluated at the frequency of interest $\omega_{\text{int}}$ with respect to the model parameters $\theta$ is defined as

$$J_{\text{int}} \triangleq \frac{\partial G(\omega_{\text{int}}, \theta)}{\partial \theta}.$$  

(2.62)
2.6 Optimization of the Frequency Spacing

Similarly, $J_{\text{exc}}$ is the Jacobian calculated at the excited frequency lines $\omega_{\text{exc}} = \{ \omega_1, \ldots, \omega_F \}$.

Finally, $C_X$ denotes the covariance matrix of $X(\omega) = G(\omega, \theta_0) - \frac{Y_0(\omega) + E(\omega)}{U(\omega)}$, evaluated at the excited frequencies $\omega_{\text{exc}}$. By assuming that the $X(\omega)$ are independently distributed over the frequency $\omega$, the variance-covariance matrix $C_X$ is a diagonal matrix (Pintelon and Schoukens, 2012):

$$C_X = \begin{bmatrix} \sigma_E^2(\omega_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_E^2(\omega_F) \end{bmatrix}.$$ (2.63)

**Remark 2.7.** To introduce only limited prior knowledge into the input design, we shall assume that the noise is white. Hence, $\sigma_E^2(\omega) = \sigma_E^2$ is constant over the whole frequency band of interest. If the noise coloring is known beforehand (or measured in a previous experiment), the design can be adapted in a straightforward manner, e.g. by coloring the amplitude spectrum accordingly.

In this chapter, we select a single frequency of interest $\omega_{\text{int}}$ to obtain a scalar performance criterion. This is allowed as the matrix $C_X$ is diagonal.

**Remark 2.8.** Goodwin and Payne, (1977) note that for optimal input designs:

The choice of criteria is often not critical since it is usually the case that a good experiment according to one criterion will be deemed good by other criteria. Naturally this will only be true for sensibly chosen criteria.

Also for the design of robust excitation signals, this is a worthwhile observation.

### 2.6.2 Relation Between the Frequency Grid and the System Resonance

To determine an appropriate frequency ratio $\alpha$ for a logarithmically spaced signal, we study the estimated FRFs of a set of prototype second-order systems excited by logarithmically spaced excitation signals.

Instead of studying the response of all possible second-order sub-systems for every possible logarithmically spaced multisine– a six-dimensional problem with parameters $(\omega_0, \alpha)$ for the grid, and $(\omega_z, \omega_n, \xi, K)$ for the sub-system– we argue to simplify the problem in such a way that the results are still valid for the more complicated setting.
2.6.2.1 Exploiting the grid misalignment between the system and the grid

Denote $G(\omega; \omega_n)$ a second-order sub-system as in (2.34) with natural frequency $\omega_n$, damping ratio $\xi$ and static gain $K$, and zero at $\omega_z$. Similarly, $G(\omega; m\omega_n)$ where $m \in \mathbb{R}_0$ is a second order sub-system that shares the same parameters, except its natural frequency which is situated at $m\omega_n$.

If one calculates their FRFs by evaluating (2.34), one sees that $G(\omega; \omega_n)$ and $G(\omega; m\omega_n)$ yield the same numerical values for any value of the pulsation $\omega$. Both FRFs share the same ‘shape’, but are ‘located’ at different frequencies.

Denote $\vec{\omega}$ to be a logarithmic frequency grid with an infinite number of grid lines

$$\vec{\omega} \triangleq \{ \alpha^k \omega_0 \mid k \in \mathbb{Z} \} \quad (2.64)$$

and consequently an infinite span. We can define a second grid as the $m$-fold multiple (with $m \in \mathbb{R}_0$) of this grid and denote it $m\vec{\omega}$:

$$m\vec{\omega} \triangleq \{ m\alpha^k \omega_0 \mid k \in \mathbb{Z} \}. \quad (2.65)$$

Both $\vec{\omega}$ and $m\vec{\omega}$ share the frequency ratio $\alpha$.

**Remark 2.9.** For $N \in \mathbb{Z}$ it is obvious that $\vec{\omega} = \alpha^N \vec{\omega}$, i.e. both grids coincide.

**Definition 2.5.** To evaluate a function $G$ over the grid $\vec{\omega}$ with infinite span, let us introduce the shorthand notation $G(\vec{\omega})$ to mean $\{ G(\omega) \mid \omega \in \vec{\omega} \}$.

**Lemma 2.1.** Consider the second-order systems $G(\omega; \omega_n)$ and $G(\omega; m\omega_n)$ with $m \in \mathbb{R}_0$. For a frequency grid $\vec{\omega}$ with infinite span, $G(\vec{\omega}; \omega_n) = G(m\vec{\omega}; m\omega_n)$.

**Proof.** Consider any $\omega_\star \in \vec{\omega}$. The frequency $m\omega_\star$ is then by definition an element of the grid $m\vec{\omega}$. Substituting $\omega = \omega_\star$ and $\omega = m\omega_\star$ into $G(\omega; \omega_n)$ and $G(\omega; m\omega_n)$ respectively, directly yields

$$-\omega_\star^2 \omega_n^{-2} + 2j\xi \omega_\star \omega_n^{-1} + 1 = -\omega_\star^2 \omega_n^{-2} + 2j\xi \omega_\star \omega_n^{-1} + 1. \quad (2.66)$$

Since all the factors “$m$” cancel in the equation above, this proves the stated.

**Theorem 2.1.** Second-order systems with equal damping $\xi$ and static (DC) gain $K$ but resonance frequencies that differ by a factor $\alpha^N$ with $N \in \mathbb{Z}$, share an identical ‘shape’ of their FRF at the lines of a logarithmically spaced frequency grid with infinite span and a frequency ratio $\alpha$. Or, formally: $\forall N \in \mathbb{N} : G(\vec{\omega}; \omega_n) = G(\vec{\omega}; \alpha^N \omega_n)$. 

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2.6 Optimization of the Frequency Spacing

Proof. This is an immediate consequence from Lemma 2.1 for \( m = \alpha^N \) and Remark 2.9.

Taking this into account allows to limit the study of second order systems to a small set of systems that should be considered in the search for a suitable grid spacing \( \alpha \). We can limit the evaluation of the theoretical uncertainty (2.60) obtained by exciting the system with a logarithmic signal with ratio \( \alpha \) to the analysis of second order systems with a resonance frequency set to

\[
\omega_n \alpha^{N+\gamma} \quad \text{with } 0 \leq \gamma < 1.
\]

The value \( \gamma \) is varied for a fixed value of \( N \in \mathbb{Z} \). The value of \( N \) will not influence the 'shape' (and uncertainty) of the FRF.

The arguments for a logarithmically spaced grid with an infinite span remain approximately valid for quasi-logarithmically spaced grids with a finite frequency support if the resonance peak of the system under test lies well within the 'bulk' of the frequency grid with finite support, i.e. \([\omega_{\ell\ell}, +\infty]\), where

\[
\omega_{\ell\ell} = \frac{2\pi \Delta f}{\alpha - 1} \quad (2.68)
\]

indicates the lowest frequency where the quasi-logarithmic grid could effectively reproduce the logarithmic spacing. In particular for lightly-damped sub-systems,

\[
\text{BW}_{3\,\text{dB}} \approx [(1 - \xi)\omega_n, (1 + \xi)\omega_n], \quad (2.69)
\]

such that the resonance peak is in the bulk of the frequency grid when

\[
\omega_{\ell\ell} = \frac{2\pi \Delta f}{\alpha - 1} \leq (1 - \xi) \omega_n. \quad (2.70)
\]

For more reliable results, however, it is advisable to also have a few logarithmically spaced frequency lines, outside of the resonance peak of the actual system for a more robust result. In practice, this means that systems with their resonance peak at the lowest and the very highest edge of the frequency band may not be observed with identical quality as systems at intermediate frequencies.

The validity of this approximation can be understood intuitively by inspecting the derivatives that constitute the Fisher information matrix (see Figure 2.2): those derivatives are significantly larger near the resonance frequency than at other frequencies.
2.6.3 Exploiting the Shape of the Derivatives

In particular, from Figure 2.2, it can be seen that the derivatives with respect to \((\omega_n, \xi)\) exhibit a sharper resonance and hence \((\omega_n, \xi)\) will prove the hardest to identify. This means that if one would design a signal to identify just \((\omega_n, \xi)\), such a signal would also be well-suited to identify the other parameters since their corresponding weight in the Fisher information matrix is very similar (especially in the vicinity of the resonance, where their amplitude is most significant).

The free parameters \((K, \xi, \omega_n, \omega_z)\) of the sub-system are reduced to only \((\xi, \omega_n)\) and the others are fixed during our experiments: \(K = 1\) and \(\omega_z \to \infty\).

Remark 2.10. Implicitly, this considers a design where each of the sub-systems have an equal gain \(K = 1\) and residue \(\omega_z \to \infty\). In practice, this is often not a reasonable assumption, but the design can be adapted in a straightforward manner. Concretely, using a first experiment with the proposed signal, one can estimate the different gains of the sub-systems and optimize the amplitude spectrum of the signal to take those into account: for sub-systems where \(K\) is lower than on average, the amplitude spectrum needs to be increased near the resonance frequency and vice versa if the uncertainty of the parameters is required to be constant. Of course, the appropriate actions to compensate for differing \(K = 1\) depend on the actual goal of the model.

The choice of the damping is more important. Instead of choosing the damping directly, we choose the peak gain \(|G(\omega_M)| \in \{5\ \text{dB}, 10\ \text{dB}, 20\ \text{dB}, 40\ \text{dB}\}\) of the transfer function (2.8) to study the effect of differently damped systems. However, do remember that \(|G(\omega_M)| = (2\sqrt{1 - \xi^2})^{-1}\) for the simplifications introduced above as in (2.8).

2.6.4 Selection of the Frequency Grid Spacing \(\alpha\)

Since the damping influences the 3 dB bandwidths, this will become important when one has to determine the grid spacing \(\alpha\). To deal with this, we introduce the normalized frequency ratio

\[
\tilde{\alpha} = -1 + \frac{\omega_{k+1} - \omega_k}{BW_{3\text{ dB}}} \cdot \frac{\omega_n}{\omega_k}. \tag{2.71}
\]

This normalized frequency ratio can be interpreted as a measure of the grid resolution, relative to the 3 dB bandwidth of the second order system. The grid resolution in a (quasi-) log grid is \((\alpha - 1)\omega_k\) while the system bandwidth is approximately \(2\xi\omega_n\). Around the resonance, \(\omega_k \approx \omega_n\), such that the last fraction in (2.71) can safely be neglected.

To determine a suitable value of the frequency spacing ratio \(\alpha\) for the logarithmic and (compensated) quasi-logarithmic grid, we construct a logarithmic multisine signal in the
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Figure 2.6: Graphical depiction of the misalignment coefficient \( \beta \) with respect to a logarithmic frequency grid with spacing \( \alpha \) for two systems with different damping. The 3 dB bandwidth (\( \text{BW}_{3\,\text{dB}} \)) around the peak frequency \( \omega_M \) of the system is highlighted.

frequency domain for a given frequency band (in this example, \([0.01 \, \text{rad/s}, 100 \, \text{rad/s}]\) was used). In the previous section we saw that systems with equal damping and DC gain, will behave very similarly when excited by a log signal with spacing \( \alpha \).

Instead of using (2.67) to establish the influence of the resonance frequency, we follow an equivalent approach that is more intuitive to interpret.

We choose the resonance frequency of the systems such that

\[
\omega_M (\omega_n) = \omega_k + \beta (\omega_{k+1} - \omega_k) \quad \text{with} \quad 0 \leq \beta < 1
\]

(2.72)

where \( \omega_k \) and \( \omega_{k+1} \) are the excited grid lines closest to the frequency \( \omega_M \) where the FRF has its peak amplitude and \( \beta \) is the misalignment coefficient that is varied over the interval \([0, 1]\) as depicted graphically in Figure 2.6. This means that \( \omega_k \leq \omega_M \leq \omega_{k+1} \) and that \( \beta \) has the same role as \( \gamma \) factor in (2.67). If we take a look at the special case when \( \beta = 0 \) (or \( \beta = 1 \)) in this expression, we see that this situation occurs when the peak value of the FRF at \( \omega_M \) coincides with the grid line \( \omega_k \) or \( \omega_{k+1} \) respectively. Intuitively, we study second order systems whose peak frequency \( \omega_M \) is shifted between two neighboring excited frequency lines. From the previous section, we know that the exact frequency matters little as long as it remains within the bulk of the frequency grid. In this particular example \( \omega_n \) was chosen at approximately 1 rad/s.
We normalize the root-mean-square (RMS) value of the excitation signal to 1 and choose $\sigma_E(\omega) = 0.1$ for all evaluations. The noise level is kept constant, which causes the signal to noise ratio (SNR) per excited line to drop for more finely spaced excitation signals.

We then calculate the theoretical variance on the magnitude of the peak in the FRF ($|G(\omega_M)|$), as shown in (2.60), for a coarse grid. A coarse grid was chosen with $\tilde{\alpha} = 10$; this means that the spacing between the excited lines around the resonance are much larger than the system bandwidth. It is obvious that such an excitation signal will not perform well in most cases, as the resonance peak is very likely to be overlooked in such an FRF.

For each set of second order systems, we determine for which value of the misalignment coefficient $\beta$ the maximum variance on the FRF is obtained.

For these worst cases, we study how the frequency spacing $\alpha$ influences the uncertainty by decreasing $\alpha$ until no further improvement can be obtained.

### 2.6.5 Discussion

As stated above, we start by evaluating the variance $\sigma_G(\omega_M)$ for a coarse frequency spacing $\tilde{\alpha} = 10$ for each of the prototype systems. This is shown by the solid lines in Figure 2.7. One immediately notices that the variance depends strongly on the misalignment $\beta$ of the system and the frequency grid.

Such signals are clearly undesirable, as a misalignment smaller than the grid spacing of the excitation grid and the system will lead to a large increase in the uncertainty of estimate.

For a large peak amplitude of the FRF (low damping), the worst case increase (marked with a cross) is approximately centered around $\beta = 0.5$ (i.e. midway between the nearest excited lines). In systems with a higher damping, this symmetry is lost. This can be explained intuitively as the 3 dB bandwidth is less symmetric for highly damped systems than in lightly damped systems with a very narrow resonance peak. Also note that the value of $\sigma_G(\omega_M)$ is practically identical for $\beta = 0$ and $\beta = 1$. This is in accordance with the claim that in the bulk of the grid, a shift over an integer number of lines will only marginally influence the uncertainty.

We explore the systems which have the worst uncertainty (as marked by a cross in Figure 2.7). For the corresponding values of the misalignment $\beta$, we evaluate the uncertainty when the frequency ratio $\alpha$ of the signal is decreased, while the total signal power is kept
2.6 Optimization of the Frequency Spacing

- Misalignment coefficient \( \beta \)
- Variance \( \sigma^2 \)
- Transfer Function \( G(\omega_M) \) [dB]

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( \sigma^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>0.2</td>
<td>0.3</td>
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<td>0.4</td>
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<tr>
<td>0.4</td>
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<tr>
<td>0.5</td>
<td>0.6</td>
</tr>
<tr>
<td>0.6</td>
<td>0.7</td>
</tr>
<tr>
<td>0.7</td>
<td>0.8</td>
</tr>
<tr>
<td>0.8</td>
<td>0.9</td>
</tr>
<tr>
<td>0.9</td>
<td>1</td>
</tr>
</tbody>
</table>

\( |G(\omega_M)| = 40 \text{ dB} \)
\( |G(\omega_M)| = 20 \text{ dB} \)
\( |G(\omega_M)| = 10 \text{ dB} \)
\( |G(\omega_M)| = 5 \text{ dB} \)

Figure 2.7: Evaluation of the variance \( \sigma^2_G(\omega_M) \) for different prototype systems as function of the misalignment \( \beta \). The full lines depict the variance for a coarsely spaced signal. The dashed lines show the variance for a signal with the optimal \( \alpha = \alpha^* \) (see (2.73)).

Figure 2.8: Worst case variance of the peak in the transfer function for logarithmic grid multisines and systems with different peak amplitudes.
identical. The results are displayed in Figure 2.8, where the marked points correspond to the marked points in Figure 2.7.

We notice that for sufficiently low $\tilde{\alpha}$, the variance on $G(\omega_M)$ is roughly independent of $\alpha$. We will choose the optimal grid spacing

$$\alpha^* = 1 + \xi$$

such that it represents the coarsest frequency spacing for which the variance is approximately equal to that constant value. This choice is indicated by the dashed line in Figure 2.8.

**Guideline 2.1: Design the logarithmic grid for the damping of the system:** $\alpha \leq 1 + \xi_{\text{min}}$

To measure a resonant system with damping $\xi > \xi_{\text{min}}$ well, a (pink) quasi-logarithmic multisine with frequency ratio $\frac{\omega_k}{\omega_{k-1}} \leq \alpha \leq 1 + \xi_{\text{min}}$ is recommended. Smaller frequency ratios provide more measurement points at the cost of a reduced SNR per frequency bin. As such, $\alpha = 1 + \xi = \alpha^*$ provides the required resolution and the best SNR per bin.

Choosing a finer grid will not improve the estimated $G(\omega_M)$ significantly, but will rather produce an FRF where the SNR per bin is decreased. On the other hand, a coarser grid will significantly decrease the quality of the measurement as it overlooks the resonance peak. Hence little information can be extracted from such a measurement (i.e. the variance on the estimate will be large).

**Remark 2.11.** From Figure 2.8, it can also be deduced that when a logarithmic multisine with $\alpha = 1 + \xi_{\text{min}}$ is used, that resonant sub-systems with a damping $\xi \geq \xi_{\text{min}}$ can be well-identified using such a signal. Hence, the signal is well-suited not only to observe sub-systems with a common damping, but also sub-systems that are more damped than $\xi_{\text{min}}$.

If we determine the effect of the misalignment factor $\beta$ for a grid with frequency ratio $\alpha^*$, we see that misalignment has a negligible effect on the uncertainty as shown by the dashed lines in Figure 2.7 for each of the prototype systems. Note that these values do indeed coincide with the limiting values found in Figure 2.8.

Now we can interpret Figure 2.7 in more detail. We see that for $\beta \in \{0, 1\}$ (i.e. perfect alignment), the variance of a coarse grid (full lines) is lower than it is for a fine grid (dashed lines). This is to be expected as a coarse grid places more power at each individual line. Thereby it yields a smaller variance at those lines. It is better tailored
2.6 Optimization of the Frequency Spacing

Figure 2.9: Depiction of the transfer function $G$ of a second order system, along with a logarithmic multisine excitation $U$ for $\alpha = \alpha^*$. The circled sine components lie within the 3 dB bandwidth of $G$.

to systems that are closely aligned to the excited frequency grid. However, there is a critical consequence: when there is a moderate misalignment $\beta$ of the system and the excitation grid, the variance increases significantly. This effect is more important for lightly-damped systems, as their 3 dB bandwidth is very narrow and hence their main response is easier to ‘miss’.

To give a more intuitive feeling how the frequency spacing $\alpha$ and the system under test react, both the input spectrum and the FRF of the system are visualized in Figure 2.9 for the worst value of $\beta$ and the optimal frequency spacing $\alpha^*$. The 3 dB bandwidth of the system is highlighted in gray together with the excited frequency lines that lie within that bandwidth.

We see that to attain this, four frequency components are needed within this bandwidth in order to be almost completely independent of misalignment.

In this section, we have shown that signals with a logarithmic frequency spacing allow to identify systems with equal damping equally well, when the frequency spacing is dense enough. Practically, four frequency lines in the 3 dB bandwidth of each second order sub-system suffice to suppress the effect of misalignment of such a sub-system and the frequency grid of the excitation signal.


2.7 Simulations

In the following set of simulations, we will study how the FRF of different second order systems with equal damping (and the corresponding uncertainty) are measured by the multisine excitation signals from Section 2.5.

2.7.1 Setup

We consider second-order systems with an equal damping and with resonance frequencies $4, 8, 20, 40, 80, 200, 400, 800, 2000$ and $4000$ Hz. Using the MATLAB® instruction `cheby1`, discrete-time equivalents of these systems are generated. These can be represented by a transfer function of the form:

$$G(z) |_{\xi, f_R} = \frac{b_2 z^2 + b_1 z + b_0}{z^2 + a_1 z + a_0}.$$  (2.74)

This form can also be obtained by applying Tustin’s method or bilinear transform (Oppenheimer, Willsky, et al., 1996) to the continuous time system (2.34).

The input signal has an equidistant frequency grid with a resolution $\Delta f = 1$ Hz between $1$ Hz and $16384$ Hz. The sampling frequency $f_s$ is set to $65536$ Hz to prevent aliasing. For the quasi-log grid multisine, the compensated and pink quasi-log multisine, we chose a frequency ratio $\alpha = 1.05$. According to (2.73), this frequency ratio is suited for second-order systems with a damping $\xi \geq 0.05$, or a peak amplitude of the FRF $|G(\omega_M)|$ less than approximately $20$ dB. Each of these signals is constructed to have an RMS value of $1$. The amplitude spectra of these signals are displayed in Figure 2.3 for identical signal power.

At the output of the systems, white Gaussian noise $e[n]$ is added such that its standard deviation is approximately $10\%$ of the RMS value of the output.

The obtained FRF is then fitted to the transfer function model (2.74) using the `elis` function of the Frequency Domain System Identification Toolbox (Kollár et al., 1994–2012) for MATLAB®. This is a maximum likelihood estimator that fits a parametric model to the measured data in the frequency domain. To improve convergence and avoid local minima, the initial values for the model are chosen equal to the parameters of the true system $G_o(z)$, so that we can focus completely on the variance of the estimations.

Such an experiment is repeated $R = 1000$ times with a different random phase spectrum $\phi_k$ and a different noise realization $e[n]$ for each experiment.
Remark 2.12. All simulations have been repeated as well by fitting with the MATLAB® System Identification Toolbox by means of the oe function (Ljung, 2010b). The observed results were similar and hence are not elaborated further in this text. For more information, please refer to Larsson et al., (2012).

2.7.2 Systems With a ‘High’ Damping

To test the signals for systems with a high damping with respect to the frequency ratio in the excitation signal, systems with a damping $\xi = 0.2$ are used. White Gaussian noise $e[n]$ with a standard deviation $\sigma_e \approx 0.0678$ added to the output.

The resulting frequency response function $\hat{G}$ for one such system is shown in Figure 2.10 together with the standard deviation $\hat{\sigma}_G$ of the estimated parametric model for each of the different excitation signals. The bias is at least 20 dB smaller than the standard deviation. Therefore the mean squared error (MSE) will be determined mainly by the variance of the estimate instead of the bias.

![Figure 2.10: FRF (---) of a second order system with $f_R = 40$ Hz and $\xi = 0.2$ along with the corresponding variance and bias of the parametric models obtained using three different kinds of multisine excitations.](image)

Figure 2.11 displays the estimated transfer functions and their uncertainty in the vicinity of their corresponding resonances for the whole set of considered systems. Using this
Figure 2.11: FRFs for different second order systems with constant damping $\xi = 0.2$ but different resonance frequencies. Comparison of the variance around the respective resonance frequency reveals the performance of each excitation signal.

condensed representation and the displayed envelopes of the uncertainty, we can deduce the frequency-dependent behavior of the model variance obtained by the different excitation signals.

For the linear grid multisine, the uncertainty is strongly dependent on the frequency, and is proportional to $1/f_R$. The uncertainty is generally higher than the uncertainty obtained with logarithmically spaced multisines, as the power has been divided over more excited lines.

The quasi-log multisine already shows a fairly flat uncertainty of about $-52$ dB at the higher frequency ranges. Yet at the lowest frequencies, the $1/f_R$ dependence can be observed again. This is caused by the linear distribution of excited frequencies in this range, as shown in Figure 2.3.

The compensated quasi-log multisine shows a more constant variance of about $-51$ dB even at these low frequencies. As more power is placed at low frequencies, less power can be placed at the higher ones. This causes a slightly increased variance compared to the regular quasi-log multisine. At the studied system with the lowest resonance frequency, the compensation fails to completely compensate the effects of the reduced number of
2.7 Simulations

Figure 2.12: FRFs for different second order systems with constant damping $\xi = 0.05$ but different resonance frequencies. Comparison of the variance around the respective resonance frequency reveals the performance of each excitation signal.

Given the optimal frequency ratio (2.73), one expects that quasi-logarithmic grids with a frequency ratio $\alpha = 1.05$ will allow to identify second order systems with damping $\xi = 0.05$ with a constant uncertainty. In this set of simulations, we check this by repeating the previous set for such systems. In this case, the output noise is chosen $\sigma_e \approx 0.1123$ to keep the SNR of 20 dB. The corresponding results are shown in Figure 2.12.

At the higher frequency bands, an approximately constant quality is still obtained for the (compensated) quasi-logarithmic signals. At lower frequencies, however, the logarithmically spaced signals perform only slightly better than the linearly spaced excitation signal. This is due to the fact that as the damping $\xi$ decreases, the 3 dB bandwidth of the resonance decreases and therefore less points can be fitted in this bandwidth at the lower frequencies. In our case, only 1 line was excited in the 3 dB bandwidth (respectively 0.4 Hz and 0.8 Hz wide) of the systems with the lowest resonance frequencies. It is
clearly unfeasible to estimate the five parameters in (2.74) using this little information in the 3 dB bandwidth of the system. In other words, the resonances of the systems at low frequencies, can no longer be considered to lie within the bulk of the logarithmic part of the quasi-logarithmic frequency grids. This can also be understood from equation (2.70). In that equation, \( \alpha = \alpha^* = 1 + \xi \) (see Section 2.6) such that the following guideline is the necessary condition for the 3 dB bandwidth of the system to lie within the bulk of the frequency grid.

**Guideline 2.2: Use a sufficient grid resolution for each resonance**

A quasi-logarithmic multisine with frequency resolution \( \Delta f \) and tuned frequency spacing \( \alpha = 1 + \xi \) is only guaranteed to properly excite resonance peaks with damping \( \xi \) and resonance frequency \( \omega_n \) when

\[
2\pi \Delta f \leq (\xi - \xi^2)\omega_n \approx \xi \omega_n \text{ if } \xi \ll 1. \tag{2.75}
\]

**Remark 2.13.** Guideline 2.2 can be interpreted either as

- a lower limit on the allowable \( \omega_n \) of the system (when the multisine has been designed (\( \Delta f \) and \( \xi \) are fixed)), or
- as a requirement on the measurement time \( T_{\text{meas}} = (\Delta f)^{-1} \) when lower bounds on \( \xi \) and \( \omega_n \) are known.

For this simulation, with \( \Delta f = 1 \text{ Hz}, \xi = 0.05 \), (2.75) evaluates to \( \omega_n \geq 132 \text{ rad/s} = 21 \text{ Hz} \). The system with \( \omega_{R1} = 20 \text{ Hz} \) is still well-identified, even though 'only' 3 lines are excited in its bandwidth. In that particular case, the frequency grid and the resonance peak are well-aligned (\( \beta \approx 0 \)), which leads to a reduced variance (see also Figure 2.7).

We do see, however, that the variance in the vicinity of the resonance of most systems is lower than in the case of a linearly spaced excitation.

### 2.7.4 Simulation Example on Higher Order Systems

To show that the suggested method also works on more general systems, we considered a system that consists of two second order systems (2.74) in parallel. This is equivalent to adding their respective transfer functions, therefore, the resulting transfer function is of the form:

\[
G(z) = \frac{b_4 z^4 + b_3 z^3 + b_2 z^2 + b_1 z + b_0}{z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0}. \tag{2.76}
\]
2.8 Measurements

2.8.1 Setup

To show that this methodology also works for more complicated systems, we measured the frequency response functions of a Brüel & Kjær Type 1613 Octave Filter Set using a National Instruments Elvis II board driven by LabVIEW®.
The Brüel & Kjær 1613 contains a set of eleven analog passive band-pass filters with center frequencies \( f_c \) that are logarithmically distributed between 31.5 Hz and 31.5 kHz inclusive. According to its manual (Brüel & Kjær, 1970), a 0.5 dB ripple with three resonances is present in the pass-band and the 3 dB break frequencies are located at \( f_c \sqrt{2} \) and \( f_c \sqrt{2} \). This leads to a sixth-order Chebyshev band-pass filter of Type I (Zverev, 1967) with transfer function

\[
G_{\text{init}}(s) \approx \frac{0.15922s^3_n}{s^6_n + 0.759s^5_n + 3.564s^4_n + 1.678s^3_n + 3.564s^2_n + 0.759s_n + 1}
\]

(2.77)

where \( s_n \triangleq \omega_c^{-1} \) and \( \omega_c = 2\pi f_c \) is the center frequency of the respective filter. It can be seen from Table 2.2 that each pole has \( \xi \geq 0.09 \). Since the different pole pairs that constitute such a filter have a different damping, this is an ideally suited system to evaluate the proposed excitation signals. The ‘shape’ of the transfer function of the different filters is nominally identical, which allows for an easy visual comparison between the model uncertainty of the different filters.

As excitation signals, a linear grid multisine, an uncompensated quasi-log multisine and pink quasi-log multisine with excited frequencies between 1 Hz and 64 kHz were used. These multisines were sampled at \( f_s = 256 \) kHz and had a frequency resolution of \( \Delta f = 1 \) Hz, a frequency spacing \( \alpha = 1.05 \) and identical RMS values of approximately 0.25 V. This amplitude was chosen such that the peak amplitude \( \max |u(t)| \) was approximately 1 V. An attempt was made to use a random-phase multisine with a limited crest factor

\[
Cr(u) \triangleq \frac{\|u(t)\|_{\infty}}{\|u(t)\|_2} = \frac{\max |u(t)|}{\sqrt{\int_0^{T_{\text{meas}}} \int_0^{T_{\text{meas}}} u(t)^2 dt}}
\]

(2.78)

to avoid excessive effects of nonlinear behavior of the filter. In particular, for each signal 1000 realizations of the phase spectrum \( \phi_k \) were sampled from a uniform distribution over \([0, 2\pi]\); from these realizations only the multisine with the lowest crest factor was retained and applied to the filters.

\( R = 50 \) periods of each multisine were measured, from which the first period was discarded to suppress transient effects. For each of the remaining periods, the FRF of the

---

Table 2.2: Poles of the initial model of the Brüel & Kjær 1613 filter around \( \omega_c \).

<table>
<thead>
<tr>
<th>POLE PAIR</th>
<th>DAMPING (( \xi ))</th>
<th>FREQUENCY (( \omega_n ))</th>
<th>TIME CONSTANT (( \tau ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.067 ± 0.733j) ( \omega_c )</td>
<td>0.091</td>
<td>0.74 ( \omega_c )</td>
<td>15.0 ( \omega_c^{-1} )</td>
</tr>
<tr>
<td>(0.190 ± 0.982j) ( \omega_c )</td>
<td>0.190</td>
<td>( \omega_c )</td>
<td>5.3 ( \omega_c^{-1} )</td>
</tr>
<tr>
<td>(0.123 ± 1.350j) ( \omega_c )</td>
<td>0.091</td>
<td>1.36 ( \omega_c )</td>
<td>8.1 ( \omega_c^{-1} )</td>
</tr>
</tbody>
</table>
filter is calculated and its average $G(\omega)$ over the periods is calculated. We also calculate the corresponding standard deviation $\sigma_G(\omega)$ of the FRF and normalize it for a single period of the excitation signal. We denote $G(\omega)$ the non-parametric FRF and $\sigma_G(\omega)$ its standard deviation, as these measures are determined completely nonparametrically from the measurements.

For the parametric model, on the other hand, a continuous time LTI model of the form

$$G(s) = \frac{\sum_{i=0}^{3} b_i s^i}{\sum_{i=0}^{6} a_4 s^i}$$

(2.79)

was estimated using elis (Kollár et al., 1994–2012). To aid convergence to a good optimum, the initial estimate was a sixth-order band-pass type I Chebyshev filter given in (2.77).

We summarize the behavior of these parametric models by their average $G(\theta, \omega)$ over the different periods and their standard deviation $\sigma_G(\theta, \omega)$.

### 2.8.2 Results

In Figure 2.14 the nonparametric FRF and the standard deviation of the corresponding parametric estimates are shown in a similar way as for the simulations in the previous section.

The uncertainty $\sigma_G(\theta, \omega)$ on the parametric model in its pass-band is observed to be nearly constant for most of the filters when excited by the (pink) quasi-log multisine. For the linear grid multisine excitation, $\sigma_G(\theta, \omega)$ in the pass-band is clearly decreasing proportional to $1/f$ for the different octave filters. One can see that the performance of both uncompensated quasi-log and pink quasi-log multisines is practically identical. This can be explained by the fact that in this experiment, $f_0$ was chosen low enough to ensure that all the second-order sub-systems had their 3 dB bandwidth within the bulk of logarithmic part of the excited grid. This observation also shows that in this practical application, the increase in uncertainty between the uncompensated and compensated quasi-log multisines is negligible, as was observed earlier in the simulations. However, if the system under test would have a lower resonance frequency, a quasi-log multisine with power compensation would perform better as was seen in the simulations.
# Chapter 2 Design of Excitation Signals

## 2.9 Conclusion

In this chapter, we recalled that the design of optimal excitation signals may require prior knowledge that is either not sufficiently accurate or even unavailable. When only the limited prior knowledge that the system resonances lie in a certain (possibly very wide) frequency band, a suitable class of excitation signals can be constructed such that the relative uncertainty of the estimated resonance is independent of the resonance frequency. We showed that such a signal should have a power spectrum that is distributed in \(1/f\).

In this chapter we have illustrated that quasi-logarithmic multisine excitations are well-suited to measure transfer functions over broad frequency ranges with a constant relative uncertainty of the present resonances. The suggested amplitude compensation methods allow the user to measure a parametric LTI model for systems in an even wider frequency band with a constant uncertainty. The variance of the estimates at lower frequencies is improved in comparison to the uncompensated case at the cost of a slightly increased (yet constant) variance at higher frequencies for excitation signals with an equal power.
content. In practice, this slight increase is likely to be negligible. The suggested compensation works well as long as the considered systems are sufficiently damped with respect to the density of the excited frequency lines.

We have also shown an effective way to choose the frequency spacing of a (quasi)-logarithmically spaced multisine for an equi-damped set of systems. The presented method allows to select a maximal value for the frequency ratio $\alpha = 1 + \xi$ for which the performance will not suffer from the unavoidable misalignment of the system with respect to the excited frequencies. In practice we advise to use an excitation signal with approximately four excited lines within the 3 dB bandwidth of each second-order sub-system.

Remark 2.14. The proposed design can be adapted quite straightforwardly to also account for the high-frequency regions of e.g. plates by increasing the number of frequency lines where the modal overlap becomes more important. In practice, this will boil down to linearly spaced multisines. For measurement of the transfer characteristics of volumes (e.g. rooms), more specialized techniques are required to describe the high-frequency behavior such as the ones discussed by Ege et al., (2009).

