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# Well-Posed Model Uncertainty Estimation by Design of Validation Experiments ⋆

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**Abstract:** In deterministic model validation approaches, model errors can be attributed to both disturbances and model uncertainty, leading to an ill-posed problem formulation. The aim of this paper is to remedy the ill-posedness in model validation for robust control. A two-stage procedure is developed, where first an accurate, nonparametric, deterministic disturbance model is estimated from data, followed by the enforcement of averaging properties through an appropriate periodic experiment design. The proposed deterministic approach results in an asymptotically correctly estimated model uncertainty and is illustrated in a simulation example.

## 1. INTRODUCTION

Model validation is an essential step in any modeling procedure, since a model should be accompanied by a quality certificate. Irrespective of how the model is obtained, its predictive power should be tested by confronting the model with measured data. In case the model can reproduce the measured data, then the model is not invalidated. In this respect, a model cannot be actually validated, since future measurements may invalidate it. Hence, model validation is often used to gain confidence in a model by collecting large data sets under similar but independent operating conditions.

Model quality is a crucial aspect in case the purpose of the model is subsequent control design. In case of an inaccurate model, the resulting model-based controller can result in performance degradation or even closed-loop instability when implemented on the true system. A suitable characterization of model quality in view of control design is by means of norm-bounded perturbations, e.g., as suggested in Zames [1981]. These perturbations can predict whether a designed controller will stabilize a certain class of perturbed systems, presumably containing the true plant. In contrast, this phenomenon cannot be predicted when the uncertainty is attributed to an additive signal, since additive signals cannot destabilize a feedback loop. Besides its impact on the description of model quality, the specific control application of the model, e.g., for flexible mechanical systems [van de Wal et al., 2002], also has certain specific properties, including the possibility to collect large data sets and a large freedom in the design of experiments.

The emergence of robust control methodologies, e.g., Francis [1987], has led to a development of model validation approaches that can deal with model uncertainty, see Poolla et al. [1994] and Smith and Doyle [1992] for time and frequency domain approaches, respectively. In essence, the key question that is addressed in these model validation for robust control approaches is whether there exists an admissible realization of the model uncertainty and disturbance in a certain predefined set that can explain the measurement data. Besides the perturbation model that represents model uncertainty, inclusion of an additive signal to represent disturbances is essential, since any real-life system is subject to such unmeasured exogenous inputs. Typically, both the perturbation and additive signal are characterized as deterministic sets, hence such approaches are classified as deterministic model validation approaches.

Although deterministic model validation approaches are directly compatible with robust control, these deterministic methodologies are generally ill-posed. In particular, in model validation the goal is generally to determine the smallest model uncertainty and disturbance that can reproduce the measured data. In a deterministic approach, the nominal model residual can be attributed to both perturbations and additive signals. The resulting non-uniqueness of the optimal solution to the model validation problem classifies the problem as ill-posed [Tikhonov and Arsenin, 1977]. Ill-posedness is also supported by the presence of trade-off curves in the model validation problem, as is illustrated in Kosut [1995], Kosut [2001].

From a robust control perspective, ill-posedness of the model validation problem is highly undesirable. On the one hand, if too much of the nominal model residual is attributed to disturbances, then the resulting model uncertainty may be too small to encompass the true plant. Hence, in this case a model-based controller may suffer from performance degradation or even closed-loop instability when implemented on the true system. On the other hand, if too much of the nominal model residual is attributed to disturbances, then an overly conservative control design will result. In the present paper, it is attempted to remedy the ill-posedness of the model validation problem.

The main mechanism that causes the ill-posedness in deterministic model validation is the poorly defined notion of disturbance. Conceptually, exogenous disturbances are signals that are independent of the input to the system. In case the model residual contains signals that are dependent on the input to the system, then these signals are a part of input-output behavior and should be considered as model uncertainty. However, in a deterministic setting, the optimal outcome of the model validation problem corresponds to the best-case disturbance signal in a

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predefined set. This best-case additive signal is typically perfectly dependent on the input signal, hence the optimal solution corresponds to the case where the disturbance can reproduce a part of the nominal model residual that is caused by an incorrect input-output model. This leads to an overly optimistically estimated model uncertainty. In contrast to deterministic disturbance descriptions, by defining disturbances in a stochastic framework, e.g., as in Ljung [1999], independence of disturbances and input signals can straightforwardly be enforced. In Liu and Chen [2005], Paganini [1996], approaches are suggested that resemble the stochastic notion of disturbances. However, these approaches are computationally infeasible for moderate data lengths, as is commonly encountered in model validation for control.