<table>
<thead>
<tr>
<th>Guideline 2.3: Use quasi-logarithmic multisines as initial excitation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quasi-logarithmic multisines with pink-colored amplitude spectrum facilitate the measurement of multiple sharp resonance peaks over a wide frequency band of interest with a constant quality when the modal overlap remains limited.</td>
</tr>
</tbody>
</table>
Chapter 3

Non-parametric Modeling

I love deadlines. I like the whooshing sound they make as they fly by.

DOUGLAS ADAMS

3.1 Introduction

Whereas many areas of system identification focus on parametric identification and/or parameter estimation, non-parametric techniques have received renewed interest from the community over the last decade. One of the reasons for this is that non-parametric techniques typically require fewer assumptions and knowledge of the system than building parametric models. As a consequence, non-parametric methods are often more robust than fully parametric methods. Besides, frequency response functions (FRFs) have been shown to provide a quick insight into the basic dynamic properties of linear time-invariant (LTI) systems. This capability is very useful for system design, manual controller tuning, model validation and model selection for parametric models (Pintelon and Schoukens, 2012).

The measurement of FRFs of dynamic systems is an important step in many technical applications, often used as a simple visualization of the system dynamics. However, obtaining a good non-parametric FRF from a measured input-output data set can be challenging due to the presence of noise, leakage and measurement time limitations to observe lightly-damped dynamics well. FRF measurement techniques are discussed, for instance, in (Antoni and Schoukens, 2007; Bendat, 1978; Bendat and Piersol, 1993, 2010; Blackman and Tukey, 1958; Box et al., 2008; Broersen, 1995; Guillaume, Kollár, et al., 1996; Jenkins and Watts, 1968; Pintelon and Schoukens, 2012; Pintelon et al., 2010a; Schoukens et al., 1998, 2006), and applied to practical devices and systems (Behjat et al., 2010; Lim, 2010; Robinson et al., 1990; White, 1969), among others.
In this chapter, we will investigate local modeling techniques to capture the FRF of resonant systems. In particular, a comparison between the Local Polynomial Method (LPM) and Local Rational Method (LRM) will be made such that it can be decided which method is to be preferred over the other in what conditions of signal to noise ratio (SNR) and frequency resolution (or, equivalently, measurement time).

Outline  Section 3.2 introduces theory of estimating FRFs using local modeling techniques. In Section 3.3, the bias of the LRM is derived. In Section 3.4 the performance of the different modeling methods is compared by means of simulations. In Section 3.5.1, some options for model order selection are discussed. In Section 3.6, an approach is developed to further reduce the variance of the LPM estimate by truncation of the impulse response. Finally, conclusions are presented in Section 3.7.

3.2 Local Modeling Approaches

Consider a discrete-time generalized output-error LTI set-up, excited by the input signal $u[n]$. For an infinite data length, such a system can be described in the time domain by

$$ y[n] = G(q)u[n] + H(q)e[n] $$

where $v[n]$ is filtered white noise, $v[n] = H(q)e[n]$ where $q^{-1}$ is the backward shift operator, i.e. $q^{-1}x[n] = x[n - 1]$. The transfer functions $G(q)$ and $H(q)$ are rational functions.

During typical measurements, however, $y[n]$ and $u[n]$ are only measured over a limited time span, i.e. $n \in \{0, 1, \ldots, N - 1\}$. This introduces transient terms $t_*$ in this relationship (Pintelon, Schoukens, and Vandersteen, 1997):

$$ y[n] = G(q)u[n] + H(q)e[n] + t_G[n] + t_H[n] $$

where $t_G[n]$ and $t_H[n]$ are due to the different conditions of respectively $G(q)$ and $H(q)$ at the beginning and end of the measurement record (Pintelon, Schoukens, and Vandersteen, 1997). Both terms can be lumped together as $t[n]$, which is often (Pintelon et al., 2010a) determined predominantly by $t_G[n]$.

**Definition 3.1.** The $N$-points discrete Fourier transform (DFT) of a signal $x(nT_s) = x[n]$ with $n \in \{0, \ldots, N - 1\}$ is

$$ X[k] = X(\omega_k) \triangleq \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(nT_s) e^{-j\omega_k nT_s} $$

where \( \omega_k \triangleq \frac{2\pi k}{NT_s}, k \in \{0, \ldots, N - 1\} \) and \( T_s \) is the sampling time.

By applying the discrete Fourier transform (DFT) (3.3) to both sides of (3.2), one obtains its frequency-domain counterpart:

\[
Y(\omega_k) = G_o(\omega_k)U(\omega_k) + H_o(\omega_k)E_o(\omega_k) + T(\omega_k)
\]

(3.4)

\[
= G_o(\omega_k)U(\omega_k) + V(\omega_k) + T(\omega_k).
\]

(3.5)

Here, we focus on using local modeling to separate the different terms, i.e.

- the transfer function \( G_o \),
- the leakage term \( T \), and
- the noise term \( V \).

The main terms of interest are the transfer function \( G_o \) and the noise variance \( \sigma_V^2 \).

### 3.2.1 Local Modeling

Local modeling methods exploit the ‘smoothness’ (or conversely ‘roughness’) of the different terms of (3.5) over the frequency \( \omega \). In particular, we will assume that \( U(\omega) \) is ‘rough’ over the frequency (Schoukens, Vandersteen, Barbe, et al., 2009), e.g. as is the case for noise excitations or random phase multisines (see further). On the other hand, it is well-known that the transfer functions \( G_o, H_o \) and hence also the corresponding transient contributions \( T = T_G + T_H \) are smooth over the frequency (see Appendix 3.A). As such, these smooth contributions can be approximated well around each frequency \( \omega_k \) using a local rational model:

\[
G_o(\omega_k + d) \approx \frac{\sum_{i=0}^{N_B} b_i(k)d^i}{1 + \sum_{i=1}^{N_A} a_i(k)d^i} \triangleq \frac{\tilde{B}(d)}{\tilde{A}(d)} = \tilde{G}_d(k),
\]

(3.6)

\[
T_o(\omega_k + d) \approx \frac{\sum_{i=0}^{N_C} c_i(k)d^i}{1 + \sum_{i=1}^{N_A} a_i(k)d^i} \triangleq \frac{\tilde{C}(d)}{\tilde{A}(d)} = \tilde{T}_d(k).
\]

(3.7)
In these expressions, we denote $\tilde{G}_k$ and $\tilde{T}_k$ for the local model for respectively transfer function and the transient. Note that such local quantities (which depend on the frequency bin $k$) are denoted in bold. To alleviate the notations, the subscript $k$ is omitted whenever this does not cause ambiguity.

To estimate the local parameters $\theta$

\[
\theta \triangleq \begin{bmatrix} \theta_A \\ \theta_B \\ \theta_C \end{bmatrix} \quad \text{with} \quad \theta_A \triangleq \begin{bmatrix} a_1 \\ \vdots \\ a_{N_A} \end{bmatrix}, \quad \theta_B \triangleq \begin{bmatrix} b_0 \\ \vdots \\ b_{N_B} \end{bmatrix}, \quad \text{and} \quad \theta_C \triangleq \begin{bmatrix} c_0 \\ \vdots \\ c_{N_C} \end{bmatrix}
\] (3.8)

in (3.6) and (3.7), we consider a local window

\[
\Omega_k \triangleq \{ \omega_{k+r} \mid r \in \mathcal{R}_k \}
\] (3.9)

with $\mathcal{R}_k \triangleq \{-n_W, -n_W + 1, \ldots, 0, \ldots, n_W\}$ such that the local window around $\omega_k$ consists of $n_W = 2n_W + 1$ bins. If one denotes the input/output spectra in such a local frequency window $\Omega_k$ as

\[
U_k \triangleq \begin{bmatrix} U(\omega_k - n_W) \\ \vdots \\ U(\omega_k) \\ \vdots \\ U(\omega_k + n_W) \end{bmatrix} \quad \text{and} \quad Y_k \triangleq \begin{bmatrix} Y(\omega_k - n_W) \\ \vdots \\ Y(\omega_k) \\ \vdots \\ Y(\omega_k + n_W) \end{bmatrix}
\] (3.10)

and similarly for the other quantities, equation (3.5) limited to $\Omega_k$ can be written as

\[
Y_k = G_k \circ U_k + T_k + V_k,
\] (3.11)

where $\circ$ denotes the element-wise product (also, Hadamard product). Substituting $G_\circ$ and $T$ with the local models $\tilde{G}$ and $\tilde{T}$ and neglecting the influence of $V$ yields

\[
Y \approx \tilde{Y} = \tilde{G} \circ U + \tilde{T}.
\] (3.12)

Note that this encompasses $n_W$ complex equations in the $n_\theta = N_A + N_B + N_C + 2$ unknown complex model parameters ($a_i$, $b_i$ and $c_i$ in (3.6) and (3.7)). Consequently, a necessary condition to compute the model parameters is that

\[
\text{DOF} = n_W - n_\theta = 2n_W - N_A - N_B - N_C - 1
\] (3.13)

is positive. To effectively compute the local model parameters, the equation error in (3.12) is minimized by formulating a quadratic cost function:

\[
L_{\text{LRIC}} \triangleq (Y - \tilde{G} \circ U - \tilde{T})^H (Y - \tilde{G} \circ U - \tilde{T}).
\] (3.14)
Due to the presence of the denominator of $\tilde{G} = \frac{\tilde{B}}{\tilde{A}}$, the equation error is not linear in the parameters $a_i$ and hence (3.14) requires starting values and time-consuming iterative optimization schemes to obtain a parameter estimate. We hence call this method the Local Rational method with Iterative Cost function (LRIC) and denote specific configurations of this method as LRIC $(n_W, N_B, N_A, N_C)$.

**Remark 3.1.** The starting values for the LRIC are obtained by means of the LRM, the details of which are explained in Section 3.2.2.

Note that the estimated FRF is given by evaluating the local model $\tilde{G}$:

$$\hat{G}_{\text{LRIC}}(\omega_k) \triangleq \tilde{G}_k(d = 0).$$

(3.15)

In Figure 3.1, the local window $\Omega_k$, the local input-output spectra ($U_k$ and $U_k$) are illustrated.

**Figure 3.1:** Illustration of local modeling. Top figures: input spectrum (left) and output spectrum (right). A small frequency window $\Omega_k$ is selected, in which the local model $\tilde{G}_k$ is estimated. Its central value $\hat{G}(\omega_k)$ is an estimate of the FRF. This procedure is repeated for all frequencies $\omega_k$. *Figure based on (Lumori et al., 2014, Fig. 2).*

**Remark 3.2.** It should be noted that (3.6) and (3.7) are essentially isomorphic to a continuous-time transfer function model with complex coefficients. The discrete-time counterparts...
(i.e. substituting \( d^n \rightarrow \exp(jdTs) \) in the expressions), have been tried as well. However, these preliminary experiments did not yield estimates that were numerically reliable. This is in correspondence with the remark from Pintelon and Schoukens, (2006, Section 2.4).

### 3.2.2 The Local Rational Method

The LRM as first introduced by McKelvey and Guérin, (2012), overcomes the computational burden of an iterative procedure by weighting the equation error by the denominator polynomial \( \tilde{A} \) akin to the procedure in (Levy, 1959). I.e. the LRM procedure tries to minimize the equation error in

\[
\tilde{A} \circ Y = \tilde{B} \circ U + \tilde{C} + V,
\]

for which the corresponding cost function is

\[
L_{\text{LRM}} \triangleq \left( \tilde{A} \circ Y - \tilde{B} \circ U - \tilde{C} \right)^H \left( \tilde{A} \circ Y - \tilde{B} \circ U - \tilde{C} \right).
\]

where \( V \) is vector consisting of independent and identically distributed (i.i.d.) complex normally distributed variables (Gallager, 2008) that each have a variance \( \sigma^2_V \), i.e. the disturbing noise is assumed white over the local frequency window. Equivalently, the last equation can be rewritten as a linear regression problem

\[
Y = K\theta + V
\]

where \( K \) is the so-called design matrix (or observation matrix):

\[
K \triangleq \begin{bmatrix} K_A & K_B & K_C \end{bmatrix}
\]

\[
K_A \triangleq \begin{bmatrix} Y \circ d^1 & \cdots & Y \circ d^{N_A} \end{bmatrix}
\]

\[
K_B \triangleq -\begin{bmatrix} U \circ d^0 & \cdots & U \circ d^{N_B} \end{bmatrix}
\]

\[
K_C \triangleq -\begin{bmatrix} d^0 & \cdots & d^{N_C} \end{bmatrix}.
\]

In this formulation, we have used \( d^n \) to denote the \( n^{\text{th}} \) Hadamard power of \( d \) (this corresponds to \( d \cdot d^n \) in MATLAB®) such that

\[
d^n \triangleq \begin{bmatrix} (\!-n_W)^n & \cdots & (n_W)^n \end{bmatrix}^T \text{ with } n \in \mathbb{Z}
\]

where every element corresponds to a value of \( d^n \) in accordance with \( R_k \).

This formulation facilitates to solve the problem in a one-step approach:

\[
\hat{\theta}_{\text{LRM}} \triangleq K^+ Y = \left( K^H K \right)^{-1} K^H Y
\]
3.2 Local Modeling Approaches

where $K^+$ denotes the Moore-Penroose pseudo-inverse of $K$. Furthermore, it is possible to define the relationship between the measured output spectrum $Y$ and the periodic output spectrum $\hat{Y}$:

$$\hat{Y} = K\theta_{LRM} = KK^+Y = HY$$  \hspace{1cm} (3.25)

where $H$ is sometimes called the ‘hat’ matrix since it projects measurements ($Y$) onto the estimates ($\hat{Y}$). It should be noted that the hat matrix is idempotent (Cook and Weisberg, 1982, Section 2.1.1).

Following the approach in Pintelon et al., (2010a, equation (12) and further) or Cook and Weisberg, (1982, Chapter 2), the (ordinary) residuals $E$ of the fit are given by

$$E = Y - \hat{Y} = Y - HY = (I_n - H)Y.$$  \hspace{1cm} (3.26)

Substitution of (3.18) in this equation yields

$$E = (I - H)(K\theta + V)$$  \hspace{1cm} (3.27)

$$= (I - H)V + (I - H)K\theta$$  \hspace{1cm} (3.28)

$$= (I - H)V + K\theta - KK^+K\theta$$  \hspace{1cm} (3.29)

$$= (I - H)V$$  \hspace{1cm} (3.30)

since $KK^+K = K$ by construction of the pseudo-inverse (Penrose, 1955). This equation relates the residuals $E$ to the noise $V$, which aids to estimate the noise variance:

$$\hat{\sigma}_V^2 = \frac{1}{\text{DOF}} E^HE$$  \hspace{1cm} (3.31)

as elaborated in (Pintelon et al., 2010a, Appendix B). Note that DOF indicates the degrees of freedom of the residuals as in (3.13) and could also be computed as either the rank or the trace of the idempotent matrix $(I - H)$.

Remark 3.3. This system can be solved reliably only if the columns in (3.19) are linearly independent. In practice, this corresponds to the requirements on the input spectrum as stated in Schoukens, Vandersteen, Barbé, et al., (2009): the input spectrum should be sufficiently ‘rough’ to separate the transient contribution from the contribution $\tilde{G}\tilde{U}$. This can be obtained e.g. using random phase multisines or random input signals.

Remark 3.4. Since it is well-known that the Vandermonde structure in the design matrix (3.19) leads to numerical ill-conditioning for higher model complexities, additional measures are taken to improve numerical conditioning. To improve numerical conditioning (Pintelon and Kollár, 2005) of the estimation problem, we substitute $d = \delta_{rk}$ in equations (3.6) and (3.7)

$$\delta_{rk} \triangleq \frac{\omega_{k+r} - \omega_k}{\Delta\omega_k}$$  \hspace{1cm} (3.32)

where

$$\Delta\omega_k \triangleq \max \left\{ |\omega_k - \omega_j| : \omega_j \in \Omega_k \right\}$$  \hspace{1cm} (3.33)

such that $|\delta_{rk}| \leq 1$ when $r \in R_k$. 

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We denote a method that solves (3.17) as LRM \((n_W, N_B, N_A, N_C)\) where \(n_W\) is the half-bandwidth, and \(N_B, N_A\) and \(N_C\) are the local model orders.

### 3.2.3 The Local Polynomial Method

The LPM is a method that predates the LRM and has been devised by Schoukens et al., (2006). The LPM approximates both the FRF and transient contribution by means of complex-valued local polynomials in the frequency domain.

The LPM can be understood as a specific case of the LRM: for \(N_A = 0\), the LRM reduces to the LPM. As such, we use the notation LPM \((n_W, N_B, 0, N_C)\) as a synonym for LRM \((n_W, N_B, N_A, N_C)\). In the remainder of this text, we refer to LRM as a group of methods which have \(N_A \neq 0\) to make the distinction with LPM more obvious. For more information regarding LPM, we refer to e.g. Gevers, Pintelon, et al., (2011), Pintelon et al., (2010a,b), and Schoukens, Vandersteen, Barbé, et al., (2009).

**Remark 3.5.** The trivial setting for local ‘modeling’ where a window with a total width of 1 bin and no transient estimation \((N_C = -1)\) are used, corresponds to the simple empirical transfer function estimate (ETFE) (Broersen, 1995; Stenman and Gustafsson, 2001; Stenman, Gustafsson, et al., 2000) when no further smoothing or windowing is applied:

\[
\hat{G}_{\text{ETFE}}(\omega_k) \triangleq \frac{Y(\omega_k)}{U(\omega_k)} \quad (3.34)
\]

As such, ETFE ≡ LRM \((0, 0, 0, -1)\) ≡ LPM \((0, 0, -1)\).

**Remark 3.6.** The methods proposed by Stenman and Gustafsson, (2001) and Stenman, Gustafsson, et al., (2000) also refer to ‘local polynomial regression’ to smoothen the FRF. It should be noted, however, that such methods differ in a few aspects from the LPM. Specifically, those methods operate directly on the raw ETFE, in contrast to the LPM which uses the input-output spectra and can hence estimate both an FRF and a transient contribution. Moreover, the methods proposed by Stenman and Gustafsson, (2001) build upon ‘local regression’ approaches such as locally weighted scatterplot smoothing (LOWESS). See e.g. Loader, (1999) for more information regarding these ‘local regression’ approaches.

### 3.3 Derivation of the Bias of the LRM

Since the local design matrix \(K\), see (3.19), of the LRM contains the noisy output spectrum \(Y\) when \(N_A > 0\), the LRM is expected to be a biased estimator. In this section, we will
derive expressions for the bias such that it can be quantified how important this bias is in different situations.

To determine the bias, an approach similar to (Guillaume, Pintelon, et al., 1995, Appendix A) is followed.

Let us denote the expected value of the cost function (3.17) over all measurements $Z = [Y, U]$ as

$$E_{LS}(\theta) \triangleq E_Z [L_{LRM}(\theta, Z)]$$

$$E_{LS0}(\theta) \triangleq \frac{1}{nW} \sum_\delta \left| \tilde{\tilde{A}}(\delta, \theta) Y_\circ \right| \left| \tilde{\tilde{B}}(\delta, \theta) U_\circ \right| \left| C(\delta, \theta) \right|^2$$

$$E_{LSn}(\theta) \triangleq \frac{1}{nW} \sum_\delta \left| \tilde{\tilde{A}}(\delta, \theta) \right|^2 \sigma_v^2$$

where $U_\circ$ and $Y_\circ$ denote the ‘true’ (noiseless) input-output spectra in the local window. Note that this function is a quadratic function of $\theta$ and hence it can be represented exactly by its second order Taylor series around $\theta_\circ$:

$$E_{LS}(\theta) = E_{LS}(\theta_\circ) + \frac{\partial E_{LS}(\theta_\circ)}{\partial \theta_\circ} (\theta - \theta_\circ) + \frac{1}{2} (\theta - \theta_\circ)^T \frac{\partial^2 E_{LS}(\theta_\circ)}{\partial \theta_\circ^2} (\theta - \theta_\circ).$$

We use the notation

$$\frac{\partial E_{LS}(\theta_\circ)}{\partial \theta_\circ} \triangleq \left. \frac{\partial E_{LS}(\theta)}{\partial \theta} \right|_{\theta = \theta_\circ}$$

for the derivative of $E_{LS}(\theta)$ with respect to $\theta$, evaluated in $\theta_\circ$. The minimizer $\hat{\theta}$ of this function can be obtained by equating the derivative

$$\left( \frac{\partial E_{LS}(\hat{\theta})}{\partial \theta} \right)^T = \left( \frac{\partial E_{LS}(\theta_\circ)}{\partial \theta_\circ} \right)^T + \frac{\partial^2 E_{LS}(\theta_\circ)}{\partial \theta_\circ^2} (\hat{\theta} - \theta_\circ)$$

to zero. This is equivalent to

$$\left( \hat{\theta} - \theta_\circ \right) = - \left( \frac{\partial^2 E_{LS}(\theta_\circ)}{\partial \theta_\circ^2} \right)^{-1} \left( \frac{\partial E_{LS}(\theta_\circ)}{\partial \theta_\circ} \right)^T.$$

which expresses the bias $\Delta \theta = \left( \hat{\theta} - \theta_\circ \right)$ of the estimated parameters. Remark that

$$\frac{\partial E_{LS}(\theta_\circ)}{\partial \theta_\circ} = \frac{\partial E_{LS0}(\theta)}{\partial \theta_\circ} + \frac{\partial E_{LSn}(\theta)}{\partial \theta_\circ} = \frac{\partial E_{LSn}(\theta)}{\partial \theta_\circ}$$

(3.43)
since $E_{LS0}(\theta)$ is minimized in $\theta_0$ by definition. As such, the bias on the parameters can be simplified to

$$
\Delta \theta = - \left( \frac{\partial^2 E_{LS}(\theta)}{\partial \theta^2} \right)^{-1} \left( \frac{\partial E_{LS} \theta}{\partial \theta} \right)^\top \cdot (3.44)
$$

Note that the estimated parameters $\theta \in C^{n_\theta \times 1}$ but that $E_{LS}(\theta)$ is a real-valued function. In contrast to a complex-valued function, this requires additional attention when the derivatives are calculated (Messerschmitt, 2006) and (Pintelon and Schoukens, 2012, Section 15.9) since $E_{LS}(\theta)$ is not an analytic function. Particularly, the function $E_{LS}(\theta)$ is regarded as a two-dimensional function $E_{LS}(\theta, \bar{\theta})$ where $\theta$ and $\bar{\theta}$ are considered as independent variables (Hjørungnes, 2011; Hjørungnes and Gesbert, 2007). The derivatives are then computed with respect to these variables.

To facilitate the derivations, we rewrite the expected value of the cost function in terms of $\theta$ and $\bar{\theta}$:

$$
E_{LS0}(\theta, \bar{\theta}) = \frac{1}{n \bar{\theta} V} \sum_\delta \bar{Q}(\delta, \theta, Y_o, U_o) \bar{Q}(\delta, \theta, Y_o, U_o) (3.45)
$$

$$
\bar{Q}(\delta, \theta, Y_o, U_o) \triangleq \bar{A}(\delta, \theta) Y_o(\delta) - \bar{B}(\delta, \theta) U_o(\delta) - \bar{C}(\delta, \theta) (3.46)
$$

$$
E_{LSn}(\theta, \bar{\theta}) = \frac{\sigma^2}{n \bar{\theta} V} \sum_\delta \bar{A}(\delta, \theta) \bar{A}(\delta, \bar{\theta}) (3.47)
$$

since $\bar{A}(\delta, \theta) = \bar{A}(\delta, \bar{\theta})$, and similar for $\bar{B}$ and $\bar{C}$ and $\bar{Q}$.

**Contributions to the first order derivative**  The last term in (3.44), i.e. the first order derivative, is given by

$$
\frac{\partial E_{LSn}(\theta, \bar{\theta})}{\partial \theta} = \left[ \frac{\partial E_{LSn}(\theta, \bar{\theta})}{\partial \theta_A} \frac{\partial E_{LSn}(\theta, \bar{\theta})}{\partial \theta_B} \frac{\partial E_{LSn}(\theta, \bar{\theta})}{\partial \theta_C} \right] (3.48)
$$

and split into the parts pertaining to respectively $\bar{A}$, $\bar{B}$ and, $\bar{C}$. The respective components are given by:

$$
\frac{\partial E_{LSn}(\theta, \bar{\theta})}{\partial a_i} = \frac{\sigma^2}{n \bar{\theta} V} \sum_\delta \delta^i \bar{A}(\delta, \bar{\theta}) \quad \forall i \in \{1, \ldots, N_A\} (3.49)
$$

$$
\frac{\partial E_{LSn}(\theta, \bar{\theta})}{\partial b_i} = 0 \quad \forall i \in \{0, \ldots, N_B\} (3.50)
$$

$$
\frac{\partial E_{LSn}(\theta, \bar{\theta})}{\partial c_i} = 0 \quad \forall i \in \{0, \ldots, N_C\} . (3.51)
$$
It can be seen that only the factors pertaining to $\theta_A$ are present. As this term occurs linearly in expression (3.44), only $\theta_A$ can incur a bias. Consequently, only the poles can be shifted by the presence of output noise. As such, the bias term $\Delta \theta$ can be written in the following sparse form:

$$
\Delta \theta = \begin{bmatrix}
\Delta \theta_A \\
\Delta \theta_B \\
\Delta \theta_C
\end{bmatrix} = \begin{bmatrix}
\Delta \theta_A \\
0 \\
0
\end{bmatrix}.
$$

**Contributions to the Hessian**  Computing the Hessian in (3.44) also boils down to recognizing the block structure. This results in the following block matrix:

$$
\frac{\partial^2 E_{LS}(\theta)}{\partial \theta^2} = \begin{bmatrix}
\frac{\partial^2 E_{LS}(\theta)}{\partial \theta_A \partial \theta_A} & \frac{\partial^2 E_{LS}(\theta)}{\partial \theta_A \partial \theta_B} & \frac{\partial^2 E_{LS}(\theta)}{\partial \theta_A \partial \theta_C} \\
\frac{\partial^2 E_{LS}(\theta)}{\partial \theta_B \partial \theta_A} & \frac{\partial^2 E_{LS}(\theta)}{\partial \theta_B \partial \theta_B} & \frac{\partial^2 E_{LS}(\theta)}{\partial \theta_B \partial \theta_C} \\
\frac{\partial^2 E_{LS}(\theta)}{\partial \theta_C \partial \theta_A} & \frac{\partial^2 E_{LS}(\theta)}{\partial \theta_C \partial \theta_B} & \frac{\partial^2 E_{LS}(\theta)}{\partial \theta_C \partial \theta_C}
\end{bmatrix}
$$

where every sub-block consists of the different second-order derivatives and exhibits a similar pattern.

The different contributions are given below:

$$
\frac{\partial^2 E_{LS}(\theta)}{\partial a_{i_1} \partial a_{i_2}} = \frac{1}{nW} \sum_{\delta} \delta^{i_1+i_2} \left| Y_o(\delta) \right|^2
$$

$$
\frac{\partial^2 E_{LS}(\theta)}{\partial a_{i_1} \partial b_{i_2}} = \frac{1}{nW} \sum_{\delta} \delta^{i_1+i_2} \left| U_o(\delta) \right|^2
$$

$$
\frac{\partial^2 E_{LS}(\theta)}{\partial c_{i_1} \partial c_{i_2}} = \frac{1}{nW} \sum_{\delta} \delta^{i_1+i_2}
$$

$$
\frac{\partial^2 E_{LS}(\theta)}{\partial b_{i_1} \partial a_{i_2}} = \frac{1}{nW} \sum_{\delta} \delta^{i_1+i_2} \overline{U_o(\delta)} Y_o(\delta)
$$

$$
\frac{\partial^2 E_{LS}(\theta)}{\partial c_{i_1} \partial a_{i_2}} = \frac{1}{nW} \sum_{\delta} \delta^{i_1+i_2} \overline{Y_o(\delta)} Y_o(\delta)
$$

$$
\frac{\partial^2 E_{LS}(\theta)}{\partial c_{i_1} \partial b_{i_2}} = \frac{1}{nW} \sum_{\delta} \delta^{i_1+i_2} U_o(\delta),
$$

Remark 3.7. Computing the inverse of this Hessian cannot, to the knowledge of the author, be reduced to a form that yields more insight. However, if the non-diagonal
blocks can be neglected, this would lead to an inverse that is also block-diagonal:

\[
\begin{bmatrix}
\frac{\partial^2 E_{\text{LS}}(\theta)}{\partial \theta_A \partial \theta_A} & 0 & 0 \\
0 & \frac{\partial^2 E_{\text{LS}}(\theta)}{\partial \theta_B \partial \theta_B} & 0 \\
0 & 0 & \frac{\partial^2 E_{\text{LS}}(\theta)}{\partial \theta_C \partial \theta_C}
\end{bmatrix}
^{-1} =
\begin{bmatrix}
\left(\frac{\partial^2 E_{\text{LS}}(\theta)}{\partial \theta_A \partial \theta_A}\right)^{-1} & 0 & 0 \\
0 & \left(\frac{\partial^2 E_{\text{LS}}(\theta)}{\partial \theta_B \partial \theta_B}\right)^{-1} & 0 \\
0 & 0 & \left(\frac{\partial^2 E_{\text{LS}}(\theta)}{\partial \theta_C \partial \theta_C}\right)^{-1}
\end{bmatrix}
\] (3.60)

**Bias on the frequency response function**  As a first order approximation, the bias on the FRF can be written as

\[
\Delta \tilde{G}(\omega_k) = \Delta \tilde{G}(\delta_{0k}) \approx \frac{\partial \tilde{G}(\delta_{0k}, \theta)}{\partial \theta} \Delta \theta.
\] (3.61)

The derivative in that expression can be evaluated easily

\[
\frac{\partial \tilde{G}(\delta, \theta)}{\partial \theta} = \left[ \frac{\partial \tilde{G}(\delta, \theta)}{\partial \theta_A} \quad \frac{\partial \tilde{G}(\delta, \theta)}{\partial \theta_B} \quad \frac{\partial \tilde{G}(\delta, \theta)}{\partial \theta_C} \right]
\] (3.62)

\[
\frac{\partial \tilde{G}(\delta, \theta)}{\partial \theta_A} = -\frac{\tilde{B}(\delta, \theta)}{\tilde{A}(\delta, \theta)} \left[ \delta^1 \quad \delta^2 \quad \cdots \quad \delta^{N_A} \right]
\] (3.63)

\[
\frac{\partial \tilde{G}(\delta, \theta)}{\partial \theta_B} = \frac{1}{\tilde{A}(\delta, \theta)} \left[ 1 \quad \delta^1 \quad \delta^2 \quad \cdots \quad \delta^{N_B} \right]
\] (3.64)

\[
\frac{\partial \tilde{G}(\delta, \theta)}{\partial \theta_C} = 0_{1 \times N_C+1}.
\] (3.65)

However, the sparsity of \( \Delta \theta \), see (3.52), can be exploited such that the bias on the FRF can be expressed as:

\[
\Delta \tilde{G}(\delta, \theta) \approx -\frac{\tilde{G}(\delta, \theta)}{\tilde{A}(\delta, \theta)} \left[ \delta^1 \quad \cdots \quad \delta^{N_A} \right] \Delta \theta_A.
\] (3.66)

Equivalently, the relative bias of the FRF can be written as

\[
\frac{\Delta \tilde{G}(\delta, \theta)}{\tilde{G}(\delta, \theta)} \approx -\frac{\tilde{A}(\delta, \theta)^{-1}}{\tilde{A}(\delta, \theta)^{-1} \left[ \delta^1 \quad \cdots \quad \delta^{N_A} \right]} \left( \frac{\partial^2 E_{\text{LS}}(\theta_\circ)}{\partial \theta^2} \right)_A^{-1} \frac{\partial E_{\text{LSn}}(\theta_\circ)}{\partial \theta_A}\n^T
\]

\[
= -\sigma_n^2 \left[ \delta^1 \quad \cdots \quad \delta^{N_A} \right] \left( \frac{\partial^2 E_{\text{LS}}(\theta_\circ)}{\partial \theta^2} \right)_A^{-1} \left[ \sum_\delta \delta^1 \frac{\tilde{A}(\delta, \theta)}{\tilde{A}(\delta, \theta)} \quad \vdots \quad \sum_\delta \delta^{N_A} \frac{\tilde{A}(\delta, \theta)}{\tilde{A}(\delta, \theta)} \right]
\] (3.68)
3.4 Simulations

where \( (\frac{\partial^2 E_{LS}(\theta_0)}{\partial \theta_0^2})^{-1}_A \) denotes the upper left \( N_A \times N_A \) sub-block of \( (\frac{\partial^2 E_{LS}(\theta_0)}{\partial \theta_0^2})^{-1} \) which pertains to the parameters of \( \hat{A} \). From the leading factor, it can be seen that the bias on the FRF is \( O(\sigma_Y^2) \). Consequently, one can expect that the bias is inherently linked to the SNR of the output signal in the frequency domain.

**3.4 Simulations**

In the simulations, we consider a discrete-time second order system with transfer function

\[
G_0(z) = \frac{0.64587z + 0.64143}{47.9426z^2 - 51.2955z + 46.9933},
\]

(3.69)

this is a resonant system with a resonance near \( \omega = 1 \) rad/sample, \( \xi = 0.05 \). The system is scaled such that its peak amplitude is 3 dB and hence its 3 dB bandwidth can be read easily from bode plots. As indicated in Figure 3.2, the noise filter \( H'_0 = 1 \) is used such that white noise is disturbing the output. The gain \( \kappa \) of this filter is adapted such that the SNR in the 3 dB bandwidth of the system, i.e.

\[
\text{SNR}_{BW} \triangleq \frac{\int_{BW} \phi_{Y_0}(\omega) d\omega}{\int_{BW} \phi_V(\omega) d\omega}
\]

(3.70)

where \( \phi_\bullet \) denotes the power spectral density (PSD) of \( \bullet \) and BW is the 3 dB bandwidth of the system, is fixed to

\[
\text{SNR}_{BW} \in \{10, 20, 40, 60, \infty\} \text{ dB}.
\]

This ensures that the excited bins in the 3 dB bandwidth of the system each receive a pre-defined SNR.

The input signal used is a random-phase multisine with a full linear grid. A number of samples of the input and output are measured, in such a way that the DFT grid excited

![Figure 3.2: Block schematic used in the simulations.](image-url)
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$n_{BW}$ bins within the 3 dB bandwidth of the system. The different values for

$$n_{BW} \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 32, 40, 64, 80, 92, 128, 256\}$$

are tried. The use of $n_{BW}$ makes that the experiment length is normalized to take the bandwidth of the resonance into account. This means that the obtained results generalize to systems with a different relative damping.

The FRF is estimated using the local modeling approaches indicated in Table 3.1. This table also indicate $\Delta \text{SNR}$ which indicates the expected gain in SNR if the error is mainly determined by the variance. Concretely,

$$\Delta \text{SNR} \triangleq \frac{n_\theta}{n_{BW}}$$

accounts for the smoothing action of local modeling.

We then compute basic statistics of the estimated FRFs $\hat{G}^{[i]}_\bullet$ over $n_{MC} = 2000$ Monte Carlo runs of the simulations above. Specifically, the following values

$$m_\bullet(\omega_k) \triangleq \frac{1}{n_{MC}} \sum_{i=1}^{n_{MC}} \hat{G}^{[i]}_\bullet(\omega_k)$$

$$b_\bullet(\omega_k) \triangleq \frac{1}{n_{MC}} \sum_{i=1}^{n_{MC}} \hat{G}^{[i]}_\bullet(\omega_k) - G_0(\omega_k) = m_\bullet(\omega_k) - G_0(\omega_k)$$

$$\sigma^2_\bullet(\omega_k) \triangleq \frac{1}{n_{MC} - 1} \sum_{i=1}^{n_{MC}} \left(\hat{G}^{[i]}_\bullet(\omega_k) - m_\bullet(\omega_k)\right) \left(\hat{G}^{[i]}_\bullet(\omega_k) - m_\bullet(\omega_k)\right)$$

$$\text{RMSE}_\bullet(\omega_k) \triangleq \sqrt{\sigma^2_\bullet(\omega_k) + b_\bullet^2(\omega_k)}$$

are computed and are the sample estimates of respectively the expected value $E[\hat{G}_\bullet]$, bias $\text{Bias}[\hat{G}_\bullet]$ and variance $\text{Var}[\hat{G}_\bullet]$ of the model. Remark, that the standard deviation on the $b_\bullet \approx \sqrt{\sigma^2_\bullet/n_{MC}}$ which limits the precision with which we can hence observe $b_\bullet$. For $n_{MC} = 2000$, this means that we can detect the bias only when it is at most $20 \log \sqrt{2000} \text{dB} \approx 33 \text{dB}$ smaller than $\sigma_\bullet$ and smaller biases cannot be detected by the simulations.

As the Monte Carlo simulations yield the estimated FRF, the bias and variance for the different methods as a function of frequency, a huge amount of data is generated. In Figure 3.3, two such sets of data are shown as an example. It can be seen that the root mean squared error (RMSE) of LRIC behaves erratically and exhibits spikes over the whole frequency band. For a short data length ($n_{BW} = 8$), the LPM and ETFE shows a
3.4 Simulations

Figure 3.3: The bias, variance and root-mean-square (RMS) error of the FRF using the different estimators is shown for SNR = 40 dB and $n_{BW} \in \{8, 64\}$ samples in the 3 dB bandwidth of the system (as highlighted). See Table 3.1 for the color legend.

Table 3.1: Local modeling methods used in the Monte Carlo analysis.

<table>
<thead>
<tr>
<th>METHOD</th>
<th>COLOR</th>
<th>$n_\psi$</th>
<th>$n_\theta$</th>
<th>DOF</th>
<th>$\Delta$SNR [dB]</th>
</tr>
</thead>
<tbody>
<tr>
<td>ETFE</td>
<td>Yellow</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>LRM (7, 1, 1, 1)</td>
<td>Red</td>
<td>15</td>
<td>5</td>
<td>10</td>
<td>4.77</td>
</tr>
<tr>
<td>LRIC (7, 2, 2, 2)</td>
<td>Green</td>
<td>15</td>
<td>8</td>
<td>7</td>
<td>2.73</td>
</tr>
<tr>
<td>LRM (7, 2, 2, 2)</td>
<td>Blue</td>
<td>15</td>
<td>8</td>
<td>7</td>
<td>2.73</td>
</tr>
<tr>
<td>LRM (6, 2, 2, 2)</td>
<td>Gray</td>
<td>15</td>
<td>8</td>
<td>5</td>
<td>2.11</td>
</tr>
<tr>
<td>LPM (7, 4, 4)</td>
<td>Red</td>
<td>15</td>
<td>10</td>
<td>5</td>
<td>1.76</td>
</tr>
</tbody>
</table>
significant bias, whereas the LRM-based methods have a very small bias and variance that lies below the noise floor of each experiment. However, the variance of the LRM is still elevated by a few decibels near the resonance frequency. For long data lengths ($n_{BW} = 64$), both LPM and LRM perform comparably.

To interpret the data from the whole Monte Carlo simulation, we combine the observed values over the frequencies to obtain the aggregated error, respectively:

\[
\hat{\sigma}^2 = \frac{1}{n_{BW}} \sum_{\omega_k \in BW} |\hat{\sigma}^2(\omega_k)| \quad (3.76)
\]

\[
\hat{b} = \frac{1}{n_{BW}} \sum_{\omega_k \in BW} |\hat{b}(\omega_k)| \quad (3.77)
\]

\[
\text{RMSE} = \sqrt{\hat{b}^2 + \hat{\sigma}^2} \quad (3.78)
\]

which can be visualized for the different values of the SNR, $n_{BW}$ and for each method. Note that this aggregation is limited to the 3 dB bandwidth of the system, which is where the worst-case behavior is typically encountered.

**Noiseless case** In the noiseless case, one can observe the approximation error that is made by using the LRM, LPM, LRIC and ETFE as shown in Figure 3.4. For the ETFE, it can be seen that the error is very high. This can be expected since the ETFE does not separate transient/leakage contributions. Consequently, the leakage contributions (which change over the different Monte Carlo runs), lead to a large variance error for the ETFE. The approximation error of the LPM for short datasets ($n_{BW} \leq 8$) is dominated by the bias error. This is to be expected since the resonance in the band of interest is modeled using only polynomials instead of the full rational form. For longer datasets, the error decays approximately as $O\left(n_{BW}^{-6}\right)$. This is in line with the theoretical derivation from Schoukens, Vandersteen, Pintelon, Emedi, et al., (2013) that state that the approximation error decays as $O\left(n_{BW}^{-2(N_B+2)}\right)$. For the LRM and LRIC, the RMS error is already very low for small datasets ($-64$ dB for the first-order method, $-114$ dB for second-order methods). For short datasets ($n_{BW} \leq 16$), the LRM ($7, 1, 1, 1$) error has a bias contribution that is slightly larger (1 to 4 dB) than the variance contribution. For larger datasets ($n_{BW} \geq 16$), this bias contribution starts to diminish significantly as $O\left(n_{BW}^{-2}\right)$. For second-order rational methods, the error is dominated by the variance and is about 50 dB smaller than for the first-order methods. Practically, this approximation error is very likely to be negligible in measurements as such an error level corresponds to the quantization noise level of a 19 bit analog-to-digital converter (ADC).
3.4 Simulations

Figure 3.4: Local modeling of a discrete-time system results in an approximation error that is small enough for many practical uses.

**Good signal-to-noise ratios** The next situation we consider is when \( \text{SNR} = 60 \text{ dB} \) as shown in Figure 3.5. Here, the LPM remains severely biased for small datasets (\( n_{BW} < 12 \)). However, for sufficiently long experiments (\( n_{BW} > 32 \)), the bias becomes negligible. For the LRM and LRIC, the bias is always much smaller than the variance. In contrast to LPM, the LRM already reaches a constant variance level when \( n_{BW} > 10 \). It should also be noted that for the LRIC, the variance is almost 10 dB more than the corresponding LRM method. It is hence not advantageous to use the LRIC as-is. For all methods (except the ETFE and LRIC), the simulations show that the RMSE reaches the noise floor minus \( \Delta \text{SNR} \).

For the more noisy situation where \( \text{SNR} = 40 \text{ dB} \) (Figure 3.6), the situation is qualitatively equal except the bias incurred by the LPM reaches the level of the noise floor for a smaller data set (\( n_{BW} = 12 \)). It can be seen that the bias of the LPM behaves independently of the noise level (40 dB versus 60 dB), while the bias of the LRM clearly depends on the noise level.

**Poor signal-to-noise ratios** In Figure 3.7 and Figure 3.8, the results are shown for \( \text{SNR} = 20 \text{ dB} \) and \( \text{SNR} = 10 \text{ dB} \) respectively. Again, it can be seen that LRIC is plagued by high variance and is hence not directly usable. However, its bias for small datasets is much...
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Figure 3.5: Comparison of bias and variance of local modeling for SNR = 60 dB. The shaded area denotes the noise floor.

Figure 3.6: Comparison of bias and variance of local modeling for SNR = 40 dB. The shaded area denotes the noise floor.
lower than for the other methods. At SNR = 20 dB the LPM and the LRM is significantly biased for small datasets ($n_{BW} < 6$), however, the LRM still perform better than the LPM. For larger datasets ($n_{BW} \geq 16$), all methods perform similarly and at a nearly constant level of (variance) error. The LRM, in contrast to the LPM, still contains an observable bias in the simulations. At even lower SNR = 10 dB, most methods perform quite poorly for short datasets ($n_{BW} < 6$). However, here it can be seen that the LPM (7, 4, 4) outperforms both the LRM (7, 1, 1, 1) and LRM (7, 2, 2, 2) and slightly worse than LRM (6, 2, 2, 2). It can be seen that the bias error of the LPM drops off more quickly ($O\left(n_{BW}^{-3}\right)$) than the LRM bias which drops in approximately $O\left(n_{BW}\right)$.

### 3.4.1 Summary

In general, it can be seen that the LRM performs better than LPM unless the SNR near the resonance is very low (10 dB). For the LPM, it should be noted that near resonances, it is more advantageous to use even model orders (Schoukens, Vandersteen, Pintelon, Emedi, et al., 2013). The results for the LPM (7, 4, 4) in the previous section are representative of the best-case performance of an LPM ($7, N_B, N_A$) technique. Concretely, for small $n_{BW}$, the dominant error in the LPM is due to the bias. Reducing the model order of the LPM further will only increase the RMS error. Increasing the model order of the LPM (e.g. to LPM (7, 6, 6)) would result in an estimation where the residuals have very few degrees of freedom (DOF = 1 for LPM (7, 6, 6)) such that estimated noise level is less reliable (Pintelon et al., 2010a).

For the LRM, it can be seen that its model order has a significant influence on the approximation error. In small datasets with high SNR (SNR = 60 dB, $n_{BW} < 3$), and in datasets with low SNR (SNR < 40 dB), selecting a better local model can reduce the bias significantly. Nevertheless, the bias of the LRM is only a significant contributor to the RMS error for short datasets with low SNR. As such, selection of a better local model is only reduces the RMS error for those settings. Practically, this means that in most other settings of SNR and frequency resolution, either a first-order or second-order LRM ($N_A, N_B \in \{1, 2\}$) will perform almost equally well. From an intuitive point of view: this works well when each local bandwidth $\Omega$ is affected by at most two resonances.

The LRIC is not practically viable due to its high variability. However, since its bias is much lower than for the other methods, LRIC offers perspectives for improving the overall performance significantly.
Figure 3.7: Comparison of bias and variance of local modeling for SNR = 20 dB. The shaded area denotes the noise floor.

Figure 3.8: Comparison of bias and variance of local modeling for SNR = 10 dB. The shaded area denotes the noise floor.
### Guideline 3.1: Use the LRM for resonant systems when $\text{SNR} > 10$ dB

For resonant systems, the LRM offers better performance than LPM when a good SNR is obtained in the frequency domain. When the SNR is poor ($\leq 10$ dB), the LPM may perform slightly better than LRM. The LRIC is unsuitable due to a high variance, but has a much smaller bias error than the non-iterative solutions.

### Guideline 3.2: Use a second-order LRM for resonant systems

For resonant systems, the LRM offers better performance than LPM when a good SNR is obtained in the frequency domain. In the assumption that different resonances are well-separated in the frequency, a second-order LRM (e.g. LRM $(7, 2, 2, 2)$) is a reliable method to estimate the FRF.

### Guideline 3.3: For long measurements, LPM and LRM are similar

For very long measurements, the difference between LPM and LRM becomes negligible if the different resonances are well-separated.

### 3.5 Trading Off the Bias and Variance

In the last section, it was illustrated that the model order of the local LRM models can affect the obtained RMS error. This is specifically the case for small and noisy datasets. For larger datasets (or when $\text{SNR} > 20$ dB), the order of the LRM plays a less significant role. Moreover, it was observed that the LRIC has a high variance which might be due to pole-zero cancellations in the LRIC, i.e. a model order that is too high. As such, it makes sense to ask ourselves the question: “can we determine a good local model order” from the data? That is the question we will investigate in this section. First, some classical approaches are compared in the context of local modeling, i.e. relatively small (local) datasets.

#### 3.5.1 Local Model Order Selection

Selecting suitable values of $n_W$, $N_B$, $N_A$ and $N_C$ affects the quality of the obtained local models. For local rational models, there are a few insights from parametric system
identification that can be retained to make these choices easier. In particular (Pintelon et al., 2010a):

\[ N_C \leq \max \{ N_B, N_A \} \]  

(3.79)

and from statistical point of view (Mahata et al., 2006), the degrees of freedom should remain positive such that

\[ n_W \geq N_B + N_A + N_C + 2. \]  

(3.80)

Different heuristics exist to select the local bandwidth \( n_W \), e.g. Fan and Gijbels, (1995), Stenman, Gustafsson, et al., (2000), and Thummala and Schoukens, (2012) in the framework of local modeling approaches. As such, we focus mainly on the selection of the model order hyper-parameters \( N_A, N_B \) and \( N_C \) for a fixed value of \( n_W \).

Here, we take an approach that decides amongst a group of candidate models \( \{ M_1, \ldots, M_N \} \), which model \( M_i \) is the best model according to some criterion: i.e. one that captures that system dynamics but remains parsimonious. To do so, one needs to define a criterion that both takes ‘goodness-of-fit’ (e.g. value of the cost function, \( R^2 \), ...) and model complexity into account to ensure that the model captures only the systematic behavior and not random variations of the particular measurement. The latter is often done using ‘cross-validation’.

In statistical learning, many relevant cross-validation approaches exist. Within the system identification community a few simple approaches have been widely adopted (Pintelon and Schoukens, 2012, Chapter 11) to ensure that one does not overmodel the data. In the following, a short overview is given of different approaches. For a more complete overview, we refer to Arlot and Celisse, (2010).

### 3.5.1.1 Plug-in methods

In plug-in methods, a goodness-of-fit criterion (e.g. the log-likelihood function \( \ell \) of the estimated local model \( \tilde{M} \) given the estimation dataset \( Z \)), is penalized with a term accounting for the complexity of the model (Burnham and D. R. Anderson, 2002). E.g. a common choice is the Akaike’s information criterion (AIC) (Akaike, 1974) or the Akaike’s information criterion corrected for small sample sizes (AICc) (Hurvich and Tsai, 1989), which are defined as

\[ L_{AIC} = -2 \ell (\tilde{M} | Z) + 2n_\theta \]  

(3.81)

\[ L_{AICc} = -2 \ell (\tilde{M} | Z) + \frac{2n_\theta n_W}{n_W - n_\theta - 1} \]  

(3.82)
where \( n_\theta \) is the number of estimated parameters in the model \( \tilde{M}(\theta) \). In the context of local modeling, the AIC is a poor choice since it relies on asymptotic arguments of the data set, i.e. \( n_W \to \infty \), which is obviously an unwarranted choice in this context. The best model among a set of \( q \) possible models \( \{ \tilde{M}^1, \ldots, \tilde{M}^q \} \) is then

\[
\tilde{M}^\star \triangleq \arg_{\tilde{M}} \min \left\{ L_{AIC(c)}(\tilde{M}^i) \mid i = 1, \ldots, q \right\},
\]

(3.83)

which is the only model that is hence retained.

**Remark 3.8.** From a theoretical point of view, equations (3.81) and (3.82), require full knowledge of the likelihood function. This means specifically that the noise variance \( \sigma_V^2 \) needs to be known beforehand. When the variance is unknown, however, it is more robust to incorporate the AIC as a penalty factor on the cost function as indicated in Ljung, (1999b, Section 7.4) and Pintelon and Schoukens, (2012, Section 11.3 and 19.7) such that the AIC criterion becomes

\[
L_{AIC,\text{cost}} = L \cdot \left( 1 + \frac{n_\theta}{n_W} \right)
\]

(3.84)

with \( L \) the original local cost function.

**Remark 3.9.** It should be noted that due to \( \ell(\tilde{M}|Z) \), the value of \( L_{AIC(c)}(\tilde{M}, Z) \) is inherently coupled with the particular dataset \( Z \). Practically, this means that such criteria can only be used to compare local models that are estimated from the same local dataset (i.e. they have the same \( n_W \) and center frequency). Consequently, the AICc criterion cannot be used to select \( n_W \), but only \( N_A, N_B, N_C \) when \( n_W \) is fixed.

### 3.5.1.2 Holdout method

The so-called ‘holdout’ cross-validation method ensures a good model by first splitting the dataset into two smaller disjoint datasets: e.g. 70% of the data is used to estimate the model and a goodness-of-fit criterion (e.g. cost function) is used on the remaining data to determine which model has the best out-of-sample performance. By using independent datasets and assessing the performance of the model on (unseen) validation data, overfitted models can be rejected.

**Remark 3.10.** The typical approaches, such as the holdout and plug-in methods, either assume that an abundantly large dataset is available or assume the models are estimated on a fixed dataset. As such, these methods are not ideal for local modeling where the (local) dataset is rather small by design and could change by means of the local bandwidth \( n_W \) over the frequency.
3.5.1.3 Leave-out cross-validation

Instead, we use a better-suited cross-validation approach known as leave-one-out cross-validation (LOOCV). That is related to the AIC (Stone, 1977). Plainly, LOOCV cross-validates the model by in turn leaving out each data point, estimating the local model parameters and observing the difference between the omitted data point and the estimate. In effect, such an approach calculates the prediction residual error sum of squares (PRESS) for each local model:

\[
\text{PRESS} = \frac{1}{nW} \sum_{r \in R} |E_{[r]}|^2,
\]

where \(E_{[r]} = Y_r - \hat{Y}_{[r]}\), i.e. the difference between the observed (local) output spectrum at the \(r\)th line and the predicted output spectrum at the same line when the corresponding data point is removed from the estimation procedure. To select the best model from a set of candidate models, only the model with the lowest PRESS value is retained. In general, this means that LOOCV requires estimating \(nW\) additional models (one for each omitted data point) and hence can be very time-consuming.

For linear models (i.e. \(Y = K\theta\), such as for the LRM and LPM), however, the PRESS can be calculated in a single step without estimating any additional models (Golub et al., 1979):

\[
\text{PRESS} = \frac{E^H W^{-1} E}{2NW + 1}
\]

where \(E\) is the residual vector of the full estimation and \(W\) is a diagonal weighting matrix with \(W_{ii} = (1 - H_{ii})^H(1 - H_{ii})\) and \(H = KK^+\) is the so-called ‘hat-matrix’ that is known to be idempotent. The proof for this can also be found in Seber and Lee, (2003, Sec. 12.3.2).

Note that since the PRESS can be interpreted similar to the residuals, this forms a convenient metric to select models on. In particular, when there is a large difference between the level of the residuals and the PRESS, this indicates that a poor model has been selected.

3.5.2 Combining Different Local Model Orders

In the previous section, different methods were introduced to select the local model order. This begs the question whether it is advantageous to use the same local model complexity globally over all frequencies or whether the local model order can be allowed
3.5 Trading Off the Bias and Variance

to vary for different frequency bands. In fact, both approaches have their merits and which of both is to be preferred, depends on the ultimate purpose of the FRF.

When the goal is to inspect the FRF visually or to obtain a smooth representation of the transfer function, one can use different model orders per frequency bin. Fan, Gijbels, et al., (1996), Stenman and Gustafsson, (2001), and Thummala and Schoukens, (2012) show that a varying model order can reduce both bias and variance depending on the characteristic of the system. Near resonances one might need a higher local model order than in regions with constant slopes (e.g. $20 \cdot n \text{ dB/decade}$). For non-parametric purposes, this means that one can select the optimal local model for each frequency bin and combine those local models to obtain the optimal FRF.

For estimating a global parametric model using the FRF obtained from local modeling, one has to take extra care. Pintelon et al., (2010b) prove asymptotic properties of the sample maximum-likelihood estimator (MLE) in conjunction with an FRF obtained from local modeling. Assuming DOF > 4 and a constant model order, Pintelon et al., (2010b) gives expressions for the expected value and variance of the (global) cost function and the asymptotic loss in efficiency for using the sample MLE instead of the actual MLE based on the work of Mahata et al., (2006). For varying local model complexity, deriving these asymptotic properties analytically becomes intractable. As such, for estimating a (global) parametric model, a fixed local model order complexity should be used over the whole frequency axis.

When an optimal model order is to be selected for the whole frequency range, the local optimality criteria need to be reformulated in the global context of all local models over the frequency. In practice, the formulas of the previous section extend quite “naturally” to this global context. Particularly for the PRESS:

\[
\text{PRESS}_{\text{global}} \propto \sum_k \text{PRESS}_k
\]  

(3.87)

where PRESS$_k$ denotes the PRESS for the $k^{th}$ local model. One should realize that for all local models together ($M = \cup_k M_k$) and the global dataset $Z$, the global log-likelihood, cost function and number of parameters can be related to their local counterparts:

\[
\ell (M | Z) = \sum_k \ell (\tilde{M}_k | Z_k)
\]

(3.88)

\[
L_{\text{global}} = \sum_k L_k
\]

(3.89)

\[
n_{\theta}^{\text{global}} = \sum_k n_{\theta}(k).
\]

(3.90)
These global quantities can be substituted in the expressions of the local AIC and AICc of the previous section. Note that one should interpret $n_W$ as the total number of data points (and not the number of points in each window) for the global case.

### 3.5.3 Example

The system as in (3.69) has been simulated for 1 200 samples such that $n_{BW} = 3$. Two different noise levels have been tried: one with poor SNR $\approx 10$ dB and a better SNR $= 40$ dB. The resulting FRF has been estimated with the different LRM and LPM estimators as in the Monte Carlo simulations. In Figure 3.9 the hence estimated noise variance $\hat{\sigma}_V$ and PRESS are shown as a function of the frequency.

For the high SNR case, it can be seen in Figure 3.9 that the LPM is heavily biased and hence results in a large overestimate of the noise variance near the resonance peak. As a result, the PRESS in the same region is also elevated for the LPM. Outside the resonance peak, the LPM and LRM estimate very similar noise variances. The PRESS of the different methods are in agreement with the observations regarding the RMS error from the Monte Carlo simulations in Figure 3.6: all tested LRM perform similarly (with a slight advantage for the first-order method), the LPM produces much larger errors near the resonance. Nevertheless, this figure illustrates that for the LRM, the error is quite flat over the frequency. Note also that the PRESS suffers from a quite high variance over the frequency, which is a well-known drawback (Arlot and Celisse, 2010).

For the low SNR case (top plots in Figure 3.9), it can be seen that both LPM and LRM result in a very similar estimate of the noise level. The values of the PRESS are again very noisy, especially for the LPM estimates. Near the resonance, the LRM $(7, 1, 1, 1)$ results in the smallest PRESS value. This is in correspondence with the observations of the variance for similar noise levels in Figure 3.8. Note however that the results do not carry over to the RMS error: since the LRM is severely biased, the PRESS becomes unreliable to detect model errors.
### 3.5 Trading Off the Bias and Variance

The transition from low to high frequencies is accompanied by a drop in the transfer function. This change is more pronounced at higher SNR levels.

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**Figure 3.9:** It is convenient to compare the PRESS (left plots) and the noise variance (right plots) when good SNR is available. The theoretical noise level is shaded and the transfer function is shown in bold.
3.6 Smoothing the FRF by Impulse Response Truncation

This section is based on Lumori et al., (2014).

The main contribution of this section is to come up with an improved FRF by developing a method for smoothing the FRF estimated via the LPM. The smoothing of the FRF happens by truncating the associated impulse response, as in Schoukens et al., (1998), but with the following additional extension: the determination of the optimal truncation time, without any user interaction, in conjunction with the use of the LPM for leakage reduction. The truncation point (cut-off index) is determined by one of two statistically inspired methods. It is then shown that a smooth LPM estimate of the FRF lowers the variance, thus improving the assessment level of the dynamic properties of a given LTI system. It should be noted, however, that such smoothing is merely of importance when inspecting the FRF visually. In particular, a slight bias might be incurred when smoothing the FRF. If one is to fit a parametric model to the smoothed FRF, one essentially ends up with a biased model (which is often undesirable).

Based on Schoukens, Vandersteen, Barbé, et al., (2009), the nonparametric method developed in this work is formulated in an output-error (OE) framework. A single-input single-output (SISO) LTI system is considered. The input signal is a known random noise signal and the output signal is corrupted by measurement noise. This is depicted in Figure 3.2 as a linear dynamic discrete-time SISO system, whose full mathematical model is of the form:

\[ y(t) = G_o(q)u_o(t) + H_o(q)e(t) = G_o(q)u_o(t) + v(t) \]  

(3.91)

where \( G_o(q) \) represents the dynamics of the system to be estimated, \( u_o(t) \) is the input signal, \( v(t) = H_o(q)e(t) \) is the noise source at the output, \( H_o(q) \) is the noise dynamics, \( e(t) \) is white noise, and \( q^{-m} \) is the backward shift operator \( (q^{-m}x(t) = x(t - m) \) with \( m \) a positive integer).

Numerous parametric identification techniques are devoted to the development of parametric plant \( G(q, \theta) \) and noise models \( H(q, \theta) \), where \( \theta \) is the model parameters vector (Ljung, 1999b; Söderström and Stoica, 1989). In this section, however, a nonparametric technique is developed, and formulated in the frequency domain, consistent with (Mahata et al., 2006; Pintelon and Schoukens, 2012).

The following choices are made:

- The work in this section is for discrete-time systems. Denote the \( k^{th} \) discretized frequency as \( \omega_k = e^{-j2\pi kf_s/N} \), with \( f_s \) the sample frequency and \( N \) the total number of sample points.
• Describe the parametric plant model $G(q, \theta)$ by the nonparametric FRF counterpart $G(\omega_k)$ at the $k^{th}$ frequency.

• Describe the parametric noise model $H(q, \theta)$ (associated with the output noise source $v(t)$) by a nonparametric noise variance contribution $\sigma_v^2(\omega_k)$ at the $k^{th}$ frequency.

The rest of the section is structured as follows. Section 3.6.1 discusses the novel smoothing method. Ultimately, in Section 3.6.5, simulation results of the following FRF estimates and their corresponding variances are compared: (i) the smooth LPM estimate $\hat{G}_{\text{trunc}}(\omega_k)$, (ii) the LPM estimate $\hat{G}_{\text{LPM}}(\omega_k)$.

These FRF estimates are also compared with the true LTI system $G_o(\omega_k)$. Discussion of the simulation results is then followed by a conclusion in Section 3.6.6.

In this section, the LPM (3, 2, 2) is used. However, since the techniques discussed in this section are not specifically geared towards the LPM, one can just as well use the concepts with other non-parametric FRF estimators such as the LRM.

### 3.6.1 Smoothing the FRF estimate

The method for smoothing (improving) the LPM estimate of the frequency response function is presented in this section together with pertinent assumptions. After obtaining the LPM estimate of the FRF (denoted as $\hat{G}_{\text{LPM}}(\omega_k)$) from Section 3.2.3, the smoothing method is decomposed into the following procedures (see Figure 3.10), which will be elaborated on later:

1. The impulse response $\hat{g}_{\text{LPM}}(t)$ is computed from the inverse discrete Fourier transform (IDFT) of $\hat{G}_{\text{LPM}}(\omega_k)$.

2. An accurate estimate of the static (DC) value of the FRF is computed by inspecting the average value of the tail of the estimated impulse response and this is subtracted from the impulse response.

3. From the estimated impulse response $\hat{g}(t)$, the noise floor is estimated.

4. Assuming that the impulse response decays with the passing of time, the noise is bound to predominate after a certain time interval. In particular, when $t > t_{\text{trunc}}$, the impulse response has decayed below the noise floor. In Section 3.6.4.1 and Section 3.6.4.2, two approaches are introduced to estimate $t_{\text{trunc}}$. 
5. A truncation is effected at the point beyond which the estimated impulse response is buried in noise. This action results in smoothing of the FRF when the truncated impulse response is transformed to the frequency domain using the DFT.

These procedures require that the following assumptions are satisfied.

**Assumption 3.1.** The estimate $\hat{G}_{LPM}(\omega_k)$ is available at all frequencies $\omega_k$ for $k \in \{1, 2, \ldots, N/2\}$.

This assumption ensures that the impulse response corresponding to the FRF can be computed (up to its mean value). It requires that the input signal excites the whole unit circle. This is satisfied, for instance, by using white noise as an excitation signal.

**Assumption 3.2.** The impulse response $g(t)$ decays exponentially over time, i.e.

$$\exists A \in \mathbb{R}^+, \exists a > 0, \exists t_0 > 0, \forall t > t_0 : |g(t)| \leq A \exp(-at).$$

**Assumption 3.3.** Within 90% of the measurement window, the impulse response decays to a level that is indistinguishable from the noise.

Note that Assumption 3.2 and Assumption 3.3 exclude the possibility of considering a system that is a pure integrator.

### 3.6.2 Obtaining the Impulse Response Function

The estimated impulse response function $\hat{g}_{LPM, DC}(t)$ is obtained explicitly via the IDFT of the LPM-estimate $\hat{G}_{LPM}(\omega_k)$ of the FRF, viz:

$$\hat{g}_{LPM, DC}(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N-1} \hat{G}_{LPM}(\omega_k)e^{\frac{j2\pi kt}{N}}$$

(3.92)
where the estimated FRF in the frequency band between the Nyquist and the sample frequencies is obtained as follows:

\[
\hat{G}_{\text{LPM}}(\omega_{N-k}) = \overline{\hat{G}_{\text{LPM}}(\omega_k)}, \text{ for } k = 1, \ldots, N/2
\]  
(3.93)

(with \(\overline{G}_{\text{LPM}}\) the complex conjugate of \(G_{\text{LPM}}\)) to ensure \(\hat{g}_{\text{LPM}}(t)\) to be real.

Smoothing of the estimated FRF requires the correct estimate of the impulse response. Unfortunately, the LPM presented in Section 3.2.2 and in (Schoukens, Vandersteen, Barbé, et al., 2009) does not estimate the DC value of the FRF, hence the subscript “\(\backslash \text{DC}\)” in equation (3.92). Consequently, the mean value of the corresponding estimated impulse response given in equation (3.92) is not correct. This limitation is lifted by developing a simple estimator of the mean value, presented below.

### 3.6.3 Estimating the DC Value of the FRF

The correct mean value of the impulse response accounts for (i.e. estimates) the DC value of the FRF. A time-domain method is proposed for estimating the mean value of the impulse response in equation (3.92).

According to Assumption 3.2, the true impulse response tends to zero asymptotically; and by Assumption 3.3, the last 10% of the estimated impulse response is noise, plus a constant value due to an inaccuracy in the average value of the impulse response. The correct estimate of the mean value is obtained by shifting the whole signal such that the last 10% of the impulse response is centered around 0. To this end, the following procedure is executed:

1. Compute the mean value \(m_{g_{10}}\) of the last 10% of the impulse response \(\hat{g}_{\text{LPM}\backslash \text{DC}}(t)\), estimated by the LPM, as

\[
m_{g_{10}} = \frac{1}{[0.1N - 1]} \sum_{t = \lfloor 0.9N \rfloor}^{N-1} \hat{g}_{\text{LPM}\backslash \text{DC}}(t)
\]  
(3.94)

2. Next, subtract the computed mean value \(m_{g_{10}}\) from \(\hat{g}_{\text{LPM}\backslash \text{DC}}(t)\) to obtain the improved impulse response \(\hat{g}_{\text{LPM}}(t)\), viz:

\[
\hat{g}_{\text{LPM}}(t) = \hat{g}_{\text{LPM}\backslash \text{DC}}(t) - m_{g_{10}}, \text{ for } t = 0, 1, \ldots, N - 1
\]  
(3.95)
3.6.4 Determining the Truncation Time of the Impulse Response

In this section, two methods will be discussed to determine the truncation time, i.e. the time beyond which the measured impulse response cannot be separated from the noise level. Specifically, the first method uses an $F$-test to distinguish between signal and noise. The second method fits an exponential through a well-chosen subset of the data to obtain $t_{trunc}$.

3.6.4.1 Truncation Via the $F$-Test

To distinguish the part of the measured impulse response where the actual impulse response dominates from the part where the noise dominates, the estimated impulse response $\hat{g}_{LPM}$ is split into smaller segments. One can then gradually determine if in each of these segments, the observed impulse response is dominated by the noise.

1. The impulse response $\hat{g}_{LPM}(t)$ is split into $N_S$ segments of approximate equal lengths $L_S = \left\lfloor \frac{N}{N_S} \right\rfloor$. If necessary, the length of the first segment is reduced to allow for the others to have equal lengths. The $i^{th}$ segment is denoted $\hat{g}[i]$, for $i \in \{1, \ldots, N_S\}$. By Assumption 3.3, the last segment, $\hat{g}[N_S]$, is noise when $L_S \leq 0.1N$.

2. Iteration is then carried out from the last segment ($i = N_S$) towards the beginning of the impulse response. To determine whether the preceding segment $\hat{g}[i-1]$ is completely dominated by noise, both an $F$-test and an absolute criterion on the RMS are used.

The $F$-test is suited to compare the variances of different segments $\hat{g}[i]$ and $\hat{g}[i-1]$ (Parsons, 1974), while accounting for the degrees of freedom in each segment. For zero-mean noise, these variances are the RMS value of each noise segment. This has the advantage that a segment in which the impulse response still has a significant amplitude is likely to have a large RMS value, which is more likely to be detected. Let $s^2(i)$ denote the mean square value of segment $\hat{g}[i]$, viz:

$$s^2(i) \triangleq \frac{1}{L_S} \sum_{j=1}^{L_S} (\hat{g}[i](j))^2 = \left( \text{RMS} \left( \hat{g}[i] \right) \right)^2. \quad (3.96)$$
3.6 Smoothing the FRF by Impulse Response Truncation

Starting with the last segment \( (i = N_S) \), the following null hypothesis and the alternative hypothesis are formulated:

\[
H_0 : s^2(i - 1) = s^2(i) \\
H_a : s^2(i - 1) > s^2(i).
\]

As segment \( \hat{g}^{[i]} \) has been tested and been classified as noise (or assumed to be noise if \( i = N_S \)), the null hypothesis effectively states that the segment \( \hat{g}^{[i-1]} \) is also noise. The alternative case occurs when \( \hat{g}^{[i-1]} \) has a significant change, which indicates the presence of the signal. Using the \( F \)-test on the RMS values of the two segments with a high significance level of \( \alpha = 0.999 \) and the aforementioned hypotheses, we can determine whether \( \hat{g}^{[i-1]} \) is likely to be part of the actual impulse response. In particular, the null hypothesis is rejected when the empirical \( F \) statistic is high, i.e.

\[
F \triangleq \frac{s^2(i - 1)}{s^2(i)} \geq \text{CDF}^{-1}_F(\alpha; n_{i-1}, n_i)
\]

where \( \text{CDF}^{-1}_F \) is the inverse cumulative density function (CDF) of the \( F \)-distribution with \( n_{i-1} \) and \( n_i \) degrees of freedom (Hazewinkel, 2002). In this case \( n_{i-1} \) and \( n_i \) are the respective number of samples in the \((i - 1)\)th and \(i\)th segment of the impulse response.