Pursuing a worst-case approach to model validation [Ozay and Sznaier, 2007], which is closely related to identification in $\mathcal{H}_\infty$, does not resolve the ill-posedness in the model validation problem. Indeed, in this case the disturbance is allowed to perfectly work against the input, see Hjalmarsson [1993], and hence is dependent on the input. The use of many data sets in model validation motivates an optimistic approach instead of a pessimistic/worst-case approach. Specifically, in case a certain data set contains a large disturbance contribution, then this data set is less informative regarding the input-output behavior of the system. Hence, the input-output model is not necessarily inadequate, and thus this should not lead to a large model uncertainty. In the present paper, ill-posedness is further analyzed and a framework is presented where independence of disturbances is appropriately enforced, leading to an asymptotically correctly estimated disturbance model and model uncertainty. Thus, the optimism is reduced for an increased measurement length, classifying the proposed approach as a moderately optimistic model validation procedure.

The main contribution of this paper is a model validation approach that attempts to resolve the ill-posedness in deterministic approaches and that is suitable for multivariable systems with large data lengths. To this end, a two-stage frequency domain approach is developed. Firstly, accurate, nonparametric, and deterministic disturbance models are estimated based on mild stochastic assumptions (Section 3). Secondly, averaging properties of disturbances in a deterministic framework are established (Section 4). Averaging is achieved by considering an appropriate input design, specifically periodic input signals. Advantages of periodic input signals are well-established in a stochastic framework [Pintelon and Schoukens, 2001]. In the present paper, it is shown how these advantages can be established in a deterministic setting. In addition, a computational solution is presented (Section 5) and the approach is illustrated in a simulation example (Section 6).

## 2. MODEL VALIDATION PROBLEM

### 2.1 Motivation for a frequency domain approach

In this paper, a frequency domain approach to model validation is considered. A frequency domain approach has the following advantages that are not found in a time domain approach.

(1) Frequency domain approaches enable the use of non-parametric disturbance models. This implies that a parameterization and numerical optimization step are not required. Consequently, no undermodeling errors are introduced during disturbance modeling.
(2) The resulting algorithm involves constant matrix computations at each frequency, where the matrix dimensions are invariant under the measurement length. In contrast, the involved matrix dimensions in a time domain approach grow with the measurement length, resulting in computational infeasibility for medium or large scale problems.
(3) The $\mathcal{H}_\infty$-norm has a clear frequency domain interpretation. In fact, necessary and sufficient conditions are available for verifying the validity of $\mathcal{H}_\infty$-norm bounded perturbation models in case a discrete frequency grid is used, see Oomen and Bosgra [2008]. This is a useful property, since experiments on real-life systems are always based on finite time data, and hence discrete frequency grids.

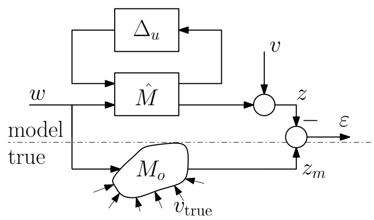


Fig. 1. Model validation setup.

### 2.2 Problem formulation

The model validation setup considered in this paper is depicted in Figure 1. The true system $M_o \in \mathcal{RH}_\infty^{n_z \times n_w}$ is given by

$$z_m = M_o w + v_{\rm true}, \tag{1}$$

where $z_m$ is the measured output, $w$ is the manipulated input, and $v_{\rm true}$ represents a disturbance term, including unmeasured inputs and measurement noise. The uncertain model is represented by

$$z = \mathcal{F}_u(\hat{M}, \Delta_u)w + v, \tag{2}$$

where $z$ is the uncertain model output, $v$ represents the disturbance model, and $\hat{M} \in \mathcal{RH}_\infty^{(n_p+n_z)\times(n_q+n_w)}$ contains the nominal model and uncertainty model interconnection structure. Both open-loop and closed-loop systems can be handled by considering appropriate uncertainty structures, see, Oomen et al. [2009] for a dual-Youla based structure that is tuned towards a certain control criterion. Structured $\mathcal{H}_\infty$-norm bounded perturbations are considered, i.e.,

$$\boldsymbol{\Delta}_u := \left\{ \Delta_u \in \mathcal{RH}_\infty \middle| \Delta_u(e^{j\omega}) \in \boldsymbol{\Delta}_u^c, \omega \in [0, 2\pi) \right\}, \tag{3}$$

$$\begin{aligned} \boldsymbol{\Delta}_u^c := \operatorname{diag}\big\{ & \left(\delta_1 I_{r_1}, \ldots, \delta_s I_{r_S}, \Delta_{S+1}, \ldots, \Delta_{S+F}\right) \big| \\ & \delta_i(z) \in \mathbb{C}, \Delta_j \in \mathbb{C}^{p_j \times q_j}, i = 1, \ldots, S,\ j = 1, \ldots, F \big\} \end{aligned} \tag{4}$$

where $\delta_i$ and $\Delta_j$ represent repeated scalar perturbations and full block perturbations, respectively, and are norm-bounded as

$$\bar{\sigma}(\delta_i) < \gamma,\ i = 1, \ldots, S, \quad \bar{\sigma}(\Delta_j) < \gamma,\ i = 1, \ldots, F. \tag{5}$$

In case the data for model validation is extracted from a real-life system, then model validation is always performed using time signals that have a certain finite length $T$. Hence, it is assumed that all signals are in $\ell_2[1, T]$. In the frequency domain validation problem, the measured discrete time signals $w(t)$ and $z_m(t)$ are transformed into the frequency domain using the DFT, resulting in $W(\omega_i)$ and $Z_m(\omega_i)$, respectively, e.g.,

$$W_N(\omega_i) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} w(t) e^{j\omega_i t}, \tag{6}$$

where $\omega_i \in \Omega$, and $\Omega$ is the DFT grid $\Omega$ given by $\frac{2\pi p}{N}$, $p = 0, 1, \ldots, N-1$. To anticipate on the results in Section 4, the frequency grid $\Omega^{\text{val}}$ is introduced, which is given by

$$\Omega^{\text{val}} = \left\{ \omega_i \in \Omega \middle| W(\omega_i) \neq 0 \right\}. \tag{7}$$

In the frequency domain case, the uncertain model residual is defined as

$$E(\omega_i) = Z_m(\omega_i) - Z(\omega_i), \tag{8}$$

where $Z(\omega_i)$ is the DFT of $z(t)$. In addition, assume that

$$V(\omega_i) \in \mathbf{V}(\omega_i), \tag{9}$$

where $\mathbf{V}(\omega_i)$ is defined more precisely in Section 3.

This leads to the following Frequency Domain Model Validation Decision Problem (FDMVDP).

*Problem 1.* (FDMVDP). Let the uncertain model (2), a norm-bound $\gamma(\omega_i)$, and measurements $W(\omega_i), Z_m(\omega_i)$, $\omega_i \in \Omega^{\text{val}}$ be given. Then, the FDMVDP amounts to verifying whether $\exists \Delta_u(\omega_i) \in \mathbf{\Delta}_u^c, V(\omega_i) \in \mathbf{V}(\omega_i)$ such that $E(\omega_i) = 0$.

To determine the minimum-norm validating $\Delta_u$, the Frequency Domain Model Validation Optimization Problem (FDMVOP) is introduced.

*Problem 2.* (FDMVOP). Let the uncertain model (2) and measurements $W(\omega_i), Z_m(\omega_i)$, $\omega_i \in \Omega^{\text{val}}$ be given. Then, the FDMVOP amounts to determining the minimum value of $\gamma(\omega_i)$ such that $\exists \Delta_u(\omega_i) \in \mathbf{\Delta}_u^c, V(\omega_i) \in \mathbf{V}(\omega_i)$ such that $E(\omega_i) = 0$.

In this paper, a solution to the FDMVOP is provided. In the pursued approach, the solution to the FDMVOP is obtained by performing a bisection over $\gamma(\omega_i)$ and solving the corresponding FDMVDPs. To ensure the model validation problem is sensible, the following assumptions are imposed for each $\omega_i \in \Omega$. Firstly, well-posedness of the LFT $\mathcal{F}_u(\hat{M}, \Delta_u)$ is ensured.