The high level of significance (\( \alpha = 0.999 \)) ensures that the probability of a Type II error is smaller than \( 1 - \alpha \) (Parsons, 1974). In our case, such an error means that a part of the actual impulse response would be falsely classified as noise, which could significantly increase the bias as information of the system is discarded. A Type I error is less detrimental since, in that case, noise is falsely classified as a signal component and kept in the impulse response, thereby causing a sub-optimal value of the variance of the estimate. As the LPM samples are correlated over a short frequency span, the actual noise present in \( \hat{g} \) may be slightly non-stationary. To cope with this, one can introduce other criteria which must be satisfied together with the outcome of the \( F \)-test. A criterion that shows good results is to check whether the segment \( \hat{g}^{[i-1]} \) has an RMS value that is at least a factor \( \kappa \) larger than the RMS of the noise. Even for a moderate \( \kappa = 1.1 \), a large improvement in the detection was observed.

3. This procedure is repeated until a segment \( \hat{g}^{[i_{\star} - 1]} \) is found that is dominated by the actual impulse response according to the outcome of the \( F \)-test and the absolute RMS value of the segment. At that point, it is very likely that the transition from noise to the actual impulse response happens within the segment \( \hat{g}^{[i_{\star} - 1]} \). One now has the choice to accept the last sample of \( \hat{g}^{[i_{\star} - 1]} \) to be the last meaningful sample \( t_{\text{trunc}} \) for the impulse response. The accuracy of this estimate is limited by the length of the segment \( L_S \).
4. A more accurate estimate can be obtained by dividing the segment $\hat{g}[^{i-1}]$ — which may contain both the signal and noise — yet again into smaller segments. The procedure described above is then repeated until a satisfactory, accurate $t_{\text{trunc}}$ is obtained, or until the subsegments have a length that is too short ($L_{S_{\text{min}}} = 10$) to guarantee the accuracy of the RMS values. To start this refinement, the last $\hat{g}[^{i}]$ should be used to compare the RMS with the $F$-test, since the last subsegment of $\hat{g}[^{i-1}]$ cannot be asserted to be dominated by noise.

This procedure is illustrated in Figure 3.11 for the system described by the following equations:

\begin{align*}
y_o(t) &= 1.5371y_o(t - 1) - 0.9025y_o(t - 2) + u(t) \quad (3.100a) \\
y(t) &= y_o(t) + e(t), \quad (3.100b)
\end{align*}

where $e(t)$ is a white noise sequence, such that the SNR of the output signal is 78.2 dB. The figure shows segments of the estimated impulse response. The last two segments are dominated by the noise, while all samples at $t < 80$ have large system contributions. The algorithm outlined above selects $t_{\text{trunc}} = 80$, beyond which all samples are set to zero, resulting in a smoothed FRF estimate.

![Figure 3.11](image_url)

Figure 3.11: Illustration of truncating the impulse response by performing an $F$-test on successive segments of the impulse response.
3.6.4.2 Truncation Via Exponential Fitting

The FRF estimate is smoothed by truncating the estimated impulse response function. The truncation is applied at the time index beyond which the signal is indistinguishable from the noise.

This is done via the fit of an exponential function on the maxima of the impulse response, implemented as follows.

1. Obtain the estimate of the impulse response $\hat{g}_{LPM}(t)$ from the LPM, corrected for its DC value as discussed in Section 3.6.3. Henceforth, $\hat{g}_{LPM}(t)$ will simply be denoted as $\hat{g}(t)$.

2. Obtain an estimate $\hat{\sigma}_n$ of the standard deviation of the noise from the last 10% of the data, viz:

$$\hat{\sigma}_n^2 = \frac{1}{|0.1N - 1|} \sum_{t = 0.9N}^{N-1} \hat{g}^2(t).$$

(3.101)

This is valid, as per Assumption 3.3.

3. Obtain $T_{HSNR}$ as the set of time instants where $\hat{g}(t)$ is significantly above the standard deviation of the noise. Only samples with absolute values of at least $5\hat{\sigma}_n$ are retained:

$$T_{HSNR} = \{t : |\hat{g}(t)| \geq 5\hat{\sigma}_n\}. \quad (3.102)$$

(Subscript HSNR stands for High SNR.)

4. Find the set $T_{max}$ of indices corresponding to monotonically decreasing maxima of the impulse response:

$$T_{max} = \{t : |\hat{g}(t)| > |\hat{g}(t')|, t < t' < N \land t' \in T_{HSNR}\}. \quad (3.103)$$

5. Fit an exponential function $Ae^{at}$ approximating $\hat{g}(t)$, in $t_{max} \in T_{max}$. This is done by solving the following expression

$$\ln |\hat{g}(t)| \approx \ln A + at, \text{ with } t \in T_{max},$$

(3.104)

for $\ln A$ and $a$ in a least squares sense. This is a quadratic problem in the parameters and, thus, amounts to a convex optimization problem that can be solved directly. Since $\ln |\hat{g}(t)|$ decreases for an increasing $t$ in $T_{max}$ (by construction in equation (3.103)), the estimated $a$ from equation (3.104) is always negative.
6. Determine the first time-instant $t_{\text{trunc}}$ at which the estimated exponential gets significantly below the noise floor, viz:

$$t_{\text{trunc}} = \min \left\{ t : Ae^{at} < \gamma \hat{\sigma}_n \right\}$$

(3.105)

where the parameter $\gamma$ can be tuned such that an appropriate trade-off between the decreased variance and the increased bias on the estimated smoothed FRF is found. This is discussed below.

7. Truncate the estimated impulse response for $t \geq t_{\text{trunc}}$.

As an illustration, this procedure was applied to the noisy impulse response of the system described by the following difference equation:

$$y_\circ(t) = 2.583y_\circ(t - 1) - 2.176y_\circ(t - 2) + 0.592y_\circ(t - 3) + u(t)$$

(3.106)

The measured signal was disturbed by random white noise, $y(t) = y_\circ(t) + e(t)$, such that the SNR was 14.2 dB. The result is depicted in Figure 3.12. An exponential function
3.6 Smoothing the FRF by Impulse Response Truncation

(black thick line) is fitted on the peaks of the absolute value of a noisy impulse response. The truncation time (vertical line) $t_{\text{trunc}}$ was selected as the time instant at which the fitted exponential fell below $0.4\sigma_n$ (i.e. $\gamma = 0.4$ in equation (3.105)).

**Considerations on the choice of $\gamma$**

- The tuning parameter $\gamma$ is application- and system-dependent. A higher value lowers the variance of the estimated FRF, but increases its bias, and vice versa.

- The bias error is highest in the vicinity of (sharp) resonance peaks. If the latter is to be estimated with a high accuracy, a value $\gamma \ll 1$ must be used.

- If one is interested in obtaining a smooth initial estimate of the FRF, a (small) bias error is acceptable, and choosing $\gamma \approx 1$ was found to be a good rule of thumb.

**3.6.5 Simulation Results**

Figure 3.13 and Figure 3.14 compare the LPM with and without truncation of the impulse response. Here, the truncation instant $t_{\text{trunc}}$ is determined using the exponential fitting method. They were obtained from simulations on the system in (3.100) where the noise variance was set such that the SNR of the output signal is 18.3 dB.

In Figure 3.13 one observes the following:

- a decrease of the variance on the truncated estimate of about 10 dB compared to the non-truncated LPM, is observed. Note also that the improvement in variance is proportional to the measurement time, i.e.

$$\frac{\tilde{\sigma}_{\text{trunc}}}{\tilde{\sigma}_{\text{LPM}}} \approx \sqrt{\frac{t_{\text{trunc}}}{T_{\text{meas}}}}$$  \hspace{1cm} (3.107)

where $T_{\text{meas}}$ is the total measurement time. Equivalently, this means that for FRFs estimated with a finer frequency resolution, are improved more by truncation than coarsely spaced FRFs.

- an increase of the bias of the truncated estimate, especially in the vicinity of the resonance frequency. Still, this bias lies below the variance of the non-truncated estimate. As such, for a single experiment, the increase in bias still yields a better estimate when truncation is invoked.
Figure 3.13: The bias $\hat{b}$ and standard deviation $\hat{\sigma}$ of the FRF estimates using both the LPM (blue) and truncated LPM (red) are shown. It can be seen that the truncation reduces the variance at the cost of an increased bias, however, the overall RMS error is reduced by the truncation.
3.6 Smoothing the FRF by Impulse Response Truncation

In Figure 3.14, the RMS error of the estimated FRF (without the DC value) as a function of the time $t_{\text{trunc}}$ at which the impulse response is truncated, is shown. To the left of the optimal truncation length, the RMS error increases steeply. In that region, one is essentially using an overly simplified impulse response to represent the system and hence the RMS error is dominated by the bias error. To the right hand side of the minimum, the RMS error increases more gradually, due to an increase of the noise variance.

This bias depends on the time instant at which the truncation is performed, as discussed below.

- the error on the truncated LPM estimate is strongly correlated over the frequency. This must be taken into account when formulating a maximum likelihood parametric estimator of the system. It is, however, advised to use the raw data instead of such smoothed FRFs to estimate a parametric model such that no bias is introduced into the parametric estimate. In Chapter 5, however, it will be shown that smoothed FRFs can lead to improved starting values.
A good practice would be to truncate the impulse response at the minimizer (black dot) of the RMS error. However, this minimizer is unknown in practice, since it depends on the true underlying system. Nevertheless, the proposed approaches provide a reasonable way to approximate the optimal truncation length.

The truncation time is determined from the data as described in Section 3.6.4.2 and Section 3.6.4.1. This was done on 1000 realizations of the noise, and depicted in Figure 3.14 by the histogram. Clearly, both methods for selecting the truncation time $t_{\text{trunc}}$ have a good overall performance, based on the mode of their distribution (around the 90th sample).

For the $F$-test based method, the obtained $t_{\text{trunc}}$ values only take a discrete set of values due to the limitation of the segment length. While most trials result in $t_{\text{trunc}} \in \{70, 80, 90\}$, a few outliers with much higher truncation times can be observed. These outliers can be attributed to the high power of the $F$-tests.

For the exponential fitting approach, however, a closely grouped set of (continuous) values for $t_{\text{trunc}}$ is obtained around the 90th sample, without significant outliers. Although this distribution has a mode that is quite a bit higher than the method using the $F$-test, due to the absence of outliers, this method is more reliable to use for the particular system that we studied.

From the plot, we can conclude that the RMSE increases rapidly when $t_{\text{trunc}}$ is smaller than the optimal. On the other hand, selection of a value of $t_{\text{trunc}}$ that is too large, is not nearly as detrimental to the modeling error. The graph also shows that the RMSE of the model can be decreased from 0.62 (without truncation) to 0.18 when the optimal truncation is applied. Also, one observes a low sensitivity of the RMSE w.r.t. $t_{\text{trunc}}$, when truncating at times beyond that optimum. Therefore, a somewhat conservative truncation method is still likely to yield a close to optimal result.

**Interaction of the truncation method and the system properties**  As seen in the previous simulation example, the exponential fitting method is well-suited to determine $t_{\text{trunc}}$ when the underlying system exhibits an exponential decay. However, if the system under test is not part of this class of systems, the exponential fitting method may provide sub-optimal results or even be ineffective.

Consider instead that we are measuring a boxcar averager, i.e. a moving-average filter. Such a system is governed by the difference equation:

$$y(t) = \sum_{i=0}^{N_{bc}-1} N_{bc}^{-1} u(t - i),$$

(3.108)
3.6 Smoothing the FRF by Impulse Response Truncation

Figure 3.15: Exponential fit of the impulse response of a boxcar averager. \( t_{\text{trunc}} \gg N_{\text{bc}} \) shows that a sub-optimal truncation point is found.

where \( N_{\text{bc}} = 88 \). Since this system is a trivial finite impulse response (FIR), it is easy to see that \( t_{\text{trunc}} = N_{\text{bc}} \) provides the optimal truncation point.

We simulate the boxcar averager for a white noise input \( u(t) \) and add white noise with standard deviation \( \sigma_e \approx 0.1 \) to the output \( y(t) \) for 2048 samples from which the first 1024 samples are discarded to randomize the state of the system under test. A single realization of the impulse response function is shown in Figure 3.15 together with the \( t_{\text{trunc}} \) and other relevant quantities of the exponential fitting method. It is easy to observe that the exponential fitting provides sub-optimal results since it truncates well past the end of the finite impulse response.

By repeating the experiment above \( n_{\text{MC}} = 1000 \) times as a Monte Carlo simulation where both the input signal and disturbing noise are varied, we gain insight into the behavior of both truncation methods. The histograms of the observed values of \( t_{\text{trunc}} \) for both methods is shown in Figure 3.16 together with the RMS error as a function of \( t_{\text{trunc}} \). It can be seen that the \( F \)-test based method mostly selects the optimal truncation point. On the other hand, the exponential fitting method is largely ineffective and doesn’t truncate the response in most of the cases.

It can be seen that both the \( F \)-test method and the exponential fitting method each have their own strengths and weaknesses. Particularly, the \( F \)-test method can only select \( t_{\text{trunc}} \) from a discrete set of values, but works well for specifically designed FIR filters.
Figure 3.16: RMSE of a boxcar averager as a function of the truncation time $t_{\text{trunc}}$. Empirical histograms of $t_{\text{trunc}}$ as determined using both the $F$-test and exponential fitting are shown. For this system, the $F$-test based method is clearly superior. The exponential fitting method is not effective in most cases.

where there is a clear jump in the amplitude of the impulse response. On the other hand, the exponential fitting method relies on the exponential decay of the impulse response to obtain sensible values of $t_{\text{trunc}}$ and is hence a better match for ‘physical’ systems. One should remember, however, that for the studied examples, both approaches only very seldomly select a $t_{\text{trunc}}$ that increases the RMS error on the impulse response compared to not truncating the response. As such, a use has little to lose by trying out both methods: the model quality is very unlikely to decrease and the additional computational effort is minimal.

**Guideline 3.4: Check and reinforce assumptions in the data.**

As illustrated by the presented smoothing operation, it pays off to check typical assumptions of the system (e.g. smoothness of the FRF, finite support of the impulse response) and even reinforce such assumptions to improve the model quality. For the truncation approaches, even if the assumptions do not hold, the methods are constructed such that the unavoidable model error is unlikely to increase.
3.6.6 Conclusion

This section introduced a time domain method to smooth the LPM-estimate of an FRF. It consisted of, after obtaining the FRF from the LPM, computing the associated estimated impulse response via the IDFT. Then, it was determined statistically at which time index the impulse response had decayed below the noise floor, yielding a point beyond which the response may be set to zero.

The different methods to determine $t_{\text{trunc}}$ presented in this section, essentially enforce different prior knowledge of the impulse response of the system. The exponential fitting technique relies on the exponential decay present in most physical systems. The $F$-test based technique is better suited for systems where a sudden jump in the impulse response is present.

The results clearly indicate that the truncation technique lowers the impact of the noise on the estimate of the FRF, resulting in a decreased variance. A bias-variance trade-off is possible by tuning the time beyond which the impulse response is indistinguishable from the noise.

3.7 Conclusions

In this chapter, the performance of non-parametric LPM and LRM estimators were compared to observe the FRF of resonant systems. Since the LRM is a biased estimator and the LPM is asymptotically unbiased, it is important to compare their behavior. Hence, expressions for the bias of the LRM were derived. For good SNRs in resonant systems, it was seen that the LRM is always superior to the LPM. For extremely poor SNR, i.e. $\text{SNR} \approx 10 \, \text{dB}$, the LPM may perform slightly better than the LRM when short measurement records are available. In this chapter, the LRIC was introduced. This is an extension of the LRM which solves the non-linear optimization problem and should thus be able to avoid a bias altogether. However, it is seen that this estimator is extremely noisy but seems to provide a smaller bias than the LRM. Since the overall RMS error for the LRIC is much larger than most other methods, the LRIC cannot be used in its current form.

Relatedly, two approaches to aid in performing the bias-variance trade-off for these local methods are illustrated. First, a short study of cross-validation for local models is given. Secondly, an approach to truncate the impulse response obtained from the LPM is illustrated. The latter can improve the performance of the LPM considerably if a large number of noisy samples is measured.
Appendix

3.A Transients and Leakage in the Frequency Domain

This appendix is based on multiple expositions regarding transient contributions in the frequency domain. For a more in-depth discussion, we refer to (Pintelon, Schoukens, and Vandersteen, 1997), (Pintelon and Schoukens, 1997), (Schoukens, Vandersteen, Pintelon, and Guillaume, 1999), (McKelvey, 2002), (McKelvey and Guérin, 2012, Section 2), and (Pintelon and Schoukens, 2012, Sections 2.6.3, 6.3.2 and Appendix 6.B).

This appendix derives the transient contributions in the frequency domain when a LTI system is observed during a finite measurement window (i.e. \( t \in 0, \ldots, N - 1 \)). The following derivations are repeated from (Pintelon and Schoukens, 2012, Appending 6.B), but have been adapted in notation.

Consider the ordinary difference equation

\[
\sum_{n=0}^{N_A} a_n y(t - n) = \sum_{m=0}^{N_B} b_m u(t - m) \quad \forall t \in \mathbb{Z} \tag{3.109}
\]

where the coefficients \( a_n \) and \( b_m \) can be either real or complex coefficients for all values of their subscripts \( n \in \{ 0, \ldots, N_A \} \) and \( m \in \{ 0, \ldots, N_B \} \). Such a difference equation can be used to describe any discrete-time LTI system (except those with arbitrary delays).

To describe this system in the frequency domain, we first introduce the (one-sided) \( Z \)-transform of the signals \( x \in \{ u, y \} \) and \( X \in \{ U, Y \} \), respectively:

\[
X(z) \triangleq Z \{ x(t) \} \triangleq \sum_{t=0}^{\infty} x(t) z^{-t} \tag{3.110}
\]

which reduces to the discrete-time Fourier transform (DTFT) when \( z = e^{j\omega} \) is examined (Oppenheim, Willsky, et al., 1996, Chapter 10).

First, we revisit two pertinent properties of the \( Z \)-transform that are necessary to allow one to compute the \( Z \)-transform of difference equation (3.109).
**Property 3.1.** The $Z$-transform is a linear transform: $Z\{ay(t) + bu(t)\} = a\ Z\{y(t)\} + b\ Z\{u(t)\} = aY(z) + bU(z)$ when $a$ and $b$ are finite constants (Oppenheim, Willsky, et al., 1996, Section 10.5.1).

**Property 3.2.** The $Z$-transform of a shifted signal can be related to the $Z$-transform of the unshifted signal (Oppenheim, Willsky, et al., 1996, Section 10.5.2) as

$$
Z\{x(t-n)\} = \sum_{t=0}^{+\infty} x(t-n)z^{-t} = \sum_{\tau=-n}^{+\infty} x(\tau)z^{-(\tau+n)} \quad \text{where } \tau = t-n
$$

$$
= z^{-n}\sum_{\tau=0}^{+\infty} x(\tau)z^{-\tau} + \sum_{\tau=-n}^{-1} x(\tau)z^{-\tau-n}
$$

$$
Z\{x(t-n)\} = z^{-n}\left( Z\{x(t)\} + \sum_{\tau=-n}^{-1} x(\tau)z^{-\tau} \right).
$$

By using both properties, we compute the $Z$-transform of the left hand side of (3.109):

$$
Z\left\{\sum_{n=0}^{N_A} a_n y(t-n)\right\} = \sum_{n=0}^{N_A} a_n\ Z\{y(t-n)\} + \sum_{n=0}^{N_A} a_n z^{-n} \sum_{\tau=-n}^{-1} y(\tau)z^{-\tau}
$$

$$
= A(z^{-1})Y(z) + I_A(z^{-1})
$$

where

$$
A(z^{-1}) = \sum_{n=0}^{N_A} a_n z^{-n}
$$

$$
I_A(z^{-1}) = \sum_{n=1}^{N_A} \sum_{\tau=-n}^{-1} y(\tau)a_n z^{-\tau-n}
$$

are polynomials in $z^{-1}$ of respective degrees $N_A$ and $N_A - 1$ (at most). Note that $I_A(z^{-1})$ depends on $y(\tau)$ where $\tau \in \{-1, \ldots, -N_A\}$, i.e. samples before the beginning of the measurement record.
For the right hand side, the derivation is analogous and yields

\[
Z \left\{ \sum_{m=0}^{N_B} B_m y(t-m) \right\} = B(z^{-1}) U(z) + I_B(z) \quad (3.119)
\]

\[
B(z^{-1}) = \sum_{m=0}^{N_B} b_m z^{-m} \quad (3.120)
\]

\[
I_B(z^{-1}) = \sum_{m=1}^{N_B} \sum_{\tau=-m}^{-1} u(\tau) b_m z^{-\tau-m}, \quad (3.121)
\]

where the degrees of the polynomials are respectively \(N_B\) and \(N_B - 1\) (at most).

Combining both expressions, one obtains the \(Z\)-transform of the complete expression (3.109):

\[
A(z^{-1}) Y(z) = \overbrace{B(z^{-1}) Y(z) + I_B(z^{-1})}^{I_{BA}(z^{-1})} - \overbrace{I_A(z^{-1})}^{E_{BA}(z^{-1})} \quad (3.122)
\]

where \(I_{BA}(z^{-1})\) is a polynomial of degree at most \(\max\{N_A, N_B\} - 1\). Since \(I_{BA}\) depends on \(u(\tau)\) and \(y(\tau)\) for \(\tau < 0\), it accounts for the initial conditions of the experiment.

Also notice that (3.122) cannot be evaluated generally without the knowledge of \(u(T)\) and \(y(T)\) for \(T \geq N\), which are not available in experimental conditions. In particular, we are interested in the evaluation on the unit circle, i.e. \(z = e^{j\omega}\) where one could hope to evaluate the DTFT. To overcome this problem, the samples \(y(T)\) and \(u(T)\) for \(T \geq N\) need to be eliminated from the equation. This can be done, as in Pintelon and Schoukens, (2012, Appendix 6.B.1), by multiplying (3.109) such that one can eventually rewrite

\[
A(z^{-1}) Y_{N,\infty}(z) = \overbrace{B(z^{-1}) Y_{N,\infty}(z) + z^{-N} E_B(z^{-1})}^{E_{BA}(z^{-1})} - \overbrace{I_A(z^{-1})}^{E_A(z^{-1})} \quad (3.123)
\]

where

\[
X_{N,\infty}(z) = \sum_{t=N}^{\infty} x(t) z^{-t} \quad (3.124)
\]

\[
E_B(z^{-1}) = \sum_{m=1}^{N_B} \sum_{\tau=-m}^{-1} u(N + \tau) b_m z^{N-\tau-m} \quad (3.125)
\]

\[
E_A(z^{-1}) = \sum_{n=1}^{N_A} \sum_{\tau=-n}^{-1} y(N + \tau) a_n z^{N-\tau-n}. \quad (3.126)
\]
Note that $E_B$ depends on the last $N_B - 1$ samples of the measurement record: it represents the final conditions of the experiment.

Subtracting (3.123) from (3.122) and denoting $X_N(z) = X(z) - X_{N,\infty}(z)$ then yields

$$A(z^{-1})Y_N(z) = B(z^{-1})U_N(z) + I_{BA}(z^{-1}) - z^{-N}E_{BA}(z^{-1})$$

which no longer depends on the samples of $u(t)$ and $y(t)$ outside the acquired measurement record where $t \in \{0, \ldots, N - 1\}$.

The latter expression can hence be evaluated on the unit circle. More specifically, by substituting the DFT frequencies $z_k = e^{j2\pi kN^{-1}}$ for $k \in \{0, \ldots, N - 1\}$, one retrieves

$$A(z_k^{-1})Y_{DFT}(k) = B(z_k^{-1})U_{DFT}(k) + T(z_k^{-1})$$

with

$$T(z_k^{-1}) = \frac{I_{BA}(z^{-1}) - E_{BA}(z^{-1})}{\sqrt{N}}$$

the transient (or leakage) contribution that is a polynomial of order $\max\{N_B, N_A\} - 1$ (at most), and the $N$-points DFT spectrum of $x(n)$ given by

$$X_{DFT}(k) \triangleq \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n)e^{j2\pi knN^{-1}} = \frac{1}{\sqrt{N}} X_N(z_k).$$

Note that since $T(z^{-1})$ is a polynomial, $T(z_k^{-1})$ is a smooth function of the frequency, hence illustrating that the transient/leakage contribution at the DFT frequencies is a smooth function that encapsulates the difference between the initial conditions $I_{BA}$ and end conditions $E_{BA}$ of the measurement record.

**Remark 3.11.** Note that in the remainder of this dissertation, the DFT spectra are typically denoted without the subscript DFT, but the subscript is added here to emphasize the fact that the final expression is valid for the DFT spectra of the input and output.

**Remark 3.12.** The derivation of the leakage has been proven here for discrete-time systems. For lumped continuous-time LTI and diffusive systems, the derivations are similar and elaborated in Pintelon and Schoukens, (2012, Appendix 6.B). Most importantly, the observation that the transient contribution $T$ is described in the frequency domain by a polynomial of finite degree (and hence smooth), remains valid for such systems.
Non-parametric Peak Modeling

This chapter is based on Geerardyn et al., (2014a,b).

Design is not just what it looks like and feels like. Design is how it works.

Steve Jobs

4.1 Introduction

For mechatronic design and many other engineering disciplines it is important to obtain good models that approximate the physical reality. Recent trends in many disciplines have led to increasingly complicated systems. Consequently, building a full-complexity parametric model rapidly becomes disproportionately expensive for a required model fidelity. Often, such parametric models are required to understand elementary properties of the system such as the resonance frequencies, damping of different poles, the maximal gain of the system. In this chapter, we will investigate the use of the local parametric approaches to extract such basic information from the system without having to build a full parametric model. The approach that is followed in this chapter is applied to estimating the $\mathcal{H}_\infty$ norm of the transfer function of a system. This is a problem that is important in the design of robust controllers (Oomen and Steinbuch, 2016) to characterize the effect of unmodeled dynamics in low-complexity models used for control. Such analysis allows one to verify that the eventually designed controller stabilizes the system.

Robustness is of key importance in feedback controlled systems, since feedback can lead to performance degradation or even closed-loop instability. An important example includes an active vibration isolation system (AVIS), where feedback is used to isolate high-precision equipment from external disturbances. The underlying feedback control principle is skyhook damping (Collette et al., 2011; Karnopp, 1995). However, the performance of skyhook damping is limited by high-frequent lightly-damped parasitic
resonance phenomena. Such model uncertainties can be taken into account explicitly in a robust control design (Zhang et al., 2005) for approaches based on $\mathcal{H}_\infty$ optimization. However, the uncertainty estimation in these references is based on rough prior assumptions and may be inaccurate. On the one hand, this can lead to potentially dangerous results, since no stability and performance guarantees can be given if the estimated uncertainty is too small. On the other hand, this can also lead to overly conservative controllers if the estimated uncertainty is too large.

Several approaches have been developed to determine bounds on the model uncertainty. First, model validation techniques have been developed (see e.g. Smith and Doyle, 1992; Xu et al., 1999). However, these results are typically overly optimistic (Oomen and Bosgra, 2009). Second, model error modeling approaches based on parametric system identification (Ljung, 1999a) with explicit characterization of bias and variance errors have been proposed, but these require a significant amount of user intervention. Third, non-parametric identification approaches have been adopted (de Vries and P. M. J. Van den Hof, 1994; van de Wal et al., 2002, see e.g.). In van de Wal et al., (2002), an identified frequency response function is used directly to evaluate the $\mathcal{H}_\infty$ norm on a discrete frequency grid. In de Vries and P. M. J. Van den Hof, (1994), an extended method is presented that bounds the error in between frequency points in a worst-case fashion. However, such worst-case methods are well-known to be overly conservative (Vinnicombe, 2001, Section 9.5.2). Fourth, recently a data-driven $\mathcal{H}_\infty$ norm estimation has been developed in Oomen, van der Maas, et al., (2014) and Wahlberg et al., (2010). These methods rely on a sequence of iterative experiments and directly deliver an estimate of the $\mathcal{H}_\infty$ norm, and combine an optimal experiment design while essentially taking inter-grid errors into account. A recent application of this method in Oomen, van der Maas, et al., (2014) has revealed that these iterative methods lead to significantly more accurate $\mathcal{H}_\infty$ estimates compared with traditional frequency response-based methods, thereby underlining the importance of the inter-grid error. Unfortunately, these methods require a sequence of dedicated iterative experiments. This increases the measurement time significantly, especially for multiple-input multiple-output (MIMO) plants. When measurement time is expensive, this is a severe impediment.

Although important steps have been made in model-error-modeling, existing methods involve a trade-off between accuracy, measurement time, and user intervention. The aim of this chapter is to develop a golden mean between traditional non-parametric techniques (third approach) and a full parametric model (second approach). The proposed $\mathcal{H}_\infty$ norm estimation procedure exploits recent developments in frequency domain estimation using the Local Polynomial Method (LPM) (Schoukens, Vandersteen, Barbé, et al., 2009) and the Local Rational Method (LRM) (McKelvey and Guérin, 2012) as introduced in Geerardyn et al., (2014a,b). Before, these LPM and LRM techniques have been used to enhance at-grid frequency response estimates. Here, these methods are extended to provide both an at-grid and inter-grid evaluation of the maximum singular value near resonances, as is required for the $\mathcal{H}_\infty$ norm and consequently the design of robust
controllers. The performance of these local parametric models in the context of robust control design is investigated, and bounds on the required measurement time to obtain a reliable estimate of the \( \mathcal{H}_\infty \) norm are obtained.

**Outline** In Section 4.2, a short introduction to robust control is linked to the work in this chapter. The problem is formulated in Section 4.3. In Section 4.4 the proposed method to estimate the \( \mathcal{H}_\infty \) norm by means of the LRM is introduced. In Section 4.5 the method is illustrated on simulations, and the required measurement time is determined. The technique is demonstrated on measurements of an AVIS in Section 4.6. Finally, the results and some challenges are discussed in Section 4.7.

**4.2 Intermezzo: Basics of Robust Control**

When designing a feedback controller \( C \) (as in Figure 4.1), the goal is to design \( C \) such that the output \( y \) of a plant \( P \) follows the command signal \( r_2 \) with some additional constraints on the performance of the whole closed-loop system (e.g. limited overshoot, short response times, good disturbance suppression, limited controller effort \( u_c \), ...). This obviously requires knowledge of the actual plant \( P \) that is to be controlled. Such knowledge can be obtained by means of modeling, be it first-principles modeling or system identification or a combination of modeling approaches, which provides an approximate model \( \hat{P} \).

![Figure 4.1: The considered feedback configuration \( T(P_0, C) \).](image)

In classical control, the working assumption is that \( \hat{P} = P \) and hence the design of the controller is carried out on the estimated plant model. This has the downside that if such a controller is not designed conservatively, errors in the model may produce instability or unwanted performance degradation.

In robust control (Skogestad and Postlethwaite, 2005; Zhou et al., 1996), the design is carried out while taking uncertainties into account. This is typically done by considering that the actual plant \( P \in \mathcal{P} \). Hence, the problem is then to construct the model set
Figure 4.2: The plant model set.

\[ P \] as illustrated in Figure 4.2. The design process thus guarantees performance and stability for the whole set instead of for a single model. However, these guarantees hinge on proper specification of \( P \) which is typically done by estimating a nominal model \( \hat{P} \) and the model error \( \Delta(\hat{P}, P_o) \). A generic way to express the plant model set (see also Figure 4.2) is

\[
P_{\text{generic}} \triangleq \left\{ P \mid \| \Delta(\hat{P}, P) \| \leq \gamma \right\} \tag{4.1}
\]

where both the norm and the exact uncertainty description \( \Delta(\hat{P}, P) \), can be chosen by the control engineer. An example of such a model set is easily imagined for additive uncertainty \( P = \hat{P} + \Delta \) when a 2-norm is used, in that case one obtains

\[
P_{\text{additive}} = \left\{ P \mid \| P - \hat{P} \|_2 \leq \gamma \right\}
\]

which is a multidimensional ball with radius \( \gamma \) centered around \( \hat{P} \). Note that in many cases, the uncertainty structure (additive, multiplicative, ...) can be converted to other representations (Douma and P. M. Van den Hof, 2005).

Building a good model set \( P \) as in (4.1), hence requires estimating a good value of the over-bound \( \gamma \) to define the size of the set. This over-bound has important practical implications for the robust controller design. Specifically, let us denote \( \gamma_o \triangleq \| \Delta_o \| \triangleq \| \Delta(\hat{P}, P_o) \| \). For an estimation procedure that yields the estimate \( \hat{\gamma} \) of the over-bound, a few situations can hence occur.

- \( \hat{\gamma} < \gamma_o \): the model set \( P \) is too small and hence \( P_o \not\in P \). Based on such an estimate, the robust controller may fail to stabilize and perform well for the actual plant \( P_o \).

- \( \hat{\gamma} \gg \gamma_o \): the model set \( P \) is too big such that the robust controller is overly conservative. Consequently, the actual plant is still stabilized by the controller, but the
performance of the system may be degraded as the controller design accounts for too many possible plants.

• \( \hat{\gamma} = \gamma_0 \): the model set is just large enough to contain \( P_0 \) such that both stability and performance are satisfied.

In robust control, it is most common to use a (possibly weighted) \( \mathcal{H}_\infty \) norm in the description of the plant model set (Skogestad and Postlethwaite, 2005; Zhou et al., 1996) such that

\[
P = \left\{ P(\Delta, \hat{P}) \mid \|\Delta\|_\infty \leq \gamma \right\}, \tag{4.3}
\]

which is also the approach we follow in this chapter. This \( \mathcal{H}_\infty \) norm can be interpreted as the peak gain of the model error \( \Delta \), i.e. its maximum amplitude over the frequencies. In control it is a common practice to use low-order models \( \hat{P} \) to keep the controller design feasible, this means that \( \Delta \) is often littered with unmodeled dynamics. When those dynamics are resonant, it can be cumbersome to detect the respective peak amplitudes reliably from a short measurement dataset. In this chapter, this problem will be tackled by exploiting the local models.

### 4.3 Problem Formulation

Robust control based on \( \mathcal{H}_\infty \) optimization requires a nominal parametric model \( \hat{P} \) and a bound on the model error \( \Delta \). After the model \( \hat{P} \) is determined, either through first principles modeling or system identification, it remains to determine a bound on the model error \( \Delta \). The \( \mathcal{H}_\infty \) norm of the error system \( \Delta \) is

\[
\gamma = \|\Delta\|_\infty \overset{\Delta}{=} \sup_\omega \sigma(\Delta(\omega)) \tag{4.4}
\]

for a stable system where \( \sigma(\bullet) \) denotes the maximum singular value of \( \bullet \). Note that for single-input single-output (SISO) systems, \( \sigma(\Delta) = |\Delta| \) and the definition above reduces to

\[
\gamma = \|\Delta\|_\infty \overset{\Delta}{=} \sup_\omega |\Delta(\omega)| \text{ for SISO systems.} \tag{4.5}
\]

In robust control design methodologies, such \( \mathcal{H}_\infty \) norms often play an important role to specify the ‘size’ of a model class for which the controller that is to be designed should meet certain requirements (Skogestad and Postlethwaite, 2005).

**Remark 4.1.** It is even more common to use a weighted norm \( \|W \Delta V\|_\infty \) in robust control designs where \( (W, V) \) are stable and minimum-phase weighting filters. Two use cases can be distinguished: either these filters are chosen by the control engineer and hence known, or, one wishes to construct such filters from data. In the case where the weighting filters are ‘known’, the approach of this chapter can be adapted in a straightforward
manner to account for these weights, i.e., by replacing $\bar{\sigma}(\Delta)$ with $\bar{\sigma}(W \Delta V)$ in the formulas. The other case is explained in Remark 4.6.

To estimate $\gamma$, only $\Delta(\omega_\star)$ is of interest, where

$$\omega_\star \triangleq \arg \sup_\omega \bar{\sigma}(\Delta(\omega)). \tag{4.6}$$

If the measurement time is finite, non-parametric approaches only give access to a finite and discrete grid of frequencies $\omega_k \in \Omega_{\text{FRF}}$, i.e. $\Omega_{\text{FRF}} \triangleq \{ \omega_1, \omega_2, \ldots, \omega_{N-1} \}$. Typically, this discrete grid is the frequency grid of the discrete Fourier transform (DFT). As such, (4.4) can be rewritten as

$$\gamma \triangleq \sup_\omega \bar{\sigma}(\Delta(\omega)) = \max \left\{ \max_{\omega \in \Omega_{\text{FRF}}} \bar{\sigma}(\Delta(\omega)), \sup_{\omega \in \Omega} \bar{\sigma}(\Delta(\omega)) \right\}. \tag{4.7}$$

For convenience, we extract both contributions above as:

$$\gamma_{\text{FRF}} \triangleq \max_{\omega \in \Omega_{\text{FRF}}} \bar{\sigma}(\Delta(\omega)) \tag{4.8}$$
$$\gamma_{\text{IG}} \triangleq \sup_{\omega \in \Omega} \bar{\sigma}(\Delta(\omega)) \tag{4.9}$$

where $\Omega \triangleq \left[ 0, \frac{2\pi}{T_s} \right]$ is the measured frequency range for a sampling time $T_s$. Direct inspection of the frequency response function (FRF) data only allows access to $\gamma_{\text{FRF}}$. Two cases can be distinguished:

1. $\omega_\star \in \Omega_{\text{FRF}}$ such that $\bar{\sigma}(\Delta(\omega_\star)) = \gamma = \gamma_{\text{FRF}}$ is observed directly on the discrete frequency grid,

2. $\omega_\star \notin \Omega_{\text{FRF}}$, i.e. the extremal value $\bar{\sigma}(\Delta(\omega_\star)) = \gamma = \gamma_{\text{IG}}$ is determined by the inter-grid (IG) behavior.

The use of $\gamma_{\text{FRF}}$ is standard in FRF-based approaches, e.g. van de Wal et al., (2002). However, when the measurement time is finite, the expected value of $\gamma_{\text{FRF}}$ for different realizations of the input signal and noise, may underestimate the actual peak gain $\gamma$, i.e. $E[\gamma_{\text{FRF}}] \leq \gamma$. This is due to the limited frequency resolution imposed by the measurement time: the frequency where the actual peak gain occurs might not coincide with the excited frequency grid. Such an underestimate of $\gamma$ leads to overly optimistic controller designs which in turn can cause unstable behavior of the plant.

The main contribution of this chapter is to estimate the inter-grid term $\gamma_{\text{IG}}$ without the need for a (global) parametric model or imposing worst-case prior knowledge that leads to conservatism. Instead, a realistic value of $\gamma$ is obtained using parametric models that are valid over a finite frequency range.
4.4 $\mathcal{H}_\infty$ Norm Estimation Procedure

4.3.1 Set-Up

Consider an output error linear time-invariant (LTI) discrete-time SISO system $\Delta(q)$ excited by an input signal $u_\Delta(n)$. This can be described in the time-domain as

$$y_\Delta(n) = y_{\Delta 0}(n) + v(n) = \Delta(q)u_\Delta(n) + H(q)e(n) \quad (4.10)$$

where $q^{-1}$ is the lag operator, $e(n)$ is white Gaussian noise with unity variance such that $v(n) = H(q)e(n)$ can be colored noise, and both $\Delta(q)$ and $H(q)$ are stable causal real-rational functions. For a limited data record ($n \in \{0, \ldots, N - 1\}$ with $N$ the number of data points), the transient contributions $t_\Delta(n)$ lead to:

$$y_\Delta(n) = \Delta(q)u(n) + H(q)e(n) + t_\Delta(n). \quad (4.11)$$

By applying the DFT

$$X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) \exp\left(-\frac{j2\pi kn}{N}\right) \quad \forall k \in \mathbb{Z} \quad (4.12)$$

to both sides of (4.11), one obtains:

$$Y_\Delta(k) = \Delta(\omega_k)U_\Delta(k) + T_\Delta(\omega_k) + V(k). \quad (4.13)$$

The $T_\Delta$ contribution models the difference between the begin and end conditions of the filters $\Delta$ and $H$ in the measurement (Pintelon, Schoukens, and Vandersteen, 1997). The index $k$ corresponds to the $k^{th}$ frequency bin with frequency $\omega_k = \frac{2\pi k}{NT_s}$ such that the frequency grid $\Omega_{FRF} \triangleq \{ \omega_k | k = 1, \ldots, N - 1 \}$.

The presented results in this paper directly extend to closed-loop systems, as is shown in Section 4.6.1. The results in this chapter are presented in discrete time, but the extension to the continuous time case is straightforward.

4.4 $\mathcal{H}_\infty$ Norm Estimation Procedure

To estimate $\gamma_{IG}$, we propose to identify local parametric models in the frequency domain. The frequency axis is partitioned into continuous segments $\Omega_k \triangleq [\omega_{k-1}, \omega_{k+1}]$ in view of the local validity of the local models. In particular, the inter-grid aspect of the $\mathcal{H}_\infty$ norm is estimated as

$$\gamma_{IG} = \sup \left\{ \sup_{\omega \in \Omega_k} \sigma(\bar{\Delta}_k(\omega)) \mid k = 1, \ldots, N - 1 \right\} \quad (4.14)$$
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\[ \tilde{\Delta}_k \]

\[ \tilde{\Delta}_2 \]

\[ \Delta_3 \]

\[ \gamma_{IG} \]

Amplitude \( \sigma(\Delta) \)

\[ \Omega_1 \]

\[ \Omega_2 \]

\[ \Omega_3 \]

\[ \Omega_4 \]

\[ \Omega_5 \]

\[ \Omega_6 \]

Frequency \( \omega \) [rad/s]

\[ k_L(\omega) = -1 \]

\[ k_R(\omega) = -1 \]

\[ \omega_1 \]

\[ \omega_2 \]

\[ \omega_3 \]

\[ \omega_4 \]

\[ \omega_5 \]

\[ \omega_6 \]

Figure 4.3: Illustration of the local models \( \tilde{\Delta}_k \) around each frequency bin \( \omega_k \) for \( k \in \{1, \ldots, 6\} \). The bottom part shows the equivalence of the frequency segments \( \Omega_k \) and the intervals \( \{ \omega \mid k_L(\omega) = k \lor k_R(\omega) = k \} \).

where the \( \tilde{\Delta}_k \) is the local model around the \((k^{th})\) frequency bin \( \omega_k \). An example of such local models is shown in Figure 4.3.

Direct implementation of (4.14) allows a simple and efficient evaluation of \( \gamma_{IG} \) that matches well to parallel computation frameworks such as MapReduce (Dean and Ghemawat, 2008). In particular, the inner \( \sup_{\omega \in \Omega_k} \) can be understood as a map operation that maps a local model to its peak value and the outer \( \sup \) is a reduce that only retains the global peak value given the peak value of each local model. Unfortunately, such an implementation offers no access to the behavior of \( \sigma(\Delta(\omega)) \), e.g. around the peak amplitude.

**Interpolation method**  To allow inspection of \( \tilde{\sigma}(\Delta(\omega)) \) in (4.5) for arbitrary values of \( \omega \in \Omega \), we propose an interpolation approach to estimate \( \gamma \) and to allow visual inspection of \( \tilde{\sigma}(\Delta(\omega)) \). It can be understood intuitively that due to the limited validity domain of the local models, an estimate of \( \Delta(\omega) \) can only be obtained from \( \tilde{\Delta}_k(\omega) \) when the distance \( |\omega_k - \omega| \) is small. As the local models are each constructed around the frequency \( \omega_k \), for an arbitrary \( \omega \in \Omega \), one can find two adjacent frequencies around which such local models are available. To this end, two functions that indicate respectively the left and
the right adjacent local models for frequency $\omega \in \Omega$ are defined:

$$k_L(\omega) \triangleq \max \{ k \mid \omega_k \in \Omega_{\text{FRF}}, \omega_k \leq \omega \} \quad (4.15)$$

$$k_R(\omega) \triangleq \min \{ k \mid \omega_k \in \Omega_{\text{FRF}}, \omega_k \geq \omega \} \quad (4.16)$$

Using the $k_L(\omega)$ and $k_R(\omega)$ functions, two candidate values for $\sigma(\Delta(\omega))$ are obtained:

$$\sigma(\tilde{\Delta}_{k_L}(\omega)) \quad \text{and} \quad \sigma(\tilde{\Delta}_{k_R}(\omega))$$

In view of obtaining the maximum singular value, these are ‘interpolated’ to obtain a single estimate:

$$\sigma(\Delta_{\text{local}}(\omega)) \triangleq \max \left\{ \sigma(\tilde{\Delta}_{k_L}(\omega)), \sigma(\tilde{\Delta}_{k_R}(\omega)) \right\} \quad (4.17)$$

by retaining the largest of both. It follows that

$$\| \tilde{\Delta}_{\text{local}} \|_\infty \triangleq \sup_{\omega \in \Omega} \sigma(\Delta_{\text{local}}(\omega)) \quad (4.18)$$

$$= \sup_{\omega \in \Omega} \max \left\{ \sigma(\tilde{\Delta}_{k_L}(\omega)), \sigma(\tilde{\Delta}_{k_R}(\omega)) \right\} \quad (4.19)$$

**Remark 4.2.** The construction of $\sigma(\Delta_{\text{local}}(\omega))$ makes it more involved to implement the interpolation approach in an efficient and/or parallel fashion.

**Remark 4.3.** The notions $\sigma(\Delta_{\text{local}}(\omega))$ and the ensuing $\| \tilde{\Delta}_{\text{local}} \|_\infty$ should be regarded as notational convenience to denote that local models are used to estimate the maximal singular value and $\mathcal{H}_\infty$ norm, respectively. In reality, there is no tangible ‘$\Delta_{\text{local}}$’ that is constructed: neighboring local models are stitched together using the ‘$\max$’ operator.

**Remark 4.4.** The optimization problems (4.19) and (4.14) each amount to one-dimensional optimization problems, hence they can be solved accurately, fast and easily by gridding the frequency $\omega$. The peak amplitudes obtained using both implementations are identical since the local models $\tilde{\Delta}_k$ are evaluated over identical frequency ranges

$$\Omega_k = \{ \omega \mid k_L(\omega) = k \lor k_R(\omega) = k \} \quad (4.20)$$

for both methods (but in different order) and aggregated using the supremum operator, which is indifferent to the order of its arguments. In Figure 4.3 the functions $k_L(\omega)$ and $k_R(\omega)$ are visualized and the equivalence with $\Omega_k$ is apparent.

### 4.4.1 The Local Rational Method

The LRM and LPM are described in Chapter 3 and (McKelvey and Guérin, 2012; Pintelon et al., 2010a,b; Schoukens, Vandersteen, Barbé, et al., 2009) in more detail. Below, we briefly re-iterate some pertinent aspects and notations.
The local models $\tilde{\Delta}_k$ are estimated using the LRM (McKelvey and Guérin, 2012). The LRM exploits the fact that the transfer function $\Delta(\omega)$ and the leakage term $T_\Delta(\omega)$ in (4.13) are smooth functions of the frequency. As such, $\Delta$ and $T_\Delta$ are approximated locally by rational functions around each bin $k$:

$$\Delta(\omega_{k+r}) \approx \frac{\sum_{i=0}^{N_B} b_i(k) r^i}{1 + \sum_{i=1}^{N_A} a_i(k) r^i} = \frac{B_k(r)}{A_k(r)} \triangleq \tilde{\Delta}_k(\omega_{k+r})$$

(4.21)

$$T_\Delta(\omega_{k+r}) \approx \frac{\sum_{i=0}^{N_T} t_i(k) r^i}{1 + \sum_{i=1}^{N_A} a_i(k) r^i} = \frac{T'_k(r)}{A_k(r)} \triangleq \tilde{T}_k(\omega_{k+r}).$$

(4.22)

Its local parameters $a_i(k), b_i(k)$ and $t_i(k)$ are estimated by considering (4.13) in the local window $\Omega = \{-N_W, \ldots, +N_W\}$ around each bin $k$. This yields

$$Y_\Delta(k+r) \approx \Delta(\omega_{k+r}) U_{\Delta}(k+r) + T_\Delta(\omega_{k+r}) \quad \forall r \in \Omega.$$  

(4.23)

By substituting the local models $\tilde{\Delta}_k$ and $\tilde{T}_k$, one obtains

$$Y_\Delta(k+r) \approx \tilde{\Delta}_k(\omega_{k+r}) U_{\Delta}(k+r) + \tilde{T}_k(\omega_{k+r}) \quad \forall r \in \Omega$$

(4.24)

$$= \frac{B_k(r)}{A_k(r)} U_{\Delta}(k+r) + \frac{T'_k(r)}{A_k(r)} \quad \forall r \in \Omega.$$  

(4.25)

To make this expression linear in the parameters $a_i(k)$, both sides are multiplied by $A_k(r)$ as in (Levy, 1959). By doing so, the local linear-least-squares cost function

$$\sum_{r=-N_W}^{N_W} |A_k(r) Y_\Delta(k+r) - B_k(r) U_{\Delta}(k+r) - T'_k(r)|^2$$

(4.26)

around each bin $k$ can be constructed.

We denote LRM $(N_W, N_B, N_A, N_T)$ to be the LRM with the orders $N_B, N_A, N_T$ and bandwidth $N_W$ defined above. For $N_A = 0$ the LRM simplifies to the LPM which we shall denote as LPM $(N_W, N_B, N_T)$.

Remark 4.5. Since the LRM is used to locally estimate the FRF based on measurements, this sets a limit on the accuracy of the ensuing estimate of the peak value. This was treated in Section 3.4.

### 4.5 Simulation

To allow the generalization of the results of the simulations, we first introduce a way to decompose complex systems into simpler systems. The behavior of these simpler systems
then allows to predict the behavior for these more complex systems. Concretely, any proper real-rational system $\Delta$ can be decomposed in a parallel connection of lower-order sub-systems by means of a partial fraction expansion as in (2.33), i.e.

$$
\Delta(s) = \sum_{i=1}^{I_1} \sum_{\nu=1}^{M_{1,i}} \frac{K_i}{(s - p_i)^\nu} + \sum_{i=1}^{I_2} \sum_{\nu=1}^{M_{2,i}} \frac{K_{i,\nu} \omega_{ni}^{2\nu} (s + \omega_{zi})^\nu}{\omega_{zi} (s^2 + 2\xi_i \omega_i s + \omega_{ni}^2)^\nu}
$$

(4.27)

with the same notations as in Chapter 2, most importantly, each resonant pole can be characterized by its frequency $\omega_{ni}$ and damping $\xi_i$. In the structural engineering community (see e.g. Gawronski, 2004, Section 2.2.2) it is well-known that around sharp resonance peaks ($\omega \approx \omega_{ni}$), the shape of a structural transfer function is mainly determined by its modal transfer function. In many practical situations, this means that the gray terms in the equation often have a negligible influence. For local modeling, the response is hence well-approximated by only considering the poles and zeros in the local frequency window. To estimate $\|\Delta\|_\infty$ mainly the resonant sub-systems are of interest as those are responsible for large gains of the system. Assuming such resonances are sufficiently separated in the frequency, (Gawronski, 2004; Geerardyn et al., 2013; Schoukens, Vandersteen, Pintelon, Emedi, et al., 2013) show that results for a single resonance are applicable to more complex systems. This is done by considering only the dominant resonance – viz. the one that is the most relevant for the $\mathcal{H}_\infty$ controller design. By normalizing the conclusions for the time constant $\tau$ (or the 3 dB bandwidth), the results are generally valid for different values of the damping $\xi$. Specifically, the time constant $\tau = (\xi \omega_n)^{-1}$ of the dominant pole dictates the required measurement time (Schoukens, Vandersteen, Pintelon, Emedi, et al., 2013), where $\xi$ is the relative damping and $\omega_n$ the natural frequency of the dominant pole.

### 4.5.1 A Concrete Simulation

Consider a discrete time system $\Delta$ (see Figure 4.4) as unmodeled dynamics in an output-error setting:

$$
\Delta(z) = \frac{0.45373 z + 0.44752}{z^2 - 1.0595 z + 0.96079},
$$

(4.28)

which has $\tau = 50$ samples (i.e. $\xi = 0.02$ and $\omega_n = 1$ rad/s). The input signal $u_\Delta$ is white Gaussian noise with unit variance and the disturbing noise variance $\sigma_v^2$ is chosen such that a signal to noise ratio (SNR)

$$
\text{SNR} = \frac{\sqrt{N^{-1} \sum_n y_{\Delta 0}^2(n)}}{\sqrt{N^{-1} \sum_n v^2(n)}} = \frac{\sigma_{y_{\Delta 0}}}{\sigma_v} \approx \frac{\|\Delta\|_2}{\sigma_u} \frac{\sigma_u}{\sigma_v}
$$

(4.29)

of 10 is obtained. In the frequency domain, this corresponds to $\text{SNR}_{BW} \approx 26$ dB in the 3 dB bandwidth of this resonance peak.
Classical spectral analysis (SA) techniques (Bendat and Piersol, 1993) can estimate the FRF of $\Delta$. This is done by first splitting the input and output signals ($u(n)$ and $y(n)$) into $n_S$ segments of $N/n_S$ samples that we will denote as

$$x^{[i]}(n) = x((i - 1)n_S + n) \text{ with } n \in \{0, \ldots, N/n_S - 1\} \text{ and } i \in \{1, \ldots, n_S\}.$$  

(4.30)

Each of these segments is windowed using a function $w_{N/n_S}(n)$ to reduce leakage such that

$$x^{[i]}_w(n) = x^{[i]}(n) \cdot w_{N/n_S}(n).$$  

(4.31)

The frequency domain counterparts are obtained by means of the DFT, and are denoted

$$X^{[i]}_W(k) = \text{DFT}(x^{[i]}_w(n)),$$  

(4.32)

where it should be noted that $k \in \{0, N/n_S\}$ such that the frequency resolution is $\frac{2\pi n_S}{N}$. By dividing the resulting auto- and cross-spectra, the FRF of $\Delta$ can be estimated

$$\hat{\Delta}_{SA}(k) \triangleq \frac{1}{n_S} \sum_{i=1}^{n_S} Y^{[i]}_W(k) \overline{U^{[i]}_W(k)}$$

$$\frac{1}{n_S} \sum_{i=1}^{n_S} U^{[i]}_W(k) \overline{U^{[i]}_W(k)}$$  

(4.33)

In this chapter a Hann window ($w_{N/n_S}(n) = \sin^2(\frac{\pi n}{N/n_S - 1})$) and $n_S = 1$ are used. Note that $n_S > 1$ increases the measurement time by a factor $n_S$ to attain the same frequency resolution (but would improve transient and noise rejection).

Figure 4.4 illustrates for $N = 97$ that $\gamma_{\text{FRF}}$ for the SA, LPM (5, 4, 4) and LRM (5, 2, 2, 2) approaches yield unreliable estimates of $\|\Delta\|_\infty$, since the frequency resolution is too coarse. The specific orders of LRM/LPM have been chosen to keep $N_W$ small while not restricting ourselves to local linear models ($N_B = 1$). Slightly different orders (e.g. LRM (4, 1, 1, 1)) yield comparable results.

It can be observed for $\gamma_{\text{IG}}$ in Figure 4.4 that the LRM is able to model the resonance well and yields a useful estimate $\|\Delta_{LRM}\|_\infty = 24$ in the simulation. Intuitively, the LRM locally approximates the resonant pole which allows for a reliable inter-grid estimate. The LPM, on the other hand, does not provide a reliable estimate.

### 4.5.2 Study of the Experiment Length

The frequency resolution, and hence the experiment length, is key to obtain a reliable $H_\infty$ norm estimate. In this section, we determine the minimal experiment length needed for the proposed approach by means of simulations.
Figure 4.4: Simulation example revealing that $\gamma_{\text{FRF}}$ (left) for both the spectral analysis (SA) and LRM underestimate the true $\gamma = \|\Delta\|_{\infty} = 24$ due to limited frequency resolution. Using the proposed LRM-based approach, both the at-grid ($\gamma_{\text{FRF}}$) and intergrid ($\gamma_{\text{IG}}$) estimate are improved. The interpolated LRM and true $\Delta$ almost coincide. Hence, $\|\Delta_{\text{LRM}}\|_{\infty}$ approaches the theoretical $\gamma = 24$. 

\[ \text{SA} \quad \text{LRM} \quad \text{LPM} \]

\[ \|\Delta\|_{\infty} \]

\[ \text{ω [rad/s]} \]

\[ \text{Amplitude } |\Delta(\omega)| \]
Figure 4.5: This Monte Carlo simulation shows that the $\mathcal{H}_\infty$ norm (- - -) is underestimated significantly for short experiments ($N < 20\tau$) by $E[\gamma_{FRF}]$ (-----) and the LPM (-----). However, the proposed $\|\Delta_{LRM}\|_\infty$ (-----) is already reliable from much shorter experiments ($N \geq 4\tau$). A single observation for $N = 97$ (-----) is shown in Figure 4.4. The shaded areas show the ±$\sigma$ intervals which are due to the noise.

We simulate the system (4.28) for 16 100 samples and discard the first $2\tau = 100$ samples to randomize the initial conditions. To mimic the effect of progressively measuring longer data records, the first remaining $N$ samples are used to estimate $\|\Delta\|_\infty$ as in Section 4.4. The values of $N \in [35, 16000]$ are chosen such that $E[\gamma_{FRF}(N)]$ is a non-decreasing function of $N$. This ensures that grid misalignment between the actual resonance frequency and the observed frequency grid is accounted for such that the worst-case estimate of the resonance is obtained. These simulations are then repeated in a Monte Carlo simulation of $n_{MC} = 200$ runs where each run has a different realization of $u_{\Delta}(n)$ and $v(n)$. The resulting sample average and standard deviation $\sigma$ are shown in Figure 4.5.

It can be seen in Figure 4.5 that $N \geq 20\tau$ is required to attain a frequency grid that is dense enough to obtain a reliable $\gamma_{FRF}$, i.e. $E[\gamma_{FRF}] \approx \gamma$. Starting from about $N = 4$ to $5\tau$, the interpolated LRM starts to yield good estimates for $\gamma_{1G}$. For long experiments ($N \to \infty$), the true $\|\Delta\|_\infty$ is recovered. Practically, this implies that the
interpolated LRM beats the on-grid estimate using a data record that is only a fraction of the length.

**Remark 4.6.** In robust control design, often the interest is not only in determining the peak value of the uncertainty, but also in obtaining a tight overbound of the uncertainty, e.g. as in Scheid and Bayard, (1995). This is closely related to the design of (parametric) weighting filters \((W, V)\) such that \(|W \Delta V|\) is almost a constant function of the frequency in the frequency bands of interest for control design. Essentially, this is done to ensure that a synthesized robust controller is largely invariant under the choice of uncertainty structure chosen in the model set.

The proposed local parametric approach can be extended straightforwardly to accommodate for this. In particular, the overbound of \(\Delta\) can be generated for an arbitrarily dense frequency grid—i.e. this includes the inter-grid behavior—such that the approach of Scheid and Bayard, (1995) can be used directly.

---

<table>
<thead>
<tr>
<th>Guideline 4.1: Measure at least (5\tau) to observe the peak value using the local models</th>
</tr>
</thead>
<tbody>
<tr>
<td>After 4 to 5 time constants of the dominant pole, the local models offer a reliable insight into the peak value of a transfer function when a sufficient SNR of approximately 20 dB is available such that the LRM can be used. For purely FRF-based estimates, on the other hand, the measurement only attains similar performance when measuring for (20\tau).</td>
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### 4.6 Application to an AVIS

The proposed method is illustrated on the AVIS shown in Figure 4.7. The broad aim of vibration isolation (Collette et al., 2011; Karnopp, 1995) is to suppress vibrations such that a certain payload—e.g. passengers in a car, sensitive equipment such as atomic force microscopes and wafer-scanners used for lithography—is not disturbed by vibrations generated by either the payload itself \((F_{d(pl)})\) or by the outside world \((F_{d(ext)})\) as indicated in Figure 4.6.

The typical set-up for vibration isolation is illustrated in Figure 4.6a. If the controller \(C\) is removed, the set-up reduces to a passive isolation system where the suspension is to be designed such that the payload velocity \(y = v_z\) is influenced as little as possible by both the internal disturbances \((F_{d(pl)})\) and the external disturbances \((F_{d(ext)})\). To suppress \(F_{d(ext)}\), a compliant suspension is required such that the payload and the floor are decoupled. However, to enable suppression of the system disturbances \(F_{d(pl)}\), a stiff suspension is desirable to minimize the deflection of the payload (Voorhoeve et al.,...
2015). This obviously leads to a trade-off in the design. In active vibration isolation, measurements of the position, velocity and even acceleration of the payload allow more freedom in tuning the suspension of the payload. Typically, the controller $C$ acts as an additional damping term. However, when such measurements are relative to the floor or other objects which are subject to the external disturbances $F_{d(\text{ext})}$ themselves, the trade-off for passive vibration isolation remains applicable.

On the other hand, when a measurement of the absolute velocity of the payload is available, the external disturbances can be reduced effectively. Absolute velocity measurements can be obtained e.g. from inertial sensors such as geophones (Munnig Schmidt et al., 2014). In particular, such sensors suspend an inertial mass using suspension with very low resonance frequency to the payload. By observing the relative movement of the inertial mass and the payload, the absolute velocity of the payload can be measured (Collette et al., 2011). Conceptually, the active controller $C$ can be seen as connected to an imaginary fixed point in the ‘sky’ (Figure 4.6b), which is not disturbed by $F_{d(\text{ext})}$ for an ideal sensor. Consequently, the trade-off that exists in passive isolation is eliminated. A compliant passive suspension can decouple the payload from external disturbances while the active controller acts as a stiff damper to the ‘sky’ such that the effect $F_{d(\text{pl})}$ is suppressed.

Figure 4.6: Conceptual illustration of (active) vibration isolation set-ups. In addition to the passive isolation (spring and damper), the active controller $C$ provides additional isolation using the measurement of the payload velocity ($y = v_z$).
The particular AVIS used for the experiments, consists of a support fixed to the ground and a payload platform, suspended on airmounts (see Figure 4.8) providing passive vibration isolation. The payload platform is able to move in six mechanical degrees of freedom. Six geophones measure the payload velocity in an inertial reference frame, hence providing absolute velocity measurements, and eight linear motors (see Figure 4.8) allow to actively compensate for vibration of the platform using the principle of ‘sky-hook damping’. Please refer to Appendix 4.A for more details regarding the mechanical set-up, placement of the sensors and the signal processing involved. To facilitate the presentation, only the vertical translative velocity \( v_z \), i.e. a SISO system, is considered.

### 4.6.1 Measurement & Control Framework

The experiments on the AVIS are carried out in closed-loop (Figure 4.9) with \( r_2 = 0 \). The controller \( C = C_{\text{exp}} \) is a diagonal proportional-integral (PI) regulator which stabilizes the AVIS but yields only moderate vibration isolation. See Appendix 4.A for detailed information regarding both the experimental setup and the experimental controller. The \( r_1 \) input consists of five periods of a random phase multisine of 65536 samples with excited bins such that \( \omega_k \approx 1.001\omega_{k-1} \) and \( T_s = 1 \text{ ms} \). See Chapter 2 and (Geerardyn et al., 2013) for more information regarding the design of such a signal.

The parametrization of the plant model set \( \mathcal{P} \) conforms to the dual-Youla-Kučera framework (B. D. O. Anderson, 1998; F. Hansen et al., 1989) that parametrizes \( \mathcal{P} \) in terms of all plants \( P \) that are stabilized by the controller \( C \):

\[
\mathcal{P} \triangleq \left\{ \frac{\hat{N} + D_c \Delta}{D - N_c \Delta} \bigg| ||\Delta||_\infty \leq \gamma \right\}, \tag{4.34}
\]

where \( C = N_c D_c^{-1} \) and \( \hat{P} = \hat{N} \hat{D}^{-1} \) are decomposed as right co-prime factorization (RCF) (see Figure 4.10). To estimate \( \Delta \) and \( ||\Delta||_\infty \), the signals \( u_{\Delta} \) and \( y_{\Delta} \) in Figure 4.10 are required. These can be computed from \( u \) and \( y \) directly (B. D. O. Anderson, 1998):

\[
u_{\Delta}(t) = \left( \hat{D} + C \hat{N} \right)^{-1} \left[ C \begin{bmatrix} 1 \\
\end{bmatrix} \begin{bmatrix} y(t) \\
u(t) \end{bmatrix} \right] \tag{4.35}
\]

\[
y_{\Delta}(t) = \left( D_c + \hat{P} N_c \right)^{-1} \left[ I - \hat{P} \right] \begin{bmatrix} y(t) \\
u(t) \end{bmatrix} \tag{4.36}
\]

In addition, \( u_{\Delta} \) is noise free under standard assumptions (F. Hansen et al., 1989) and uncorrelated to \( y_{\Delta} \). Hence the method of Section 4.4 applies directly.
Chapter 4  Non-parametric Peak Modeling

Figure 4.7: Active vibration isolation system (AVIS).

Figure 4.8: Isolation module of the AVIS. The left photo shows one of the air mounts providing passive vibration isolation. The right photo shows two linear motors that play a role in the active vibration isolation.
The non-uniqueness of the RCF can be exploited to satisfy additional conditions. In view of the robust control criterion

\[
J(P, C) \triangleq \| W T(P, C) V \|_{\infty}, \tag{4.37}
\]

the RCFs are constructed such that

\[
J_{\text{wc}}(P, C_{\text{exp}}) \leq J(\hat{P}, C_{\text{exp}}) + \gamma \tag{4.38}
\]

where \( J_{\text{wc}}(P, C) \triangleq \sup_{P \in \mathcal{P}} J(P, C) \) (Oomen and Bosgra, 2012). \( T(P, C) \) denotes the closed loop transfer function from the loop inputs \([r_2 \ r_1]^T\) to the output and input of the plant \([y \ u]^T\) as schematically in Figure 4.9. This can be expressed symbolically, as in Oomen and Bosgra, (2012), by

\[
T(P, C) \triangleq \begin{bmatrix} P \\ I \end{bmatrix} (I + CP)^{-1} \begin{bmatrix} C \\ I \end{bmatrix}. \tag{4.39}
\]

I.e. even for a weighted control criterion (4.37), there is no need to incorporate weights \( W' \) and \( V' \) into the estimation of the uncertainty bound as \( \| W' \Delta V' \|_{\infty} \). In this chapter, an eighth-order \( \hat{P} \) (Figure 4.11) is estimated. Some dynamics are left unmodeled and are hence part of \( \Delta \) (Figure 4.12).
4.6.2 Results

To estimate $\Delta(\omega_k)$ and $\|\Delta\|_\infty$ using the proposed approach with LRM (6, 2, 2, 2), a short estimation dataset is used. This estimation dataset is sampled from a larger dataset that contains four times more excited frequency lines (and hence produces a four times more densely sampled FRF). The full dataset, i.e. including the bins not seen during the estimation phase, is used to validate the interpolation results.

The measured $\Delta$ is shown in Figure 4.12 for the whole frequency band and Figure 4.13 for a frequency subrange containing $\omega_\star$. Two aspects of the proposed method are examined. First, the interpolation results for a continuous frequency range are validated. Second, the peak amplitudes of $\Delta(\omega)$ are inspected as these are indicators for $\|\Delta\|_\infty$.

Interpolation performance Figure 4.12 and Figure 4.13 show that the estimation and validation data for LRM are generally in agreement. In the frequency range shown in Figure 4.13, the relative difference between both is at most 2%.

Peak value estimates Near 296 Hz, a modest improvement of the peak estimate from $-15.3$ dB (at-grid) to $-14.0$ dB is obtained by the interpolation. The estimate near
4.6 Application to an AVIS

Figure 4.12: Measured (dual-Youla) $\Delta$ for the AVIS using the LRM and the proposed interpolation. The data in the rectangle is also shown in Figure 4.13.

Figure 4.13: Measured $\Delta$ for the AVIS using the LRM (---) and the proposed interpolation. Clearly, the LRM leads to a higher peak value (----), especially at 328 Hz. The validity of the LRM local parametric model is confirmed by the validation measurement at the dense frequency grid (-- --), which reveals excellent interpolation properties and a similar peak value (---).
\( \omega_\ast \approx 328 \text{ Hz} \) is improved more substantially. The proposed method (4.19) yields \( \gamma_{1G} = -2.5 \text{ dB} \). \( \gamma_{\text{FRF}} \approx -9.9 \text{ dB} \) for the estimation data, which means that a 7.4 dB improvement is achieved by the proposed method. This peak agrees well with the validation data, where the nearest (at-grid) value has an amplitude of -3.2 dB (i.e. 8% difference).

This suggests that the local rational models with interpolation enable accurate estimation of \(|\Delta|\) and hence also \( \|\Delta\|_\infty \) even if the actual peak does not coincide with the discrete frequency grid.

### 4.7 Conclusions

A local parametric \( H_\infty \) norm estimation technique for high performance, non-conservative robust control design is presented. In this chapter, a new approach is presented that employs so-called local parametric LRM models that lead to an enhanced estimate of \( \|\Delta\|_\infty \) of a SISO error system by interpolating between neighboring local models. It is also shown that the LPM is not a worthwhile alternative to the LRM to estimate \( \|\Delta\|_\infty \), at least for lightly-damped systems.

The technique is illustrated on a simulation example where the LRM-based interpolation yields an accurate estimate of the \( H_\infty \) norm. It is observed that the measurement time is reduced by almost an order of magnitude compared with techniques that only examine \( \Delta(\omega) \) on the DFT frequency grid. Using measurements on an AVIS, it is illustrated that the interpolation-based results can substantially improve \( \|\Delta\|_\infty \) and/or reduce the measurement time four-fold. Furthermore, the results are validated by a detailed validation measurement. Ongoing research focuses on extending the technique to MIMO systems.
Appendix

4.A AVIS Measurement set-up

The measurements of the AVIS are carried out using a Simulink® model that configures the data acquisition (DAQ) card, implements a digital controller and processes the signals. Simulink® executes the code, as shown in Figure 4.14 and Figure 4.17, using a constant time step $T_s = 1 \text{ ms}$ on a real-time computer running xPC Target (nowadays: Simulink® Real-Time) equiped with a Quanser Q8 DAQ card that is connected to the physical AVIS setup.

In the top-level model (Figure 4.14), all signals are six-dimensional and each element corresponds to a mechanical degree-of-freedom of the AVIS, i.e.

1. $v_x$: the velocity in $x$ direction (horizontal),
2. $v_y$: the velocity in $y$ direction (horizontal),
3. $v_z$: the velocity in $z$ direction (vertical),
4. $\phi_x$: the angular velocity around the $x$ axis,
5. $\phi_y$: the angular velocity around the $y$ axis,
6. $\phi_z$: the angular velocity around the $z$ axis.

The coordinate frame and axes are indicated in Figure 4.15.

In the main text, we have simplified this system to consider only $v_z$, the vertical translation (i.e. the third signal). For the other directions, the applied signals ($r1$ and $r2$) were set to 0. However, inside the feedback loop, all directions are controlled by the experimental controller $C^{\text{exp}}$.

Specifically, $C^{\text{exp}}$ is a $6 \times 6$ transfer function matrix that is implemented as a discrete state-space model. This controller was constructed by Rademakers, (2005) and is given
Figure 4.14: Simulink® model used during the AVIS measurements. All signals are six-dimensional real values.

Figure 4.15: AVIS with \((x, y, z)\) coordinate frame indicated and bird’s eye view of the approximate locations of the actuators and sensors at each of the four corners. For each of the actuators and sensors, the color indicates the principal direction.
by the state-space matrices below

\[
C^{\text{exp}} \triangleq \begin{cases} 
  x[n+1] = Ax[n] + Bu_c[n] \\
  y_c[n] = Cx[n] + Du_c[n]
\end{cases} \tag{4.40}
\]

\[
A \approx 828.8 \cdot 10^{-3} \cdot I_6 \tag{4.41}
\]

\[
B = C \approx \begin{bmatrix} 93.83 & 0 & 39.57 \\ 0 & 10^{-3} \end{bmatrix} \otimes I_3 \tag{4.42}
\]

\[
D = \begin{bmatrix} 4.982 & 0 & 0.886 \\ 0 & 10^{-3} \end{bmatrix} \otimes I_3 \tag{4.43}
\]

with \( I_n \) the \( n \times n \) identity matrix. It can be seen that the controller is diagonal, i.e. the directions are decoupled, and that the dynamics are common among the translational directions and the rotational directions respectively, such that

\[
C^{\text{exp}} = \begin{bmatrix} C^{\text{exp}}_{\text{tran}} & 0 \\ 0 & C^{\text{exp}}_{\text{rot}} \end{bmatrix} \otimes I_3 \tag{4.44}
\]

with

\[
C^{\text{exp}}_{\text{tran}} \approx \frac{4.9821z + 4.6787}{z - 0.8282} \cdot 10^{-3} \tag{4.45}
\]

\[
C^{\text{exp}}_{\text{rot}} \approx \frac{0.88595z + 0.83201}{z - 0.8282} \cdot 10^{-3} \tag{4.46}
\]

as visualized in Figure 4.16.

For multisine excitations the \texttt{RepeatingSequence} block was used to load the \( r2 \) signal from \texttt{MATLAB®}, for noise excitations, the \texttt{RandomNumber} block of Simulink® was used instead. The signals \( r2, u \) and \( y \) are returned from Simulink® to \texttt{MATLAB®} where further processing of the data is carried out. This allows to compute the transfer function from \( r2 \) to \( u \) and \( y \) respectively, or, the transfer function of the AVIS block in the Simulink® model.

The AVIS block (Figure 4.17), conceptually, contains:

- static transformations \((K_{\text{act}}, K_{\text{sens}})\), as derived from first principles in (Rademakers, 2005), which translates the sensor and actuator signals to physical velocities at the center of the payload,

- logic to communicate with the Quanser Q8 DAQ board which is physically connected to the AVIS, and

- signal conditioning.
Figure 4.16: Bode plots of the elements of $C^{\text{exp}}$ and sensor filter $F$ present in the set-up.

Figure 4.17: Simulink® model of the AVIS sub-block.
In Rademakers, (2005, Appendix A.4), linear static relationships are derived to link the velocities at the center of the payload to the corresponding actuator voltages. This yields the matrix:

\[
K_{\text{act}} \approx \begin{bmatrix}
0 & -0.5 & 0 & 0 & 0 & -0.6485 \\
0.0974 & 0.1233 & -0.25 & -0.6667 & 0.5263 & 0 \\
-0.5 & 0 & 0 & 0 & 0 & -0.5119 \\
0.0974 & -0.1233 & -0.255 & 0.6667 & 0.5263 & 0 \\
0 & 0.5 & 0 & 0 & 0 & -0.6485 \\
-0.0974 & -0.1233 & -0.25 & 0.6667 & -0.5263 & 0 \\
0.5 & 0 & 0 & 0 & 0 & -0.5119 \\
-0.0974 & 0.1233 & -0.25 & -0.6667 & -0.5263 & 0
\end{bmatrix}
\]

(4.47)

Note that \( K_{\text{act}} \in \mathbb{R}^{8 \times 6} \) and hence produces signals for the 8 actuators on the AVIS. These signals are limited to the range \( \pm 5 \text{ V} \) before they are sent to the digital-to-analog converters (DACs) on the Q8 DAQ board to avoid overdriving the DACs.

The 6 velocity signals that are digitized by the Q8 DAQ are each filtered to transform the voltage induced in the coils of the geophones on the AVIS into a velocity using a filter \( F \) with transfer function

\[
F(s) \approx \frac{6.51s^2 + 122s + 11.3}{6.4s^2 - 16.6s - 2.2}
\]

(4.49)

which is shown in Figure 4.16. More information regarding this design is given in (Rademakers, 2005, Appendix A.3). This filter is discretized automatically by Simulink® for the actual implementation. These velocities are then linked by a linear static transformation \( (K_{\text{sens}}) \) to the (angular) velocities.

The output \( y \) is clipped to \( \pm 20 \) for easier detection of errors. However, during normal operation and all the measurements, the signal levels were well within the linear region of this saturation block.

**Remark 4.7.** It should be noted that while the benchmark paper by Voorhoeve et al., (2015) deals with the same physical AVIS, the experimental data of the benchmark is fundamentally different from the data presented here. Voorhoeve et al., (2015) only consider the ‘physical’ AVIS with 8 inputs and 6 outputs as indicated by the QuanserQ8 block in Figure 4.17 as the system under test. The measurements of the benchmark data were carried out in open-loop. Hence, the experimental controller \( (C_{\text{exp}}^{\text{benchmark}} = 0) \), the saturations, the filter \( F \) and the static matrices \( K \) are different. Moreover, the sample rate of the benchmark data is much higher \( (f_s = 20 \text{ kHz}) \) compared to the data in this work \( (f_s = 1 \text{ kHz}) \) after an upgrade of the real-time computer responsible for acquiring the signals.
Initialization of Parametric Estimators

This chapter is based on Geerardyn, Lumori, et al., (2015).

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5.1 Introduction

The task of interpreting measurement data from dynamic systems often involves the estimation of a transfer function (TF). Typical measurement techniques and identification strategies of TFs or frequency response functions (FRFs) are presented in (Antoni and Schoukens, 2007; Broersen, 1995; Guillaume, Kollár, et al., 1996; Pintelon et al., 2010a; Schoukens et al., 1998, 2006). Application of these methods to real devices and systems is well-represented in the literature, e.g. (Behjat et al., 2010; Lim, 2010; Robinson et al., 1990).

Parametric identification of linear time-invariant (LTI) systems from either input/output data or non-parametric frequency response data, has been well developed as evidenced by published literature (Brillinger, 1981; Goodwin and Payne, 1977; Ljung, 1999b; McKelvey, 2002; Peeters et al., 2004; Pintelon, Guillaume, et al., 1998; Pintelon and Schoukens, 2012; Sanathanan and Koerner, 1963; Schoukens, Vandersteen, Pintelon, and Guillaume,
Many of such parametric identification algorithms, however, require solving non-linear or non-convex optimization problems. This often involves an iterative algorithm to produce the optimal model. The resulting model, however, depends on which initial estimates are used to start up the optimization process: typically one can end up in local minima, leading to models of sub-optimal quality. Hence, good initial estimates are essential to obtain a high quality model estimate. In this chapter, the maximum-likelihood estimator (MLE) is used as a proxy for any parametric transfer function estimator, since the MLE is commonly used and has desirable properties such as efficiency and consistency. Numerous other parametric identification techniques are devoted to the development of parametric plants $G(q^{-1}, \theta)$ and parametric noise models $H(q^{-1}, \theta)$, where $\theta$ is the model parameters vector (Ljung, 1999b; Pintelon and Schoukens, 2012; Söderström and Stoica, 1989). Since the initial value generation methods are not specifically tailored towards the MLE, other parametric identification frameworks are likely to benefit from these improved starting values as well. Hence, the presented approach is complementary to initialization schemes such as the work of van Herpen et al., (2014) that focuses on improving the optimization algorithm, rather than the starting values.

In the perspective of developing identification tools that require minimal user interaction, it is important that good initial estimates are produced for a wide variety of systems and experimental settings; i.e. to increase the probability of the optimization algorithm to produce good estimates, or preferably even the global optimum. However, when noisy input-output measurements are to be processed, the presence of noise obstructs a clear view of the actual system behavior, especially for poor signal to noise ratios (SNRs). Such problems can be mitigated using non-parametric techniques: the presence of noise is typically reduced by averaging multiple measurements or by using more advanced smoothing techniques.

The main aim and contribution of this chapter is to demonstrate that smoothing a measured FRF non-parametrically, helps to avoid local optima during the parametric estimation of the MLE of a transfer function, using classical (deterministic) optimization algorithms. Hence, such techniques make it possible to increase the “success rate” (i.e. the probability that a good model, or even the global optimum, is obtained) of such a system identification step considerably.

In particular, two different smoothing techniques:

- the time-truncated Local Polynomial Method (LPM) (Lumori et al., 2014), and
- the regularized finite impulse response (RFIR) (Chen, Ohlsson, et al., 2012; Pil-lonetto and De Nicolao, 2010)
are tested for different SNRs and different measurement record lengths of the input/output data of a few single-input single-output (SISO) LTI systems. These are compared with the existing initialization schemes, namely: (i) the generalized total least squares (GTLS), and (ii) the bootstrapped total least squares (BTLS).

Outline The rest of the chapter is structured as follows. Section 5.2 covers the problem formulation. Section 5.3 presents the methodology for obtaining the initial estimates, and their influence on the success rate of the MLE. This is followed by Section 5.4 and Section 5.5, which are demonstrations of the improved initial estimates on simulation and experimental data, respectively. In Section 5.7 the generality of the results is investigated briefly. The computational effort and recommendations for practical implementations are given in Section 5.8. Ultimately, concluding remarks are presented in Section 5.9.

5.2 Problem Formulation

In this section, the assumptions on the system and the noise are presented, together with the associated MLE (formulated in the frequency domain), which turns out to be a non-convex function in the parameters, requiring initial estimates. The procedure for obtaining the initial estimates is elaborated in the subsequent sections.

5.2.1 System Framework and Model

Figure 5.1 depicts a schematic of the output error framework for a generalized (single- or multiple-order) resonating, dynamic LTI discrete-time SISO system, subjected to a known white random noise input. The full model of the system is

\[
y(t) = G_{\circ}(q^{-1})u_{\circ}(t) + H_{\circ}(q^{-1})e(t)
\]

(5.1)

where \(G_{\circ}(q^{-1})\) represents the dynamics of the system to be estimated, \(u_{\circ}(t)\) is the input signal, \(v(t) = H_{\circ}(q^{-1})e(t)\) is the noise source at the output, \(H_{\circ}(q^{-1})\) is the noise dynamics, \(e(t)\) is white Gaussian noise, and \(q^{-m}\) is the backwards shift operator \((q^{-m}x(t) = x(t - mT_s))\) with \(m\) a positive integer and \(T_s\) the sampling time. We only treat the discrete-time case (i.e. \(t = n \cdot T_s\) for integer values of \(n\)) theoretically in this chapter. However, the generalization to continuous time is straightforward (Pintelon and Schoukens, 2012, Chapter 6) and has been demonstrated in Section 5.5.

In this chapter we assume that the output signal is disturbed by random noise, resulting in noisy error-prone data. For the simulations, white noise is used \((H_{\circ} = 1\) and \(e(t) = \)

$v(t))$. It is also possible to apply the estimation procedure in this chapter to a system that is disturbed by colored noise, as will be demonstrated in Section 5.5.

With reference to equation (5.1), the relation between the noiseless input and the output signals ($v(t) = 0$) is assumed to be of the form

$$A(q^{-1})y_{o}(t) = B(q^{-1})u_{o}(t) \quad \forall t \in nT_s \text{ with } n \in \mathbb{Z} \quad (5.2)$$

where $A$ and $B$ are polynomials in $q^{-1}$. Thus, it follows from equation (5.1) that

$$G_{o}(q^{-1}) = \frac{B(q^{-1})}{A(q^{-1})}. \quad (5.3)$$

From Pintelon and Schoukens, (2012, Section 6.3.2.), in the frequency domain, and for the discrete Fourier transforms (DFTs) of the windowed signals, equation (5.2) is of the form

$$A(e^{-j\omega k})Y_{o}(k) = B(e^{-j\omega k})U_{o}(k) + T(e^{-j\omega k}) \quad (5.4)$$

with $q^{-1}$ sampled in $q_{k}^{-1} = e^{-j\omega k}$, where $\omega_k = \frac{2\pi k}{N}$ are the DFT frequencies, and $T$ is a polynomial of order $\max(N_a, N_b) - 1$, which depends on the difference between the initial and end conditions of the filters $G_{o}$ and $H_{o}$ during the measurement record.

Remark 5.1. The knowledge of the true model orders makes the analysis of the smoothers easier to carry out as we thus avoid extra calculation time required to perform model selection. However, these smoothers can be used in conjunction with model order selection procedures.

Assumption 5.1. It is assumed that the orders of the polynomials $A$, $B$ and $T$ are known.

5.2.2 Parametric Identification Algorithm

The maximum likelihood estimate of the parameter vector $\theta$ containing the coefficients of the $A$, $B$ and $T$ polynomials is obtained by solving the optimization problem

$$\hat{\theta} = \arg \min_{\theta} V(\theta). \quad (5.5)$$
5.3 Methodology

Since the noise \( v(t) \) is assumed to be Gaussian, \( V(\theta) \) accords to the weighted least squares cost function (Pintelon and Schoukens, 2012, Section 9.11):

\[
V(\theta) = \frac{1}{2} \sum_{k=1}^{N/2-1} \frac{|\varepsilon(k, \theta)|^2}{\sigma^2_{\varepsilon}(k, \theta)}
\]

(5.6)

for \( N \) measurements and when \( \varepsilon \) denotes the error in equation (5.4), viz.:

\[
\varepsilon(k, \theta) = A(k, \theta)Y(k) - B(k, \theta)U(k) - T(k, \theta).
\]

(5.7)

The variance of the error is of the form:

\[
\sigma^2_{\varepsilon}(k, \theta) = |A(k, \theta)|^2 \sigma^2_Y(k) + |B(k, \theta)|^2 \sigma^2_U(k) - 2 \text{Re} \left( A(k, \theta) \overline{B(k, \theta)} \sigma_{YU}(k) \right)
\]

(5.8)

where \( \sigma^2_Y \) is independent of \( k \) for white noise, and the input error variance is assumed to be zero, i.e. \( \sigma^2_U = 0 \), and similarly, the covariance \( \sigma_{YU} = 0 \). Here, \( \overline{B} \) denotes the complex conjugate of \( B \).

Consequently, \( V(\theta) \) is a non-quadratic function of \( \theta \) which, in general, results in a non-convex optimization problem. The Levenberg-Marquardt algorithm (Marquardt, 1963), is used to solve this optimization problem deterministically and is shown in Algorithm 5.1. Such an approach requires good initial estimates of \( \theta \) to avoid inherent local optima, which is the focus of this chapter.

**Remark 5.2.** Alternatively, stochastic optimization algorithms (Press et al., 2007; Spall, 2012) are less likely to get stuck in local optima since such algorithms use randomization either to select initial values or to iterate towards the optimum (or both). However, due to this random influence, their results are not always exactly reproducible and can incur a huge computational cost compared to ‘classical’ deterministic schemes where only a single or a few initial estimates are used to start the expensive iterative optimization procedure. Stochastic optimization is not discussed in the remainder of this chapter.

### 5.3 Methodology

In this section, the considered methods for obtaining the initial estimates of \( \theta \) are briefly explained. Then their influence on the success rate of the maximum likelihood estimator is described.
Figure 5.2: Flow chart depicting the different estimation procedures, from left to right: ➊ an approach using the true model as initial estimate; ➋ an existing approach using BTLS and GTLS starting values, and the novel initialization strategies outlined in this chapter (➌, ➍) that make use of smoothers to generate improved starting values for the non-convex optimization problem. The flow of non-parametric data is depicted by full arrows, with the FRF data marked by asterisks. Parametric models are indicated by dashed, unfilled arrows.
5.3 Methodology

Algorithm 5.1: Levenberg–Marquardt (Marquardt, 1963), (Pintelon and Schoukens, 2012, Sec. 9.L.4)

Require: Cost function (5.6) is rewritten as \( V(\theta) = \epsilon(\theta)^T \epsilon(\theta) \).

1. \textbf{function} LEVENBERGMARQUARDT(\( \epsilon(\theta) \), \( \theta^{\text{init}} \)) \( \triangleright X^{[i]} \) denotes \( X \) at iteration \( i \).
2. \( \theta^{[0]} \leftarrow \theta^{\text{init}} \)
3. \textbf{for} \( i \) in \( 1 \rightarrow \infty \) \textbf{do}
4. \hspace{1em} \( U \Sigma W^T \leftarrow \text{svd} \left( \frac{\partial \epsilon(\theta^{[i-1]})}{\partial \theta} \right) \) \( \triangleright \) Singular value decomposition of Jacobian
5. \hspace{1em} \textbf{if} \( \lambda \) is undefined \textbf{then}
6. \hspace{2em} \( \lambda \leftarrow \max(\{ \Sigma_{11}, \ldots, \Sigma_{n_\theta n_\theta} \}) / 100 \)
7. \hspace{2em} \( \Sigma_{kk} \leftarrow \frac{\Sigma_{kk} + \lambda^2}{\Sigma_{kk} + \lambda^2} \) \( \forall k \in \{ 1, \ldots, n_\theta \} \)
8. \hspace{2em} \( \delta \theta \leftarrow -W^T U^T \epsilon(\theta^{[i-1]}) \)
9. \hspace{2em} \( \theta^{[i]} \leftarrow (\theta^{[i-1]} + \delta \theta) \)
10. \hspace{2em} \textbf{if} \( V(\theta^{[i]}) > V(\theta^{[i-1]}) \) \textbf{then} \( \triangleright \) Current step is a deterioration
11. \hspace{3em} \( \theta^{[i]} \leftarrow \theta^{[i-1]} \) \( \triangleright \) Restart loop from previous estimate
12. \hspace{2em} \( \lambda \leftarrow 10\lambda \)
13. \textbf{else}
14. \hspace{2em} \( \lambda \leftarrow 0.4\lambda \)
15. \hspace{2em} \textbf{if} \( \| \delta \theta \|_2 < n_\theta \cdot 10^{-6} \| \theta^{[i]} \|_2 \) \textbf{then}
16. \hspace{3em} \textbf{return} \( \hat{\theta} \leftarrow \theta^{[i]} \)

5.3.1 Initial Estimates Procedure

The ultimate aim is to find initial estimates that are good enough to steer clear of the local optima during the MLE optimization process in the parametric identification of each system. The requisite procedures for the estimators are as follows, from left to right of the flowchart in Figure 5.2:

1. \textbf{Using the true model} (\( G_o \)) as initial estimate

The estimates from this procedure are for comparison purposes with those from the other procedures. They will be crucial to the computation of the global optimum of the MLE.

2. \textbf{Quadratic approximations of the MLE}

The GTLS and the BTLS have been presented in (Pintelon, Guillaume, \textit{et al.}, 1998). These are modifications of equation (5.6), which still take into account the noise information while retaining the quadratic nature w.r.t. \( \theta \). These estimators preserve consistency, such that the estimates converge asymptotically to the true
parameters for $N \to \infty$. Their finite sample behavior, however, is suitable for improvement by the FRF smoothing tools as described below.

3. Local polynomial method (LPM) with truncation The time-truncated LPM has been explained in Section 3.6 in more detail. The pertinent details are repeated here to aid readability. A good estimate of the FRF of the chosen system can be obtained via the truncated LPM, which is summarized as follows (Lumori et al., 2014) and Section 3.6 of this thesis. The LPM is first applied to estimate the FRF from a full input/output data record of the SISO system. The smooth characteristics of both the exact FRF $G_o$ and the transient term $T$ allow for application of the LPM, leading to a smooth FRF estimate with the transient and noise suppressed.

With reference to a detailed exposition in (Lumori et al., 2014) and Chapter 3, the LPM utilizes local polynomials to approximate the transfer function and transient contribution (in least squares sense) and, thus, smooth $G_o$ and $T$ around a central frequency $\omega_k$ in a local frequency band $\omega_{k+r} \in \Omega$. In this chapter, we limit the notations to quadratic polynomials ($N_B = N_T = 2$) in a window with half-width $n_W = 3$, such that

$$G_o(\omega_{k+r}) \approx \hat{G}_k + g_{1,k}r + g_{2,k}r^2$$
$$T(\omega_{k+r}) \approx T(\omega_k) + t_{1,k}r + t_{2,k}r^2$$

where $r \in \mathbb{R}_k$ with $\mathbb{R}_k = \{-n_W, -n_W + 1, \ldots, n_W\}$. In general (see Chapter 3), $n_W, N_B,$ and $N_T$ are tunable parameters.

The LPM estimate of the FRF at frequency index $k$ is the first estimated local parameter, viz:

$$\hat{G}_{LPM}(\omega_k) = \hat{G}_k.$$  

This procedure is repeated for all $k$ in the frequency band of interest.

Impulse response truncation. The estimate is smoothed further by truncating its impulse response function $g_{LPM}(t) = \text{IDFT} \left( \hat{G}_{LPM}(\omega_k) \right)$ as originally presented in (Lumori et al., 2014) and Section 3.6.

The impulse response is truncated after the time $t_{\text{trunc}}$ where the signal becomes indistinguishable from the noise, i.e.,

$$g_{\text{trunc}}(t) = \begin{cases} 
g_{LPM}(t) & t \leq t_{\text{trunc}} \\
0 & t \geq t_{\text{trunc}}. \end{cases}$$

To this end, an estimate of the envelope of impulse response is determined by fitting an exponential function $g_{\text{exp}}(t) = Be^{\beta t}$ to the peaks of $|g_{LPM}(t)|$ using a
5.3 Methodology

The linear least-squares approach. Then, \( t_{\text{trunc}} \) is determined as the time instant where this envelope function sinks below the noise level \( \sigma_g \) of the impulse response, i.e.

\[
t_{\text{trunc}} = \min \{ t : g_{\exp}(t) < \gamma \sigma_g \}
\]

where \( \gamma = 1 \) was used in this chapter for simplicity. By changing \( \gamma \), the user can fine-tune the bias/variance trade-off of the estimated FRF \( G_{\text{trunc}}(\omega_k) = \text{DFT}(g_{\text{trunc}}(t)) \) further. This can lead to a significant improvement over the classical LPM (Lumori et al., 2014).

4. Regularized finite impulse response (RFIR)

The RFIR method is a special case of the regularized auto-regressive exogenous (RARX) method.

The RFIR estimator is formulated in the time domain. It estimates the impulse response of a discrete time system as the minimizer of the following regularized least squares objective function:

\[
\hat{g}_{\text{RFIR}} = \arg\min_g \| y - g * u \|^2_2 + \sigma^2 g^T P^{-1} g
\]

where \( g \) is the vectorized impulse response \( g(t) \), with \( t = 0, 1, \cdots, n_{h} - 1 \), assuming that the impulse response is \( n_g \) samples long. Furthermore, \( g * u \) is the convolution of \( g \) with the input signal \( u \), and \( P \in \mathbb{R}^{n_g \times n_g} \) is the kernel matrix (Chen, Ohlsson, et al., 2012). Note that (5.13) is quadratic in \( g \), such that \( \hat{g}_{\text{RFIR}} \) can be computed analytically in a single step. The kernel matrix \( P \) embodies prior knowledge on the system to be estimated. Here, the diagonal correlated kernel (DC kernel) (Chen, Ohlsson, et al., 2012; Pillonetto and De Nicolao, 2010) is used. Specifically, the element at \( t_1, t_2 \) of \( P \) is given by

\[
P(\alpha, \beta)_{t_1, t_2} = e^{-\alpha |t_1 - t_2|} e^{-\beta (t_1 + t_2) / 2}
\]

for \( t_1, t_2 \in \{ 0, 1, \cdots, n_h - 1 \} \) which makes the estimated impulse response to be an exponentially decaying function of \( t \) with decay rate \( \beta^{-1} \), and correlation length \( \alpha^{-1} \). The latter hyperparameters \( \alpha, \beta \) and \( \sigma \) are determined using Empirical Bayes (Carlin and Louis, 2000; Gelman et al., 2014), i.e. as the maximizers of the log marginal likelihood (LML) of the measured output signal (Chen, Ohlsson, et al., 2012):

\[
\text{LML}(y) = -y^T \Sigma(\alpha, \beta, \sigma)^{-1} y - \log |\Sigma(\alpha, \beta, \sigma)|,
\]

where \( \Sigma(\alpha, \beta, \sigma) = \sigma^2 I + \phi^T P(\alpha, \beta) \phi \) and \( \phi \) is a Toeplitz matrix constructed with \( u \). This problem is solved using \texttt{fmincon} in MATLAB®.
An implementation of the RFIR estimator is available as the `arx` function in the System Identification Toolbox since MATLAB® 2013b with \( na = 0, nb = n_g, nk = 0 \) and regularization enabled in `arxOptions`. The hyperparameters can be determined via `arxRegul`. More information about the RFIR estimator can be found in (Chen, Ohlsson, *et al.*, 2012; Pillonetto and De Nicolao, 2010). The DFT of the obtained regularized estimate of the impulse response \( \hat{g}_{RFIR} \) of the system yields a smoothed estimate of the FRF.

**Remark 5.3.** In this chapter, all results are obtained using \( n_g = 200 \) since this exceeds the length of most impulse responses used. However, for general use, the model complexity \( n_g \) might need to be increased. One could use \( n_g \approx N \) and rely on regularization to reduce the effective model complexity. Unfortunately, this scales very poorly for long measurement records as explained in Section 5.8.

**Remark 5.4.** The Local Rational Method (LRM) has also been tried to generate better initial values. However, given the discussion in Chapter 3, it is clear that the LRM performs poorly in low SNR situations. The same was apparent from the quick trial we ran on LRM starting values. As such, the use of LRM over LPM offers no advantage for generating initial estimates in the settings used in this chapter.

**Remark 5.5.** Note that methods 3) and 4) result in a non-parametric estimate of the transfer function, represented as the FRF. A corresponding parametric estimate is obtained as follows. A parametric estimator (GTLS, BTLS, and MLE) is invoked, where the input spectrum is considered to be 1 at all frequencies (Dirac in the time domain), and the output spectrum is set equal to the estimated FRF \( G_\bullet(\omega_k) \), with the transient term \( T \) set to zero:

\[
U_{FRF}(\omega_k) \triangleq 1 \quad \text{and} \quad Y_{FRF}(\omega_k) \triangleq G_\bullet(\omega_k). \tag{5.16}
\]

It is important to note that invoking the GTLS, BTLS and MLE on the FRFs from methods 3) and 4) will yield a different result from applying the GTLS, BTLS and MLE on the raw data. This is because the FRFs from 3) and 4) have been smoothed and, thus, have a significantly reduced noise variance, but might be biased slightly.

### 5.3.2 Success Rates of the Initial Estimates

The initial estimates described above are then fed to the MLE together with the raw data to obtain the final estimates.

Formally, let \( \hat{G}_\bullet \) be the final parametric estimate of the system, where the subscript \( \bullet \) denotes the methods (from left to right in Fig. 5.2) when an initial estimate \( \hat{G}^{\text{init}}_\bullet \) is used, viz:

- \( \hat{G}_\circ \) via the true model \( G_\circ \) as an initial estimate,
5.3 Methodology

- \( \hat{G}_{\text{exist}} \) is obtained via GTLS and BTLS,
- \( \hat{G}_{\text{trunc}} \) by the use of the LPM with truncation as an initial estimate,
- \( \hat{G}_{\text{RFIR}} \), initialized by means of the RFIR method.

A particular initial estimate is deemed successful if the optimization of the MLE cost with the raw input-output data inserted in equation (5.6), approximately reaches the best local optimum when iteration is initiated with the selected initial value/estimate.

**Conjecture 5.1.** The true model \( G_\circ \) as an initial estimate is the best possible initial estimate one can use for the nonlinear optimization algorithm (the leftmost path in Figure 5.2) and will allow the optimization to converge to the best local optimum, i.e. \( \hat{G}_{\circ} \).

**Remark 5.6.** The use of \( G_\circ \) as an initial estimation not necessarily entails in converging to the global optimum since it is not guaranteed that \( G_\circ \) lies within the attraction region of the global optimum.

From this conjecture, use of the true system as an initial estimate would engender the best final estimate or hopefully even the global optimum. Since the true system is not really known in practice, this chapter compares the capability of different possible initial estimates to emulate the result that would be obtained by using the true model as an initial estimate.

The capability is quantified by defining the success rate of the initial estimate as the probability that the identification algorithm reaches the best local optimum when the selected initial estimate is used. Note that probability in this context should be understood with respect to different realizations of the input signal and disturbing noise.

The success rate \( \eta \bullet \) of an initialization scheme \( \bullet \) is defined as the probability that it reaches the best local optimum. In practice, and since the iterative algorithm usually does not reach the local optimum precisely, this is implemented practically as

\[
\eta \bullet \equiv P \left( \left\| \hat{G}_\bullet - \hat{G}_\circ \right\|_2 < \varepsilon_A \right)
\]

with \( \varepsilon_A \) a numerical tolerance that will be specified later on.

**Remark 5.7.** For an automated approach, it is better to define the success rate in terms of relative errors on the transfer function, i.e.

\[
\eta \bullet = P \left( \left\| \hat{G}_\bullet - \hat{G}_\circ \right\|_2 < \varepsilon_R \left\| G_\circ \right\|_2 \right).
\]

**Remark 5.8.** In this chapter, the absolute criterion is used. However, the results are equivalent for a relative criterion when the separation between successful and failed
estimates is large since then observed success rate \( \eta \) is not sensitive towards the tolerance \( \varepsilon_A \) (or \( \varepsilon_R \)) when the tolerance is chosen with care. Consequently, one can easily determine the equivalent relative tolerance from the absolute one using \( \varepsilon_A = \varepsilon_R \| G_c \|_2 \).

The \( L_2 \) norm used in equation (5.17) is defined (Skogestad and Postlethwaite, 2005) as

\[
\| G(z) \|_2 \triangleq \sqrt{\frac{1}{2\pi} \int_0^{2\pi} |G(e^{-j\omega})|^2 \, d\omega} \quad \text{for discrete time,} \tag{5.19}
\]

\[
\| G(s) \|_2 \triangleq \sqrt{\frac{1}{2\pi} \int_{-\infty}^{+\infty} |G(j\omega)|^2 \, d\omega} \quad \text{for continuous time} \tag{5.20}
\]

which, in practice, is computed by using the adaptive global quadrature algorithm provided by the MATLAB® function \texttt{integral}. The success rate in equation (5.17) is estimated via Monte Carlo simulations in Section 5.4.3.

Remark 5.9. The specific norm that is used in (5.17) can be adapted to suit the eventual purpose of the models the user wants to obtain. E.g. in robust control, it often makes sense to rather have a \( H_\infty \) norm as in Chapter 4 than a \( L_2 \) norm. In this chapter, the \( L_2 \) norm has been chosen since this is more closely related to the cost function (5.6).

### 5.4 Demonstration

#### 5.4.1 The System Under Consideration

Two systems are considered. The first, with a quality factor \( Q = 6 \) dB, has the transfer function

\[
G_c(z) \approx 1.74 \cdot 10^{-3} \frac{z^2 + 2z + 1}{z^2 - 1.93z + 0.94} \quad \text{i.e. } \tau \approx 32 \text{ s for } Q = 6 \text{ dB}, \tag{5.21}
\]

where \( \tau = (\xi \omega_n)^{-1} \) denotes the time constant of the system (with \( \xi \) the damping and \( \omega_n \) the frequency of the dominant pole). The second system, with a quality factor \( Q = 20 \) dB, has the transfer function

\[
G_c(z) \approx 309 \cdot 10^{-6} \frac{z^2 + 2z + 1}{z^2 - 1.98z + 0.989} \quad \text{i.e. } \tau \approx 180 \text{ s for } Q = 20 \text{ dB}. \tag{5.22}
\]

Both systems are low-pass Chebyshev filters which have been generated using the MATLAB® command \texttt{cheby1(2, Q, 0.05)}. Their transfer functions are shown in Figure 5.3. Their inputs are excited by zero mean white Gaussian noise with unit variance, and the
5.4 Demonstration

outputs are disturbed by different realizations of white Gaussian noise, as in Figure 5.1, with a variance $\sigma_v^2$, such that a prescribed SNR defined as

$$\text{SNR} = \frac{\sigma_v}{\text{RMS} (y_o)} \approx \frac{\sigma_v}{\|G_o\|_2 \text{RMS} (u_o)} = \frac{\sigma_v}{\|G_o\|_2}.$$  (5.23)

is attained. $\text{RMS} (x) = \sqrt{N^{-1} \sum_t x^2(t)}$ denotes the root-mean-squared value of $x$.

5.4.2 Comparison of Successful and Failed Estimates

Figure 5.3 depicts the Bode plot of the actual system $G_o$, in black and different estimates $\hat{G}_{\text{exist}}$ that may or may not approximate $G_o$ well, according to the distance defined by equation (5.17). The systems shown here are drawn randomly from the successes and failures observed in the Monte Carlo simulations, which are discussed later on in this chapter.

It is clear from Figure 5.3 that the successful estimates virtually coincide with the true system. This means that their distances from the true system, e.g. as given by $\| \hat{G}_* - G_o \|_2$, have small values. On the other hand, the ‘failed’ estimates are inaccurate descriptions of the true system. It is apparent that their associated distances from the true system are larger by, at least, an order of magnitude than for the successful estimates.

Estimation ‘failures’ can be attributed to the optimization procedure getting stuck in local optima, resulting in very poor estimates of $G_o(\omega)$ over broad frequency ranges, as seen in Figure 5.3. Consequently, such estimates are approximations of $G_o$ that are practically worthless for most applications.

5.4.3 Computation of the Success Rates

5.4.3.1 Simulation

A Monte Carlo simulation was set up to determine the success rate of the estimator with different initial estimates. In each run, both the realization of the excitation signal $u$ and the disturbing noise realization ($e$) are varied. Each run is repeated for the following ranges:

- of SNR: $-20$ dB, $-16.7$ dB, $-13.3$ dB, $-10$ dB, $-7.8$ dB, $-5.6$ dB, $-3.3$ dB, $-1.1$ dB, $1.1$ dB, $3.3$ dB, $5.6$ dB, $7.8$ dB, $10$ dB, $13.3$ dB, $16.7$ dB and $20$ dB
Figure 5.3: Illustrative Bode plots of a few successful (---) and ‘failed’ (---) estimates $\hat{G}$ for both systems $G_0$ (- - -) under test. The ‘failed’ estimates are caused by local minima in the cost function. Success or failure is determined using the criterion in equation (5.17). The $3\,\text{dB}$ bandwidth of each system is highlighted (■■).
• of number of samples \( N \) is 3 368 \( \approx 104\tau \) or 4 462 \( \approx 138\tau \) for the first system whose quality factor \( Q = 6 \) dB,

• and of number of samples \( N \) is 20 250 \( \approx 113\tau \) or 26 792 \( \approx 149\tau \) for the second system whose quality factor \( Q = 20 \) dB.

One can, equivalently, express the experiment length as the number \( n_{BW} \) of frequency bins that lie within the 3 dB bandwidth of the system. The 3 dB bandwidth is defined as the frequency band where the magnitude of the transfer function is, at most, 3 dB below its maximum as indicated in Figure 5.3. For both systems, \( N \) was chosen such that \( n_{BW} \) is respectively 36 or 48. For each value-pair (SNR, \( n_{BW} \)) and both systems, \( n_{MC} = 200 \) simulations and estimations were performed. Each individual simulation yields an estimate of the model based on which one can determine whether the estimation was successful or not. This allows to empirically estimate the success rate \( \eta \) in equation (5.17) as the proportion of the successes from the \( n_{MC} = 200 \) runs.

5.4.3.2 Illustrative Example

In particular, the simulation for each pair (SNR, \( n_{BW} \)) yields 200 estimates per initial value strategy. This is illustrated graphically in Figure 5.4 for the value-pair (−1.1 dB, 48) of the system with \( Q = 6 \) dB, where the obtained distances \( \| \hat{G}_\bullet - \hat{G}_o \|_2 \) are shown for the different strategies, and sorted by an ascending norm. This effectively shows an empirical cumulative distribution function of the distance measure \( \| \hat{G}_\bullet - \hat{G}_o \|_2 \) as sampled by the Monte Carlo algorithm.

5.4.3.3 Pertinent Observations

With reference to Figure 5.4, the following observations are worth noting:

• The actual parametric estimates, based on the different initial estimates, have distinct behaviors with respect to their distance from \( \hat{G}_o \). Two major classifications can be made:

  1. ‘good’ estimates which have \( \| \hat{G}_\bullet - \hat{G}_o \|_2 < \varepsilon_A \approx 6 \cdot 10^{-5} \),

  2. ‘poor’ estimates which are much further from \( \hat{G}_o \). This is caused by the local optima.
Chapter 5 Initialization of Parametric Estimators

\[ Q = 6 \text{ dB} \quad N = 4{,}462 \quad \text{BW} = 48 \quad \text{SNR} = -1.1 \text{ dB} \]

The distance \( \| \hat{G} - \hat{G}_o \|_2 \) between the different estimates (\( \hat{G}_{\text{exist}} \), \( \hat{G}_{\text{trunc}} \) and \( \hat{G}_{\text{RFIR}} \)) and the ‘best theoretical’ estimates (\( \hat{G}_o \)) shows that the smoothers help the estimate to converge to \( \hat{G}_o \). The same measure is shown for the initial values \( \hat{G}_{\text{init}}_{\text{RFIR}} \) and \( \hat{G}_{\text{init}}_{\text{trunc}} \).

\[ \| G_o - \hat{G}_o \|_2 \] shows how far the true model and ‘best theoretical’ estimates are apart to put the other distances into perspective.

- The obvious jump of the observed distances of the final parametric estimates shows that both classes can be separated reliably. The index at which this jump occurs is a direct (visual) indication of the success rate \( \eta \) of the corresponding strategy to obtain initial values. This indicates \( \frac{147}{200} \approx 74\% \) success for the existing methods, \( \frac{189}{200} \approx 95\% \) for the truncation method and 100\% for RFIR in Figure 5.4.

### 5.4.4 Improvement Over Existing Techniques

In Figure 5.5, the success rate (and its 95\% confidence interval CI) obtained from the Monte Carlo simulations is shown to compare the proposed method with previously existing approaches. Note that the empirical count of successes \( n_{\text{success}} \) out of the \( n_{\text{MC}} \) total observations is binomially distributed with success rate \( \eta \). The computation of confidence bounds of such a variable is not obvious due to the discrete nature of the variable: \( n_{\text{success}} \) can only attain a discrete set of values: \( \{0, 1, \ldots, n_{\text{MC}}\} \) whereas the true success rate \( \eta \) is real-valued (but limited to the interval \([0, 1]\)). The inherent
discreteness of \( \hat{\eta} \triangleq \frac{n_{\text{success}}}{n_{\text{MC}}} \) and the difficulties this presents has sparked a fair amount of research to compute confidence bounds (Barnard, 1947; Clopper and Pearson, 1934; Mato and Andrés, 1997; Ross, 2003). A simple way to deal with this problem, is to make use of the central limit theorem which states that \( \frac{n_{\text{success}}}{n_{\text{MC}}} \) is approximately normally distributed when \( n_{\text{MC}} \to \infty \). In doing so, one e.g. obtains the Wald confidence bounds (Ross, 2003):

\[
\text{CI}_{\text{Wald}} \triangleq \hat{\eta} \pm z_{1-\alpha/2} \sqrt{\hat{\eta}(1-\hat{\eta})/n_{\text{MC}}}
\]

with \( \alpha = 1 - 0.95 \) for 95% confidence bounds and \( z_{1-\alpha/2} \) is the \( 1 - \alpha/2 \) quantile of a standard normal distribution, i.e. \( z_{\beta} \triangleq \text{CDF}^{-1}_N(\beta) \) which requires inversion of the cumulative density function (CDF) of a normal distribution. However, such an approximation is overly optimistic when \( n_{\text{success}} \) approaches the boundaries of its domain (i.e. when \( \eta \approx 0 \) or \( \eta \approx 1 \)) since normal confidence regions are always symmetrical whereas the actual success rate is inherently bounded as \( 0 \leq \eta \leq 1 \) (Ross, 2003). An alternative method that does not approximate the binomial distribution by a normal one, is the Clopper-Pearson method (Clopper and Pearson, 1934). In particular, this method relies upon the exact binomial distribution, which is why the method is often quoted as the ‘exact’ method. In particular, the \( (1 - \alpha) \) confidence bounds on \( \eta \) are given by (Clopper and Pearson, 1934)

\[
\text{CI}_{\text{ClopperPearson}} \triangleq [\hat{\eta}, \bar{\eta}]
\]

\[
\hat{\eta} \triangleq \text{CDF}^{-1}_B(\alpha/2; n_{\text{success}}, n_{\text{MC}} - n_{\text{success}} + 1)
\]

\[
\bar{\eta} \triangleq \text{CDF}^{-1}_B(1 - \alpha/2; n_{\text{success}} + 1, n_{\text{MC}} - n_{\text{success}})
\]

where \( \text{CDF}^{-1}_B(\alpha; n, m) \) denotes the inverse CDF of a Beta\( (n, m) \) distribution (Hazewinkel, 2002). This is known to yield somewhat conservative bounds, especially if only failures or successes are observed (Ross, 2003).

In this chapter, the Clopper-Pearson method—as implemented in MATLAB®’s \textit{binofit}—is used whenever confidence bounds are shown. Consequently, the improvements observed in Figure 5.5 underestimate the actual improvement slightly due to the conservatism in these bounds. However, since there is often a clear separation between the different techniques, no efforts have been made to use confidence interval estimators that offer less conservative results.

By aggregating the results for the different simulations and retaining the success rate \( \eta \) only, a clearer view of the performance of the different methods is obtained.

The success rates of \( \hat{G}_{\text{exist}} \), \( \hat{G}_{\text{trunc}} \) and \( \hat{G}_{\text{RFIR}} \) are depicted in Figure 5.5 for two different numbers of sample points within the 3 dB bandwidth, and for a range of SNR values. A close scrutiny of Figure 5.5 reveals the following observations:
• The success rates increase with increasing SNR values, as expected.

• For very low SNR values (below 0 dB), \( \hat{G}_{\text{exist}} \) and \( \hat{G}_{\text{trunc}} \) are unreliable, whereas \( \hat{G}_{\text{RFIR}} \) is reliably successful.

• For relatively high SNR (20 dB), all three estimators perform equally well.

• For moderate SNR values, the success rate of \( \hat{G}_{\text{trunc}} \) lies above that of \( \hat{G}_{\text{exist}} \).

• The RFIR-based estimate exhibits a success rate of, at least, 97% for all studied conditions. This clearly indicates that regularization makes it possible to obtain far more reliable initial estimates than both the LPM-Truncation-based ones and the existing ones. This is an important conclusion.

As far as \( \hat{G}_{\text{trunc}} \) and \( \hat{G}_{\text{exist}} \) are concerned, the observations on Figure 5.5 clearly reveal the extent to which the estimator \( \hat{G}_{\text{trunc}} \) allows for the SNR to be decreased before the estimator attains too high a fail rate. In other words, it shows that, at low SNR values \( \hat{G}_{\text{trunc}} \) is more reliable than \( \hat{G}_{\text{exist}} \).

Figure 5.5 shows that, typically, for the system with \( Q = 6 \) dB when \( n_{BW} = 48 \) and \(-1.1 \) dB SNR, the existing initial values will fail to deliver a good transfer function estimate for one in four trials (\( \approx 74\% \) success rate). However, by using the truncated LPM’s initial estimates, the number of failures is reduced fivefold (\( \approx 95\% \) success rate) over the existing initial estimates. Moreover, the RFIR initial values lead to poor estimates in less than 2% (> 98% success rate) of the cases, since none were observed in the Monte Carlo simulation.

**Extension of the SNR range for \( \hat{G}_{\text{trunc}} \) over \( \hat{G}_{\text{exist}} \)** — A close inspection of Figure 5.5 reveals that the SNR range associated with an acceptable success rate (e.g. \( \eta \geq 90\% \)) is improved for \( \hat{G}_{\text{trunc}} \) over \( \hat{G}_{\text{exist}} \). The range of improvement is approximately 5 dB where the success rate is above 90%. This result is very useful for a user working with noisy data.

**Remark about the model quality** — In principle, convergence of the estimate \( \hat{G}_\star \) to the ‘best’ estimate \( \hat{G}_\circ \) does not imply that \( \hat{G}_\star \) will be suitable for a specific purpose as the estimate \( \hat{G}_\circ \) itself might be ill-suited. A straightforward tool to analyze the error associated with \( \hat{G}_\circ \) is to examine its relative root-mean-square (RMS) error, with respect to the true system \( G_\circ \), over different realizations of the experiment. In (Ljung, 1999b), a rule-of-thumb is given that describes the relationship between the SNR of the signals and the quality of the estimated parametric model. In this context this heuristic boils
Figure 5.5: The success rate $\eta$ obtained using the smoothers, i.e. $\hat{G}_{\text{RFIR}}$ (---) and $\hat{G}_{\text{trunc}}$ (--), are significantly higher than for the existing methods $\hat{G}_{\text{exist}}$ (---) in most of the studied range of SNRs, number of samples and for both test systems. The 95% confidence intervals (5.25) are indicated by bars.
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Figure 5.6: The accordance between the observed root mean squared error (RMSE) of $\hat{G}_o$ (•••) (with $\pm 2\sigma$ interval) and the rule-of-thumb (5.28) (−−−) shows how well $\hat{G}_o$ approximates the true system $G_o$. In the bottom plot, the difference between both is shown. Consequently, it makes sense to examine the behavior of the models in terms of $\hat{G}_o$ as long as this RMSE meets the user's requirements.

down to:

$$\text{RMSE}(\hat{G}_o) \approx \sqrt{\frac{n_\theta \| G_o \|_2}{N \text{ SNR}}}$$  \hspace{1cm} (5.28)

where $n_\theta = 5$ is the number of estimated parameters in this case. In Figure 5.6, the empirical RMSE($\hat{G}_o$) over the different Monte Carlo runs is displayed against this rule-of-thumb. It can be seen that these runs align almost perfectly, which shows that this rule-of-thumb can be used to approximate the RMSE of the parametric estimate. When the RMSE is then divided by $\| G_o \|_2$, this graph essentially shows the anticipated relative error on the transfer function and, hence, gives an indication on its usability.

5.5 Experimental Results

Measurements were performed on an electrical filter to assess the performance of the initial value generating methods in a physical setting.
5.5 Experimental Results

Figure 5.7: Schematic representation of the measurement set-up for the Brüel & Kjær 1613 filter. The and data acquisition (DAQ) hardware share the same clock and are thus synchronized. Blue arrows indicate coax cables. At the output of the bandpass filter, a noise generator is added that buffers the incoming signal using a TL071 opamp and adds white noise before passing the signal to the DAQ.

5.5.1 Experimental Setup

The Brüel & Kjær 1613 Octave Filter Set consists of sixth-order Chebyshev band-pass filters (Brüel & Kjær, 1970), from which we only examined the filter with a center frequency of 4 kHz.

To measure the transfer function, the Brüel & Kjær 1613 Filter was excited by a signal \( r(t) \) and both its input \( u(t) \) and output \( y(t) \) were measured as shown in Figure 5.7. The output \( y(t) \), however, was disturbed by a noise generator that added white noise over the frequency range [DC, 20 kHz]. The excitation signal \( r(t) \) was white random noise with a standard deviation \( \sigma_r = 0.25 \) V, consisting of \( N_S = 8\,192 \) samples, and sampled at 92 kHz. The same noise sequence was repeated \( N_R = 100 \) times in succession, such that a non-parametric estimate of the noise could be obtained and the performance of the methods could be gauged over these different repetitions. To distinguish the different repetitions, we denote the \( r \)th repetition of the \( u \) signal as \( u^{(r)}(t) \) (and similarly for \( y(t) \)). The signal generation and acquisition was by means of a National Instruments (NI) Elvis II, using the respective AO and AI pins on the breadboard, which were wired to BNC connectors. Although acquisition and generation were synchronized, there is a small delay \( \tau_{MUX} \approx 9 \) µs between the acquisition of the input \( u(t) \) and the output \( y(t) \) due to the hardware architecture of the NI Elvis II. Particularly, the involved channels are captured by means of a single analog-to-digital converter (ADC) which is preceded by a multiplexer to select the active channel. This obviously introduces a slight delay between the acquisition of a sample of input and output channels that is not part of
Chapter 5  Initialization of Parametric Estimators

Figure 5.8: Transfer function of the considered Brüel & Kjær 1613 filter. The empirical transfer function estimates $Y^{[r]}(\omega)/U^{[r]}(\omega)$ (●) and their average $\bar{Y}(\omega)/\bar{U}(\omega)$ (—) obtained from the Elvis measurements are shown together with the estimated reference model $G_{VXI}$ (—).

system dynamics of the Brüel & Kjær filter, but rather of the experimental set-up. As such, the output spectrum $Y(\omega)$ was multiplied by $\exp(-j\omega\tau_{MUX})$ such that this inter-channel acquisition delay of the NI Elvis is compensated; the time-domain counterpart $y(t)$ is obtained using the inverse discrete Fourier transform (IDFT).

5.5.2 Identification Procedure

The repeated nature of the experiment makes it possible to estimate the noise level from the signals non-parametrically. The mean signal from the input $u$ is of the form:

$$\bar{u}(t) = \frac{1}{N_R} \sum_{r=1}^{N_R} u^{[r]}(t) \quad (5.29)$$
Thus, the reduced noise influence and the approximate noise co-variances are, respectively:

\[
\hat{\sigma}_u^2(t) = \frac{1}{N_R - 1} \sum_{r=1}^{N_R} \left( u^{[r]}(t) - \tilde{u}(t) \right)^2 
\]

\[
\hat{\sigma}_{yu}(t) = \frac{1}{N_R - 1} \sum_{r=1}^{N_R} \left( y^{[r]}(t) - \tilde{y}(t) \right) \left( u^{[r]}(t) - \tilde{u}(t) \right) 
\]

Similar calculations apply to \( y(t) \) and as such, \( \hat{\sigma}_u(t) \), \( \hat{\sigma}_y(t) \) and \( \hat{\sigma}_{yu}(t) \) can be estimated. Their frequency domain counterparts \( \tilde{U}(\omega) \) and \( \tilde{Y}(\omega) \) are obtained by using the DFT. The resulting empirical transfer function estimate \( \tilde{Y}^{[r]}(\omega)/\tilde{U}^{[r]}(\omega) \) and its periodic average \( \tilde{Y}/\tilde{U} \) are shown in Figure 5.8. The noise covariances \( \hat{\sigma}_U^2(\omega) \), \( \hat{\sigma}_Y^2(\omega) \) and \( \hat{\sigma}_{YU}(\omega) \) are calculated as the sample covariance, akin to equation (5.31), and their values are shown in Figure 5.9. Note that the SNR at the output \( \text{SNR}_y \approx 14 \, \text{dB} \) is much smaller than at the input \( \text{SNR}_u \approx 50 \, \text{dB} \).

For each measurement repetition, the signals \( u^{[r]}(t), y^{[r]}(t) \) and the variances \( \sigma_U^2(\omega) \), \( \sigma_Y^2(\omega) \) are used in the maximum likelihood cost function given by equation (5.6). The model order was chosen according to the specifications of the sixth-order (Chebyshev)
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5.5.3 Reference Model Measurements

Since the true model $G_\circ$ is not known for real-life systems, a practically viable reference model is needed. Additional measurements were performed using a VME extensions for instrumentation (VXI) measurement setup, which allowed for a signal-to-noise ratio of more than 60 dB. Virtually noiseless, these measurements provided a very high quality model of the system, denoted as $G_{\text{VXI}}$, and used as a reference model. The VXI measurement setup is summarized as follows.
5.5 Experimental Results

- Signal Generator card: VXI HP E1445A.
- Acquisition cards: VXI HP E1430A.
- Sampling frequency: \( f_s = 156.250 \text{ Hz} \).
- A total of 558 frequency bins were used for the estimation, in the excited frequency band \([0.1, 20]\) kHz, with a frequency resolution of 35.7 Hz, giving a measurement time of 28 ms. The excitation signal was band-limited periodic noise whose RMS value was 100 mV.
- The input and output signals were buffered and anti-alias filtered.

This yielded the parametric model \( G_{\text{VXI}} \) (shown in Figure 5.8) with a relative error of less than 0.3\% in the pass-band with respect to the measured FRF. Denote \( \hat{G}_{\text{VXI}} \) to be the parametric estimate obtained using \( G_{\text{VXI}} \) as an initial estimate. Furthermore, \( G_{\text{VXI}} \) was used to play the role of \( G_o \) from the previous section.

5.5.4 Model Estimation

The measurements on the NI Elvis II indicate that the high-quality starting value \( G_{\text{VXI}} \) led to low cost function values as shown in Figure 5.10 and Table 5.1. The existing methods exhibit a high spread and a high median cost function, showing that a good estimate is obtained only in about 25\% of the cases. The truncation method provides better estimates in many cases, but still suffers from a high variability. On the other hand, the RFIR-based initial values have both a low median cost function and spread, and thus provide better fits in almost all cases. Obviously, \( \hat{G}_{\text{best}} \) has the lowest cost function values of all methods. The similarity of the results for \( \hat{G}_{\text{RFIR}} \) and \( \hat{G}_{\text{best}} \) suggests that the RFIR provides the best estimate in most of the cases. This can indeed be confirmed by inspecting the different estimates per repetition of the experiment.

Remark 5.11. In the simulations (e.g. Figure 5.4), a 60 dB difference between ‘good’ and ‘bad’ estimates was observed. Such a large gap is not observed in the measurements (Figure 5.10 and Table 5.1). Hence, defining a success rate based on a threshold tolerance would be very sensitive to the specific value of the threshold and hence unreliable. Instead, the statistical location and dispersion are inspected to assess the relative performance of each method. Practically, the median is used as a measure of location and the inter-quartile range (IQR) is used to inspect the spread as these are far more robust to outliers than e.g. the sample mean and variance. These measures also have an easy interpretation: the median (50\% percentile) indicates the cost function value that 50\% of the repetitions...
attain. On the other hand, the IQR (25% through 75% percentile) contains exactly half of the observations.

## 5.5.5 Model Validation

### 5.5.5.1 Cost Function Limitations

In the previous section, the cost function was studied to determine the effectiveness of the starting values. However, inspecting the cost function $V(\theta)$ only accounts for how well an estimated model fits the measured data, which may be misleading. For an example, overfitting a model may result in the absorption of both the systematic behavior and the noise into the model. Consequently, an arbitrarily small cost function may ensue although the estimated model may be virtually useless to predict the system behavior.

### 5.5.5.2 Validation Criterion

To objectively assess the quality of an estimated model, a different criterion than the cost function is inspected. To this end, the model is validated using the 2-norm (or distance) on the model error:

$$D_{\bullet} \triangleq \left\| \hat{G}_{\bullet} - G_{\text{VXI}} \right\|_2. \quad (5.34)$$

This criterion indicates how well the obtained estimates $\hat{G}_{\bullet}$ are able to describe the transfer function of the bandpass filter as observed in the validation measurement on the VXI.

### 5.5.5.3 Validation Performance

The observed median and IQR of the distances in Figure 5.11 and Table 5.2 have a similar qualitative interpretation as the cost function on the estimation data. The existing methods provide good estimates in only 25% of the cases. The truncation method provides a considerable improvement but still suffers from 25% poor estimates. Compared to the existing BTLS and GTLS techniques, the RFIR and hence also $\hat{G}_{\text{best}}$ entail an overall reduction in the observed distance by a factor 8, in most cases. On the other hand, the reference $\hat{G}_{\text{VXI}}$ shows the best global performance. This indicates that the proposed methods do no converge to the same local optimum. Nevertheless, the proposed methods improve the model quality by almost an order of magnitude.
5.5 Experimental Results

![Graph showing cost function values $V(\theta)$ for different methods.]

Figure 5.10: Cost function values $V(\theta)$ obtained during the different realizations of the measurement. The bottom plot shows a linear zoom of the top plot. For each method, the inter-quartile range (solid), median (dashed) and individual values (•) are shown. This implies that a lower (local) minimum of the cost function can be attained using these smoothing techniques.

<table>
<thead>
<tr>
<th>Method</th>
<th>Min. 0%</th>
<th>25%</th>
<th>Median 50%</th>
<th>75%</th>
<th>Max. 100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{G}_{VXI}$</td>
<td>4355</td>
<td>4442</td>
<td>4494</td>
<td>4707</td>
<td>9634</td>
</tr>
<tr>
<td>$\hat{G}_{exist}$</td>
<td>4345</td>
<td>4526</td>
<td>8741</td>
<td>12413</td>
<td>29237</td>
</tr>
<tr>
<td>$\hat{G}_{trunc}$</td>
<td>4336</td>
<td>4476</td>
<td>4842</td>
<td>8130</td>
<td>88308</td>
</tr>
<tr>
<td>$\hat{G}_{RFIR}$</td>
<td>4329</td>
<td>4422</td>
<td>4469</td>
<td>4716</td>
<td>9607</td>
</tr>
<tr>
<td>$\hat{G}_{best}$</td>
<td>4329</td>
<td>4415</td>
<td>4458</td>
<td>4644</td>
<td>8570</td>
</tr>
</tbody>
</table>

Table 5.1: Observed percentiles of the cost function $V(\hat{G}_\bullet)$. 
Chapter 5 Initialization of Parametric Estimators

As the criterion in equation (5.34) no longer depends on the cost function, the initial estimates can also be investigated. The difference between the distance of $\hat{G}_{\text{init}}$ and $\hat{G}_*$ indicates how much the final ML estimate from the raw data, in each respective branch of Figure 5.2, improves over the initial estimate. Remarkably, on average, the final ML estimation provides only a marginal improvement for the RFIR. However, this final step reduces the spread and hence yields a more reliable estimate. E.g. note in Table 5.2 that this step reduces the worst-case distance from 18.21 to 4.55.

5.6 Consequences of Selecting the ‘Best’ Model

To determine how important it is to retain the ‘best’ model based on the cost function values, we shall inspect how much worse (or better) the second best model $\hat{G}_{\text{2nd}}$ performs when $\hat{G}_{\text{best}}$ is not selected. To do so, we inspect the performance by means of the cost function (used during estimation) and by means of the validation distance. The datasets used in this analysis are those from the measurements, but similar results can be obtained from the other data sets.

In Figure 5.12 we can compare $\hat{G}_{\text{2nd}}$ and $\hat{G}_{\text{best}}$. By inspecting the different estimates for each of the repeated experiments (Figure 5.12), it can easily be seen that choosing $\hat{G}_{\text{RFIR}}$ while it is not $\hat{G}_{\text{best}}$ only leads to a very modest performance degradation. In Figure 5.13, the different methods are compared by subtracting the performance of the $\hat{G}_{\text{best}}$ from the performance of the respective methods. As such, each column in the figure indicates the performance of this method when it is not the ‘best’. Obviously, when we consider the cost function used during estimation (left subfigure), we can only degrade the performance by construction of $\hat{G}_{\text{best}}$. For the validation data, however, this is no longer guaranteed as can be seen in the right subplot. On average, there is very little performance that can be gained from not using $\hat{G}_{\text{best}}$, however, for two or three data points, there is a considerable improvement. From that figure it can also be seen that there is little performance to lose from choosing $\hat{G}_{\text{RFIR}}$ over $\hat{G}_{\text{best}}$ when RFIR is not the ‘best’ method. The outcome is less favorable for the alternative methods. For $\hat{G}_{\text{exist}}$ it can be seen that selecting those instead of $\hat{G}_{\text{best}}$ degrades the validation performance by 10 or even more in 75% of the cases. The situation is already less severe for $\hat{G}_{\text{trunc}}$ (in 75% of the cases, the degradation is larger than 0.5).

5.7 Remark on the Generality

In the previous sections, the usability of the smoothers to produce initial values has been studied on very specific examples. In this section, an attempt will be made to
5.7 Remark on the Generality

Figure 5.11: Validation of the measured models. For each method, the distance (5.34) between the estimates and $G_{\text{VXI}}$ is shown (○) together with the median (---) and inter-quartile range (----). $\|G_{\text{VXI}}\|_2$ (-----) is shown as a reference. The bottom plot shows a linear zoom of the top plot. The proposed methods yield models that are closer to $\hat{G}_{\text{VXI}}$ than the existing $\hat{G}_{\text{exist}}$. The results are in line with the values of the cost function in Figure 5.10.

Table 5.2: Observed percentiles of the validation distance $\|\hat{G}_\bullet - G_{\text{VXI}}\|_2$.

<table>
<thead>
<tr>
<th>Method</th>
<th>Min. 0%</th>
<th>25%</th>
<th>Median</th>
<th>50%</th>
<th>75%</th>
<th>Max. 100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{G}_{\text{VXI}}$</td>
<td>0.47</td>
<td>0.92</td>
<td>1.06</td>
<td>1.29</td>
<td>2.41</td>
<td></td>
</tr>
<tr>
<td>$\hat{G}_{\text{exist}}$</td>
<td>1.48</td>
<td>1.88</td>
<td>13.72</td>
<td>19.31</td>
<td>26.80</td>
<td></td>
</tr>
<tr>
<td>$\hat{G}_{\text{init trunc}}$</td>
<td>1.35</td>
<td>2.72</td>
<td>14.94</td>
<td>24.72</td>
<td>85.34</td>
<td></td>
</tr>
<tr>
<td>$\hat{G}_{\text{trunc}}$</td>
<td>1.27</td>
<td>1.72</td>
<td>2.35</td>
<td>13.03</td>
<td>63.84</td>
<td></td>
</tr>
<tr>
<td>$\hat{G}_{\text{init RFIR}}$</td>
<td>1.11</td>
<td>1.61</td>
<td>1.78</td>
<td>2.00</td>
<td>18.21</td>
<td></td>
</tr>
<tr>
<td>$\hat{G}_{\text{RFIR}}$</td>
<td>1.21</td>
<td>1.57</td>
<td>1.72</td>
<td>1.85</td>
<td>4.55</td>
<td></td>
</tr>
<tr>
<td>$\hat{G}_{\text{best}}$</td>
<td>1.21</td>
<td>1.56</td>
<td>1.72</td>
<td>1.85</td>
<td>15.01</td>
<td></td>
</tr>
</tbody>
</table>
Figure 5.12: Cost function $V(\hat{G}_\bullet)$ and validation distance of the model estimated in each repetition of the measurement. Based on the cost function, $\hat{G}_{\text{RFIR}}$ (■) is not always the best estimate $\hat{G}_{\text{best}}$ (•). Especially in the validation plot (bottom), $\hat{G}_{\text{RFIR}}$ performs (almost) as well or even better than $\hat{G}_{\text{exist}}$ (○) and $\hat{G}_{\text{trunc}}$ (○). Both latter methods often produce models that perform poorly compared to $\|G_{\text{VXI}}\|_2$ (—-••) and $\hat{G}_{\text{VXI}}$ (○). This means that in our limited experimental study, the performance degradation of choosing RFIR to produce initial values over the studied alternatives is small.
Figure 5.13: Performance degradation/enhancement \((X_\bullet - X_{\text{best}})\) for choosing a given method \(\bullet\) instead of the ‘best’ method for the measurement example in this chapter. The markers show individual observations and the boxes show the IQR. The dashed lines are the minimum, median and maximum of \(X_\bullet\) to give a sense of scale. On average there is little to gain or lose from using \(\hat{G}_{\text{RFIR}}\) instead of \(\hat{G}_{\text{best}}\). For \(\hat{G}_{\text{exist}}\) and \(\hat{G}_{\text{trunc}}\), the situation is a lot less favorable.
illustrate the usefulness of these smoothers for the generation of starting values for other systems. However, due to the intricate relationship between the attraction regions in the cost function, the location of the actual system poles, the SNR level and the choice of a particular excitation signal, we think it is intractable to construct rigid bounds on where the smoothers produce effective starting values.

5.7.1 Limitations of the Smoothers

On the one hand, an obvious limitation to the presented initialization techniques is that they share the limitations of the smoothing techniques: if the non-parametric estimate is a worse representation than the raw data (e.g. heavily biased), it is very unlikely that the initial estimate will outperform the existing estimates. However, it is beyond the scope of this chapter to determine formally which particular smoother is optimal in some specific experimental circumstances.

In particular for RFIR, a simulation study in (Chen and Ljung, 2013) suggests that RFIR handles systems with model orders up to at least 30, even for small datasets $N \leq 500$. For LPM (with or without time-truncation), no comparable studies are available to our knowledge, but the LPM itself has been used successfully in diverse practical applications.

5.7.2 Stress Test of the Smoothers

To test the effectiveness of the initial estimates obtained from these particular smoothers, an extra set of $n_{MC} = 100$ Monte Carlo simulations were performed. In particular, all combinations of

- low-pass, high-pass, band-pass and band-stop
- Chebyshev Type I, Chebyshev Type II, Elliptical and Butterworth

discrete-time filters (Zverev, 1967) of tenth degree generated by the MATLAB® code in Listing 5.1 are simulated. Each of these filters $G_c$ have been normalized such that $\|G_c\|_2 = 1$. The low-pass and high-pass filters have a cross-over frequency of $\frac{\pi}{2}$ rad/s. The band-pass, respectively band-stop, filters have a passband, respectively stopband, in the frequency range $\left[\frac{2\pi}{5}, \frac{3\pi}{5}\right]$ rad/s. Note also that both the band-pass and band-stop filters have a McMillan degree of 20 as per the MATLAB® conventions. See Table 5.3 for an overview of the filters and Figure 5.14 for the corresponding bode plots.
function G0s = systemsForStressTest()
    order = 10; % degree of the filter
    Qp = 3; % [dB] peak-to-peak ripple in pass-band
    Qs = 60; % [dB] attenuation in stop-band
    wPass = 0.5; % [pi rad/sample] cross-over frequency
    wBand = [0.4 0.6]; % [pi rad/sample] edges of the pass/stop band
    G0s = {normFilterDT(@() cheby1(order, Qp, wPass, 'low'))
          normFilterDT(@() cheby1(order, Qp, wBand, 'bandpass'))
          normFilterDT(@() cheby1(order, Qp, wPass, 'high'))
          normFilterDT(@() cheby1(order, Qp, wBand, 'stop'))
          normFilterDT(@() cheby2(order, Qp, wPass, 'low'))
          normFilterDT(@() cheby2(order, Qp, wBand, 'bandpass'))
          normFilterDT(@() cheby2(order, Qp, wPass, 'high'))
          normFilterDT(@() cheby2(order, Qp, wBand, 'stop'))
          normFilterDT(@() ellip(order, Qp, Qs, wPass, 'low'))
          normFilterDT(@() ellip(order, Qp, Qs, wBand, 'bandpass'))
          normFilterDT(@() ellip(order, Qp, Qs, wPass, 'high'))
          normFilterDT(@() ellip(order, Qp, Qs, wBand, 'stop'))
          normFilterDT(@() butter(order, wPass, 'low'))
          normFilterDT(@() butter(order, wPass, 'bandpass'))
          normFilterDT(@() butter(order, wPass, 'high'))
          normFilterDT(@() butter(order, wPass, 'stop'))};

function G = normFilterDT(func)
    [B, A] = feval(func);
    G = tf(B, A, 1); % transfer function B/A and Ts = 1 [s]
    G = G/ norm(G, 2); % enforces norm(G,2) == 1

Table 5.3: Overview of the test cases for the initialization stress test.

<table>
<thead>
<tr>
<th>Case</th>
<th>Filter</th>
<th>$\eta_0$ [%]</th>
<th>$\eta_{exist}$ [%]</th>
<th>$\eta_{func}$ [%]</th>
<th>$\eta_{RFIR}$ [%]</th>
<th>$\eta_{best}$ [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Chebyshev I low-pass</td>
<td>100</td>
<td>100</td>
<td>95</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>Chebyshev I band-pass</td>
<td>100</td>
<td>14</td>
<td>7</td>
<td>16</td>
<td>34</td>
</tr>
<tr>
<td>3</td>
<td>Chebyshev I high-pass</td>
<td>100</td>
<td>100</td>
<td>96</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>4</td>
<td>Chebyshev I band-stop</td>
<td>100</td>
<td>40</td>
<td>28</td>
<td>74</td>
<td>90</td>
</tr>
<tr>
<td>5</td>
<td>Chebyshev II low-pass</td>
<td>100</td>
<td>98</td>
<td>73</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>6</td>
<td>Chebyshev II band-pass</td>
<td>100</td>
<td>23</td>
<td>16</td>
<td>70</td>
<td>79</td>
</tr>
<tr>
<td>7</td>
<td>Chebyshev II high-pass</td>
<td>100</td>
<td>97</td>
<td>74</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>8</td>
<td>Chebyshev II band-stop</td>
<td>16</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>Elliptical low-pass</td>
<td>100</td>
<td>83</td>
<td>34</td>
<td>95</td>
<td>98</td>
</tr>
<tr>
<td>10</td>
<td>Elliptical band-pass</td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>Elliptical high-pass</td>
<td>100</td>
<td>83</td>
<td>33</td>
<td>97</td>
<td>100</td>
</tr>
<tr>
<td>12</td>
<td>Elliptical band-stop</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>Butterworth low-pass</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>14</td>
<td>Butterworth band-pass</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>15</td>
<td>Butterworth high-pass</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>16</td>
<td>Butterworth band-stop</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>
Chapter 5 Initialization of Parametric Estimators

The input excitation and the disturbing output noise were \( N = 1024 \) samples of white Gaussian noise such that an SNR of 20 dB was attained at the output. Note that due to \( \|G_0\|_2 = 1 \), the validation distance \( \left\| \hat{G}_\bullet - G_0 \right\|_2 \) is exactly the relative RMS estimation error of the transfer function.

5.7.3 Observations

The transfer functions are shown in Figure 5.14 while the empirical cumulative validation distance is shown in Figure 5.15. In Table 5.3, also the success rates per method are given when a tolerance \( \varepsilon_A = 0.02 \) is used when success is determined as \( D_\bullet = \left\| \hat{G}_\bullet - G_0 \right\|_2 < \varepsilon_A \). This is equivalent to the relative criterion of success since \( \|G_0\|_2 = 1 \) for all systems.

A few remarks regarding these results are appropriate.

- In many of the tested situations (cases 1, 3, 13–16), all methods yield a good model quality.

- In some situations (cases 8, 10, 12), the model quality is generally poor. The fact that for cases 8 and 12, the ‘ideal’ starting value \( \hat{G}_0 \) also provides poor models means the data is not informative enough to reasonably fit a good model.

- In all cases except 10 and 12, the RFIR performed as well or better than the existing methods.

- In case 10, the model quality of the RFIR is slightly worse than the already poor existing techniques. This is an underlying limitation of the RFIR used: \( n_g = 200 \) taps was used, however the true impulse response has not decayed significantly in that interval. Essentially, the particular RFIR used introduces a significant bias in the non-parametric description of the filter and hence the obtained initial value is unreliable.

- The RFIR performs (significantly) better than the existing methods in various cases (4–7, 9, 11).

- Since the ‘best’ method (- - - ) yields better models more often than a single method, e.g. in cases 2, 4 and 6, this illustrates that combining initialization schemes improves the model quality. Particularly, case 2 implies that \( \hat{G}_{\text{best}} \) contains initial estimates of all different techniques.
Figure 5.14: Bode plots of the tested filters (---) in the stress test. For each filter, two (randomly selected) instances of the parametric estimates \( \hat{G}_* \) are shown per method.
Figure 5.15: Empirical cumulative distribution function of the validation distance $D_\bullet = \| \hat{G}_\bullet - G_\circ \|_2$ for the different tested systems (different numbered plots) and the different initial values (colors). Note that since $\|G_\circ\|_2 = 1$ is chosen, $D_\bullet$ immediately indicates the relative RMS error of the estimates.
5.8 Computational Effort

• For all of the tested systems (except possibly for case 2), the truncated LPM is not advisable over the existing methods.

In general, one can see from these simulations that RFIR either performs as well as (or even better than) the existing techniques when a good estimate is obtained. There are no cases where using the smoothing to obtain initial values worsens the obtained model quality when the candidate model with the lowest cost function value is used (as in (5.33)). On the contrary, significant improvements were often achieved by including the initial values from the smoothed FRF and by combining different strategies.

5.8 Computational Effort

While improved initial values yield better models of an improved quality, their generation does require some computational effort. Although the amount of available computing power has been steadily increasing over the last decades, it remains important to keep an eye on the computing effort required to estimate a model or, in this case, to generate the starting values required to estimate a model. E.g. in the field of embedded processors, which has been attracting a lot of attention over the last few years leading to “Internet of Things”, it remains an important factor since computing power and physical power consumption are still relatively scarce.

Here, we will discuss two aspects of the computational effort very briefly. On the one hand, the computation time, as measured on a physical computer with a practical implementation of the algorithms, gives a rough idea of how these methods may perform in practice. On the other hand, the asymptotic complexity of the different smoothers is investigated. While such asymptotics give little insight in practical running times, they are valuable tools to see how well (or poor) such methods scale up towards larger datasets.

5.8.1 Practical Runtime

In practical measurements of the computation time for any piece of software, one typically distinguishes between different kinds of ‘time’.

• The wall-clock time $t_{\text{wall-clock}}$, sometimes called ‘real time’, is the time difference between the start and end of the program under test.
Chapter 5  Initialization of Parametric Estimators

- The CPU time $t_{CPU}$, is the amount of time that the central processing units (CPUs) actually spends executing the program under test.

This distinction is necessary for multi-tasking environments (such as modern computers): during the execution of the program under test, the CPU may spend some time executing other programs in the background. The latter obviously requires a small amount of wall-clock time, but it does not increase the CPU time of the program:

$$t_{wall-clock} = t_{CPU} + t_{other\ processes} \tag{5.35}$$

for a single-core CPU.

In summary, the $t_{CPU}$ is an indication of the computational effort required to execute the algorithm, whereas $t_{wall-clock}$ indicates the amount of time a user has to wait for the program. In this case $t_{CPU}$ is hence the most relevant of both, since $t_{wall-clock}$ is also determined by what other programs the user is running at the same time.

Unfortunately, MathWorks®, like many other interpreted languages, does not provide any feature to measure $t_{CPU}$ of a segment of code. Instead, the typical tic and toc functions measure the wall-clock time. To reduce the effect of other running programs and one-time actions (e.g. just-in-time compilation, caching, ...), it is a common practice (McKeeman, 2008) to repeatedly measure $t_{wall-clock}$ and retain the minimum (as an approximation for $t_{CPU}$) and/or the median (to reduce the effect of other programs claiming the CPU for extended amounts of time).

In this case, the MathWorks® function timeit (Eddins, 2010) has been used to measure the timing of the different functions. This function reports the median wall-clock time and chooses the number of repetitions automatically. The timings have been measured on an Early 2013 MacBook Pro Retina 15-inch with 2.7 GHz Intel Core i7 CPU and 16 GB random-access memory (RAM) running Mac OS X 10.11.3 and MathWorks® R2015b. The recommendations from the white paper by McKeeman, (2008) have been followed during these measurements.

To measure the timing of the different methods, essentially the same settings were used as in Section 5.4: the systems from equations (5.21) and (5.22) were simulated such that $n_{BW} \in \{36, 48\}$ and SNR = 0 dB. For each of the branches in Figure 5.2, three different timings are recorded (when applicable):

- $t_{smooth}$ is the time it takes to perform the smoothing (all blocks before GTLS/BTLS),
- $t_{init}$ is the time it takes to fit $\hat{G}_{init}$, and
- $t_{final}$ is the time it takes to fit $\hat{G}_{*}$. 

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Table 5.4: Observed timing of the different estimation steps.

<table>
<thead>
<tr>
<th>Q [dB]</th>
<th>$n_{BW}$</th>
<th>Method</th>
<th>$t_{\text{smooth}}$ [ms]</th>
<th>$t_{\text{init}}$ [ms]</th>
<th>$t_{\text{final}}$ [ms]</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>36</td>
<td>$\hat{G}_{\text{trunc}}$</td>
<td>745</td>
<td>62</td>
<td>163</td>
</tr>
<tr>
<td>6</td>
<td>36</td>
<td>$\hat{G}_{\text{RFIR}}$</td>
<td>6799</td>
<td>77</td>
<td>175</td>
</tr>
<tr>
<td>6</td>
<td>48</td>
<td>$\hat{G}_{\text{trunc}}$</td>
<td>1004</td>
<td>90</td>
<td>208</td>
</tr>
<tr>
<td>6</td>
<td>48</td>
<td>$\hat{G}_{\text{RFIR}}$</td>
<td>6186</td>
<td>91</td>
<td>221</td>
</tr>
<tr>
<td>20</td>
<td>36</td>
<td>$\hat{G}_{\text{trunc}}$</td>
<td>4837</td>
<td>226</td>
<td>455</td>
</tr>
<tr>
<td>20</td>
<td>36</td>
<td>$\hat{G}_{\text{RFIR}}$</td>
<td>6926</td>
<td>254</td>
<td>482</td>
</tr>
<tr>
<td>20</td>
<td>48</td>
<td>$\hat{G}_{\text{trunc}}$</td>
<td>6597</td>
<td>313</td>
<td>461</td>
</tr>
<tr>
<td>20</td>
<td>48</td>
<td>$\hat{G}_{\text{RFIR}}$</td>
<td>8052</td>
<td>312</td>
<td>469</td>
</tr>
</tbody>
</table>

The median timings as returned by \texttt{timeit} are reported in Table 5.4.

From the timings in Table 5.4, it can be seen that the smoothers require a considerable amount of time that overshadows the fitting time. However, it should be noted that all these implementations rely on code that has not been optimized for performance. As such it is not possible to draw conclusions about the performance one may expect from optimized implementations that one would use in a commercial toolbox.

In particular the RFIR smoother takes an exorbitant amount of time. The fact that for small problem sizes (top of the table) the runtime is only slightly shorter than for the larger problems (bottom of the table), suggests that there is a considerable fixed overhead in the implementation we used. Particularly, the construction of the DC kernel matrix happens to be the culprit for these long execution times. The truncated LPM, on the other hand, requires a more reasonable amount of time for the short data lengths. For longer datasets, it appears that the \texttt{kron} function is one of the major culprits for the long runtimes of the LPM.

### 5.8.2 Asymptotic Runtime

For the asymptotic runtime, we are mainly interested in how the runtime of the smoothers scale with the size of the dataset, i.e. the number of samples $N$, and the model complexity $n_{\theta}$ of the smoother.
Overall, the time-truncated LPM operates in $O\left(N n^3_\theta\right)$. First, the LPM solves $O\left(N\right)$ linear regression problems of small dimensions ($2n_W + 1$ data points and a few degrees of freedom, i.e. $n_\theta \leq 2n_W + 1$ or $O\left(n_W\right) = O\left(n_\theta\right)$), hence the runtime will be $O\left(N n^3_\theta\right)$ since solving a linear regression problem with $C$ parameters and $D$ data points requires $O\left(C^3 + DC^2\right)$ operations\(^1\). The truncation of the impulse response relies on linear regression which takes $O\left(N\right)$ operations since only 3 parameters are estimated during this step. As such, carrying out the first $O\left(N n^3_\theta\right)$ step and then the $O\left(N\right)$ step, the whole procedure remains a $O\left(N n^3_\theta\right)$ or, equivalently, $O\left(N n^3_{W}\right)$.

The RFIR relies on fitting a finite impulse response (FIR) filter with $n_\theta = n_g$ taps to $N$ datapoints. Using the same logic as above, this requires $O\left(N n^2_g + n^3_g\right)$ operations.

The computational complexity of both methods scales linearly in the length of the dataset. However, since the required length $n_g$ of a FIR filter is typically a lot more (e.g. 200 in this chapter) than the number of local parameters (or bandwidth) of the LPM (e.g. $n_W = 3$), it can be seen that eventually the RFIR method will scale less favorably for the model complexity of the smoother. Intuitively, the local modeling methods benefit from their locality since that enables them to be flexible without a huge amount of parameters.

### 5.8.3 Implementation Aspects

This section presents some implications and recommendations for the implementation of a system identification toolbox.

Whereas RFIR is shown to provide reliable starting values, it remains advisable to have multiple initialization strategies available to improve the likelihood of attaining a good starting value. This is also the approach that toolboxes such as (Kollár et al., 1994–2012) and (Ljung, 2010b) employ. Luckily, trying out multiple initialization strategies is embarrassingly parallelizable, i.e. one can easily try out a different strategy in parallel without incurring a lot of overhead in terms of wall-clock time. As such, computing multiple starting values can happen without a significant increase in wall-clock time and consequently a sluggish user-experience.

<table>
<thead>
<tr>
<th>Guideline 5.1: Try regularization to improve convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regularized estimators, such as RFIR, sometimes provide better initial estimates than the ‘true’ model parameters.</td>
</tr>
</tbody>
</table>

\(^1\)See http://math.stackexchange.com/questions/84495
5.9 Conclusion

Guideline 5.2: Try different initial values for non-convex optimization

Trying different initial values is quite inexpensive, but the quality of the final model may improve by several orders of magnitude if a local minimum can be avoided. As such, using multiple initial values, especially ones that have been derived with good judgment, pays off.

5.9 Conclusion

The simulations have demonstrated that use of a smoothed non-parametric estimate of the FRF can improve the success rate of minimizing the maximum-likelihood (ML) cost function for the estimation of the transfer functions of LTI systems, subjected to noisy signals. Specifically, two smoothing methods were used and compared: 1) the truncated LPM method, and 2) the RFIR method. The simulation results clearly show that the RFIR method is superior to the truncated LPM method and the existing BTLS and GTLS methods.

The usefulness of the initial values obtained via the proposed smoothing techniques was confirmed on a measurement of a band-pass filter in a noisy environment. In most cases, the proposed initial values in the example made it possible to reduce the cost function by a factor two compared to existing methods. Moreover, the obtained models were validated against a high-quality measurement where the RMS error on the transfer function could be reduced by a factor of eight for most cases, due to the improved initial values.

Thus, the work in this chapter has demonstrated the effectiveness of the studied FRF smoothing techniques in enhancing the initial values. Consequently, the quality of parametric estimates can be increased considerably by using non-parametric FRF models to initialize the non-convex optimization problem of fitting a rational transfer function model. Hence, the ease-of-use of parametric model fitting can be increased significantly by having high-quality non-parametric FRF models. This comes at a limited cost for the end-user: in principle these smoothers can be tried without any additional intervention by the user. The only disadvantage is the additional computational effort required and hence also the increase in wall-clock time to obtain a parametric estimate. However, thanks to the recent trend of parallel computing and the fact that this problem is embarrassingly parallel, we expect that increases in wall-clock time can be kept very modest if performance-tuned implementations are used.
Conclusions

Life can only be understood backwards; but it must be lived forwards.

Søren Kierkegaard

In this thesis, user-friendly system identification methods have been developed. These methods require little to no user interaction and are designed such that the methods work for a broad range of systems. As such, many of the obtained improvements can be obtained without bothering the user for further information and tuning. Throughout this dissertation, so-called 'Guidelines' have been scattered that offer a bird’s eye view of the work from a practical point of view. In the following sections, the main results of this work are summarized and perspectives are sketched for further (and ongoing) research paths.

6.1 Results of This Research

6.1.1 Excitation Design

In this thesis, multisine signals are proposed to measure single-input single-output (SISO) systems within a linear time-invariant (LTI) framework. Particularly, the proposed multisine signals employ a quasi-logarithmic frequency grid and a power spectrum in $1/f$ akin to pink noise to ensure a good quality of the frequency response function (FRF) over the whole excited frequency band. As such, for a constant total signal power and duration, they are a more robust excitation signal than e.g. typical general-use signals such as linear grid multisines or white noise excitation. In this thesis it has been proven that when the user knows the minimal damping $\xi_{\text{min}}$ of the poles of the system, this sets the required logarithmic frequency grid spacing to attain a guaranteed (relative)
uncertainty of the identified FRF near the resonance peaks. Concretely, the grid spacing\[ \alpha \triangleq \frac{f_{k+1}}{f_k} \]of the (quasi-)logarithmic grid with frequency resolution $\Delta f$ should conform to
\[ \alpha \leq 1 + \xi_{\text{min}} \]to attain a reliable estimate of the resonances present in the system. The choice to use the equality $\alpha = 1 + \xi_{\text{min}}$ is the optimal choice within the framework presented in Chapter 2 such that resonances with $\xi \geq \xi_{\text{min}}$ and $\omega_n \geq \omega_{\ell\ell} = \frac{2\pi \Delta f}{\alpha - 1} = \frac{2\pi \Delta f}{\xi_{\text{min}}}$ are well-represented in the FRF. Decreasing $\alpha$ further comes at the cost of more excited frequency bins and consequently a lower power per bin, or, a reduced signal to noise ratio (SNR) per bin if the overall signal power is limited.

In practical terms of a layman user, this means that the design of quasi-logarithmic multisines—as discussed in this thesis—requires only limited prior knowledge:

- the available measurement time (or equivalently, the minimal relevant frequency $\omega_{\ell\ell}$),
- the allowable total excitation power,
- a broad frequency band of interest that could span multiple decades, and,
- the minimal expected relative damping $\xi_{\text{min}}$ in the system,
- the assumption that the resonances are well-separated in the frequency domain.

### 6.1.2 Non-Parametric Estimation

**Leakage reduction** Different non-parametric estimators, specifically from the family of local estimators, *i.e.* Local Rational method with Iterative Cost function (LRIC), Local Rational Method (LRM) and Local Polynomial Method (LPM) are compared in this work. It has been shown that the most general LRIC is very sensitive to the use of the ‘correct’ local model order. The pre-existing LRM and LPM, however, are not as sensitive. It has been shown that the inherent bias of the LRM becomes the dominant source of errors, in contrast to the variance, when an SNR of less than 20 dB is available in the frequency bands of interest. For resonant systems, the LRM is superior to the LPM in terms of both bias and root-mean-square (RMS) error in almost all cases. The LPM is slightly superior for very noisy datasets (SNR < 20 dB) that have a very limited frequency resolution.
6.1 Results of This Research

Smoothing (noise reduction) In addition, an approach has been developed that allows to reduce the variance of the estimated FRF when it is captured in a long data record. This method works by examining the estimated impulse response, i.e. the inverse discrete Fourier transform (DFT) of the FRF, and estimating a truncation time $t_{\text{trunc}}$ beyond which the impulse response measurement is dominated by the noise. By truncating past this point, and transforming the result back into the frequency domain, an FRF is obtained that has a reduced RMS error by reducing the variance due to the noise. Two complementary approaches to determine $t_{\text{trunc}}$ have been developed. Both methods estimate the noise level on the impulse response by assuming the last 10% of the samples are only noise. The first method exploits the fact that many physical systems exhibit an exponential decay. This exponential decay is estimated roughly, such that the truncation point is determined as the instant where the decay is below the noise level. The second method divides the impulse response into segments. Starting from the last segment, it is tested statistically whether the preceding segment is significantly different from the noise, such that $t_{\text{trunc}}$ is estimated. For physical systems that exhibit an exponentially decaying impulse response, the exponential fitting method is to be preferred.

Peak value estimation It has also been illustrated that the use of such local models allows to extract the resonance peak and thereby the $\mathcal{H}_\infty$ gain of a system. In that setting, the LPM is not reliable due to its limited model flexibility. The LRM, on the other hand, has been shown to yield estimates of the $\mathcal{H}_\infty$ gain that improved greatly compared to pure FRF measurements. For systems that consist of complex dynamic behavior, this allows one to extract reasonable peak values without the hassle of building a global parametric model. The use of such local models, in practice, has been validated and can provide good estimates using a dataset that is four times smaller than what would be required for other FRF-based methods. As such, the measurement time (and hence cost) can also be reduced by the same amount.

6.1.3 Initialization of Parametric Estimation

To obtain a parametric model estimate from measured data often requires solving a non-convex optimization problem. However, for such optimization problems, often an initial parameter estimate is required to allow convergence to the global optimum, or a local optimum nearby. In this thesis, it was investigated whether convergence to a good optimum can be attained more easily by means of FRF smoothers; in particular when many noisy data points are available. A few practical smoothers have been studied, i.e. the time-truncated LPM, and the regularized finite impulse response (RFIR). It has been shown that especially the RFIR provides smoothed FRFs which improves convergence to a good optimum compared to the existing state-of-the-art initializers (bootstrapped...
total least squares (BTLS) and generalized total least squares (GTLS)). These smoothers allow one to obtain good models from noisier data than what existing techniques can achieve. The improvement of the initial values has been verified both on simulations, measurements, and a simulation stress test consisting of different filters. While overall the RFIR provides the best starting values from the different smoothers, combining multiple initial values remains a very valuable approach to make a system identification toolbox robust for the measurement data.

6.2 Future Perspectives

6.2.1 Excitation Design

In this thesis, quasi-logarithmic multisines were proposed as a robust excitation signal that provide efficient estimation of sharp resonance peaks in the SISO case. Obviously, a first important extension of this research is towards the multiple-input multiple-output (MIMO) case which is more involved. To measure MIMO systems using multisines, one of two typical approaches is often followed when measuring all channels at the same time (Pintelon and Schoukens, 2012, Section 2.7):

- Zippered multisines, where the frequency grids of the different input channels are interspersed. This has the downside that frequency resolution is traded in for computational simplicity and that the different entries in the FRF matrix are known on different frequency grids.

- Orthogonal multisines, which are computationally more involved to process, require multiple experiments but allow for finer frequency resolution.

Due to the intrinsic sparsity of quasi-logarithmic grids, however, zippered quasi-logarithmic multisines might offer an easy and elegant approach to measure MIMO systems with a moderate number of inputs. The use of LRM/LPM may help to overcome that the frequency grids for the different input/output channels do not coincide for zippered signals.

Also, in the related chapter, it has been assumed that the system is an LTI system. For weakly nonlinear systems, the proposed approach is likely to yield approximately valid results.
6.2 Future Perspectives

6.2.2 Non-Parametric Estimators

The LRM has only been explored for SISO systems in this thesis. This technique can obviously be extended towards MIMO systems, e.g. as the work by van Rietschoten, (2015) has shown. As explored there, the choice of parametrization remains an open question for the MIMO case. Also, unlike the SISO case, it becomes more important and tedious to select good local models since the dimensionality of the local estimation problem grows with the system dimensions. Extension to the MIMO case, moreover, may directly lead to an approach that is able to deal with concatenated data records similarly to what the LPM is able to provide.

As has been seen in Chapter 3, the LRIC is affected by a high variance. This is due to pole-zero cancellations that arise when the local bandwidth does not contain data that is informative enough to estimate models of the required complexity. As such, future research needs to be undertaken to overcome such hurdles, by means of model order selection or reduction of those local models. If these hurdles can be overcome, the bias of local modeling can be reduced significantly. This is important for short datasets with very noisy data.

Specifically for the LRM, currently a model structure that is equivalent to rational forms in continuous time, i.e. the (local) transfer function model is isomorphic with

\[ \tilde{G}_{CT}(\delta) = \frac{\tilde{B}(\delta)}{\tilde{A}(\delta)} = \frac{\sum_i b_i \delta^i}{\sum_i a_i \delta^i} \quad (6.2) \]

There have already been some experiments with other model structures such as local discrete time models where

\[ \tilde{G}_{DT}(\delta) = \frac{\tilde{B}(\delta)}{\tilde{A}(\delta)} = \frac{\sum_i b_i \exp(ji\delta T_s)}{\sum_i a_i \exp(ji\delta T_s)} \quad (6.3) \]

but these have proven more difficult to estimate numerically and hence unreliable (Pintelon and Schoukens, 2006, Section 2.4). A possible approach for this would be to use orthogonal polynomials (Bultheel et al., 2005; Forsythe, 1957) with complex coefficients. The LRM (and hence also LPM) could be extended to other model structures that are fine-tuned for specific applications as is also sometimes done for fully parametric system identification in the frequency domain (Pintelon and Schoukens, 2012, p. 179). E.g. in electrochemistry it is common (Wang, 1987) to describe diffusion phenomena using Warburg impedances which are proportional to \( \sqrt{s} \) (with \( s \) the Laplace variable). In such diffusion problems, it can be shown (Pintelon and Schoukens, 2012, p. 181) that the system transients dampen as an \( O(t^{3/2}) \), which is considerably slower than for lumped
continuous-time systems and discrete-time where the impulse decays as an \( O(e^{\alpha t}) \). As a result, estimating diffusion transients is an even more important task than the setting discussed in this dissertation. On the other hand, in microwave engineering (Pozar, 2005; Rizzi, 1988), commensurate filters are easily described using Richard’s variable \( R = \tanh(\tau s) \) with \( \tau \) the delay of the considered transmission lines. For these altered models, however, it should be verified whether numerical conditioning becomes a similar nuisance as for discrete-time models.

As the non-parametric methods such as LPM and LRM were originally intended to serve the same purpose as windows in classical spectral analysis, this naturally leads to other applications of these methods where windows are otherwise used to reduce the effect of spectral leakage.

In particular, for slowly time-varying systems, windows can be used in combination with the short-time Fourier transform (STFT) to obtain a rough approximation of either the output spectrum or the transfer characteristic of (slowly) time-varying systems. In this context, the LPM has already been used (Lataire et al., 2012). Preliminary tests using the LRM in lieu of the LPM have proven fruitful. Similarly, for linear parameter-varying (LPV) systems, the LRM allows to extract non-parametric models from LPV systems where the scheduling parameter is fixed during the experiment (van der Maas et al., 2015). Given the successful application of LRM in a frozen parameter-varying system, it seems worthwhile to investigate if the LRM can be extended to work in more challenging experimental settings. E.g. it could prove useful to extend the LRM towards measurements of LPV systems where the scheduling parameter is allowed to vary continuously during the experiment.

Another application where the LPM is developed further, is the handling of missing data (Ugryumova et al., 2015). In particular, the frequency domain relationship between the input \( U \) and output \( Y \) is augmented by a term \( M_Y \) that accounts for the missing output data samples:

\[
Y(\omega_k) = G(\omega_k)U(\omega_k) + T(\omega_k) + M_Y(\omega_k) + \text{noise},
\]

since \( G(\omega_k) \) is captured by local polynomials, this allows to construct a design matrix as discussed in Chapter 3 which only depends on \( U \) and the time instances where the data \( y(t) \) is missing. For the LRM, on the other hand, the design matrix depends on the output \( Y \) which is not known. Consequently, extending the LRM to handle missing data is non-trivial and could be an interesting topic for further research.

Finally, a comparison between different approaches for non-parametric modeling (Govers, Pintelon, et al., 2011; Hägg and Hjalmarsson, 2012; Stenman and Gustafsson, 2001) could be made. This could be done from a theoretical point of view to find correspondences between different approaches (e.g. both Transient Impulse Modeling (TRIMM) (Hägg
and Hjalmarsson, 2012) and LRM are generalizations of the empirical transfer function estimate (ETFE) for specific settings. A comparison from a qualitative point of view would be of more practical use, as this would allow to find the best operating regions of each approach. Eventually, this could even lead to hybrid methods that either select the best method for a specific setting or use results from different methods to overcome weaknesses of individual methods such as high computational complexity, high bias, high variance, ... When regarding the LRM as a specific form of local regression (Loader, 1999), this allows to reuse the ideas from that community. In particular, local regression often uses a so-called kernel that weights the least-squares fit (where the LRM implicitly uses a uniform kernel). In view of FRF estimation, a worthwhile endeavor could be to study the influence of these kernels on both the FRF estimate and the estimated noise level. Moreover, this could even lead to the design of new kernels.

6.2.2.1 Impulse Response Truncation

With respect to the impulse response truncation approaches introduced in Section 3.6, a few paths could be investigated further to broaden the class of systems where these methods are applicable and have a more subtle smoothing behavior of the transfer function.

The crude truncation approach that is discussed in this thesis, can be interpreted as equivalent to windowing the impulse response function (IRF) with a rectangular window. In the frequency domain, this causes a very obvious correlation in the FRF. By choosing a different window function (e.g., an exponential window, Hann window, ...), one could try to reduce this adverse effect.

Another alternative for this crude rectangular windowing approach, which still uses a DFT for the full size of the measurement record, could be to reduce the size of the DFT. The introduced algorithms that determine a truncation length $t_{\text{trunc}}$ could also give a good indication of what size of DFT would be suitable to represent the system well for the given measurements. However, further studies are required to compare the performance of these different alternatives.

Currently, the truncation approach works based on the assumption that the last fragment of the IRF is dominated by noise influences. The consequence of this is that the system is assumed to be stable and causal. It would be a worthwhile extension to relax this assumption to deal with unstable systems as well. For those systems, the last fragment of the IRF will be dominated by the unstable (or anti-causal) part of the system. Hence, the problem will become much harder to deal with, since a segment in the middle of the IRF needs to be found where the noise level can be estimated. Afterwards, one could
try to split up the IRF in a part attributed to the stable poles and one caused by unstable poles and apply truncation on both parts separately and recombining the results.

### 6.2.3 Non-Parametric Estimation and Interpolation

As with the regular LRM, ongoing research focuses on the estimation approach of the $\mathcal{H}_\infty$ gain towards the MIMO case. In that context, the design of weighting filters also becomes even more important (Boeren et al., 2013). Hence, the link between local models and the design of weighting filters and parametric overbounds is essential and should be confirmed further in a detailed study.

Another possibility for future research is to approach the problem of $\mathcal{H}_\infty$ gain estimation purely from the aspect of local regression such that combining multiple local models is no longer necessary.

### 6.2.4 Construction of Initial Values

In Chapter 5 it was already illustrated that smoothers can provide improved initial estimates of a system. Intrinsically, this stems from the fact that the effective SNR of the transfer function is improved by means of the smoothness assumption, and on the other hand by the reduction in the dimensionality in the parameter space. By no means were the smoothers studied in Chapter 5 chosen in an optimal way, such that it remains an open topic to determine which smoother has the best chance to produce good initial values in practical settings.

On the other hand, the LRM and its variants discussed in Chapter 3 provide some local information regarding the shape of the FRF and hence also the transfer function. The usefulness has already been illustrated in Chapter 4 in connection to the computation of the $\mathcal{H}_\infty$ norm of a system. Intrinsically, these local rational methods approximate the transfer function by a local model, or, equivalently a set of local poles and zeros. Especially for resonant poles, one can expect that these local poles are closely linked to the global (physical) poles of a system. As such, those could provide a valuable initialization strategy if the redundant local pole/zero information of neighboring windows can be reduced to a single global pole/zero initial value. Techniques in model order reduction are likely to provide a good starting point to simplify the local pole/zero estimates to a global parametric model which could then be used to serve as initial estimate for the non-convex optimization steps.
As noted in Remark 5.3, the selection of the model order used in RFIR is still an open topic. There are a few approaches that are worthwhile to investigate. On the one hand, the truncation methods explained in Section 3.6 could lead to an initial estimate of the impulse response length and hence the required model complexity for RFIR. On the other hand, one could devise crude model order selection procedures, e.g. estimating RFIR models with increasing complexity until increasing the model order offers no improvement in the non-parametric model. Alternatively, more flexible model structures (e.g. regularized auto-regressive exogenous (RARX)) could be tried. This does not do away with the model order selection of the initial model, but it will improve the computational efficiency since fewer parameters are needed to approximate a particular system.
Publications

The digital revolution is far more significant than the invention of writing or even of printing. It offers the potential for humans to learn new ways of thinking and organizing structures.

— Douglas Engelbart

During the preparation of my PhD, I have co-authored the following publications.

Journal articles


Conference papers


Conference abstracts, presentations, and posters


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Summary

To understand, simulate and control physical systems, good mathematical models are required. System identification is a set of powerful tools that distill mathematical models from measurements of the physical world. To this end, one has to:

1. perform experiments to observe the system in a meaningful way,
2. suppress unwanted effects and extract the features of interest from the measurements,
3. fit a model to the measurements.

Unfortunately, the effective use of system identification hinges strongly on the skill of the user. This doctorate focuses on easy-to-use system identification tools. These should allow both beginners to obtain good models, and seasoned practitioners to obtain better models with ease.

The first part of this work focuses on the design of a robust experiment. In contrast to optimal designs, only a few assumptions about the system, e.g. the minimal damping and the frequency bands of interest, are made. Therefore, such a constructed multisine signal can be used to measure many different systems efficiently.

The second part of this thesis describes non-parametric estimators for the frequency response function (FRF) that estimate and suppress leakage and noise effects. Extensions of the Local Polynomial and Local Rational Method (LPM and LRM) are investigated. These approximate the input-output measurements in local frequency domain windows by either polynomial or rational models.

Such local models can be used to great effect e.g. for the flexible dynamics of mechanical structures. These local models have been used to obtain a more detailed view of the many resonance peaks of an active vibration isolation system than typical FRFs provide and unlike high-order models, they don’t incur laborious model-order selection. Consequently, the measurement time and modeling effort can be reduced significantly.

The final part of the thesis considers the use of FRF smoothers to obtain better initial values to fit parametric models to the measurement data. In particular for low signal-to-noise ratios, local modeling or regularization provide considerable improvements. This allows users to obtain better models, even from poor measurements.
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- perform experiments to observe the system in a meaningful way,
- suppress unwanted effects and extract the features of interest, and
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Unfortunately, the effective use of system identification hinges strongly on the skill of the user. This doctorate focuses on easy-to-use system identification tools. These should allow both beginners to obtain good models, and seasoned practitioners to obtain better models with more ease.

First, the design of a robust all-purpose experiment is investigated. In contrast to optimal designs, only a few assumptions about the system, i.e. the minimal damping and the frequency bands of interest, are made. Therefore, such a constructed multisine signal can measure different systems efficiently.

The second part of this thesis describes non-parametric estimators for the frequency response function (FRF) that estimate and suppress leakage and noise effects. Extensions of the Local Polynomial and Local Rational Method are investigated. These methods approximate the input-output measurements in local frequency domain windows by either polynomial or rational models.

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The final part of the thesis considers the use of FRF smoothers to obtain better initial values to fit parametric models to the measurement data. In particular for low signal-to-noise ratios, local modeling or regularization provide considerable improvements by reducing the influence of noise in the estimation process. This allows users to obtain significantly better models, even from poor measurements.