*Assumption 3.* $\det(I - \hat{M}_{11}\Delta_u) \neq 0 \; \forall \Delta_u \in \mathbf{\Delta}_u, \|\Delta_u\|_\infty < \gamma$.

The following assumption implies that the model uncertainty affects the relevant model outputs.

*Assumption 4.* For a certain $\Delta_u \in \mathbf{\Delta}_u$, it holds that $Z_m - \hat{M}_{22} W \in \operatorname{Im}\left(\hat{M}_{21}\Delta_u(I - \hat{M}_{11}\Delta_u)^{-1}\hat{M}_{12}\right)$.

Assumptions 4 requires an appropriate selection of the perturbation model structure. Finally, it is ensured that the model validation problem is not trivially solved. Thereto, define the nominal model residual

$$E^{\text{nom}} = Z_m - \hat{M}_{22} W, \tag{10}$$

leading to the following assumption.

*Assumption 5.* $E^{\text{nom}} \neq 0$.

### *2.3 A motivating example*

The following example reveals that a deterministic disturbance model, as is employed in, e.g., Smith and Doyle [1992], can lead to an ill-posed problem formulation and as a consequence to a poor estimation of both the disturbance model and model uncertainty.

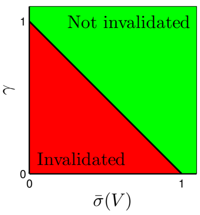


Fig. 2. Example 6: invalidated models.

*Example 6.* Consider the frequency domain model validation problem for a certain $\omega_i \in \mathbb{R}$. Given the measurement

$$W(\omega_i) = 1, \; Z_m(\omega_i) = 3 + 2j, \tag{11}$$

and the uncertain model

$$\hat{M}(\omega_i) = \begin{bmatrix} 0 & 1 \\ 1 & 3 + 3j \end{bmatrix}, \tag{12}$$

i.e., a SISO system equipped with an additive perturbation model. In Figure 2, the result of the FDMVDP is depicted for different norm-bounds of $\Delta_u$ and $V$, i.e., $\gamma$ and $\bar{\sigma}(V)$, respectively. A trade-off curve between the disturbance $V$ and model uncertainty can clearly be observed, which contains solutions to the FDMVOP for different sets $\mathbf{V}$. In case $\bar{\sigma}(V) = 1$, then the nominal model residual $E^{\text{nom}}(\omega_i)$ can be fully attributed to additive disturbances. In contrast, in case $\gamma = 1$, then $E^{\text{nom}}(\omega_i)$ can be fully attributed to model uncertainty. As explained in Section 1, this trade-off is undesirable from a robust control perspective.

## 3. DISTURBANCE MODEL

In this section, the disturbance model $\mathbf{V}$ is elaborated on in detail. As motivated in Section 2.1 and Section 1, nonparametric, deterministic disturbance models are employed.

In contrast to deterministic models, stochastic models provide an accurate description for many realistic situations. In this section, a procedure is presented where nonparametric disturbance models are estimated based on mild stochastic assumptions. Subsequently, these stochastic models are converted to deterministic models for each frequency $\omega_i \in \Omega^{\text{val}}$.

Firstly, the approach is conceptually explained for the single-variate case, which is generalized to the multivariate case at the end of this section. Throughout, the analysis is performed for the infinite time case, at the end of the section the implications for the finite time case are discussed. The following assumption is imposed in the time domain.

*Assumption 7.* Let $v_s = H_o e$, where $e \in \ell_2$ is a sequence of independent, identically distributed random variables with zero mean, unit variance, and bounded moments of all orders, and $H_o \in \mathcal{RH}_\infty$.

Assumption 7 represents the additive, stochastic disturbance model in the time domain. This model provides a suitable representation for many real-life disturbances.

The time domain Assumption 7 leads to the following frequency domain result.

*Theorem 8.* Consider $v_s$ given by Assumption 7. Then, for $N \to \infty$, the DFT of $v_s$, i.e., $V_{s,N}(\omega_i)$, converges in distribution to $\mathcal{N}_c(0, C_{v_s}(\omega_i))$, for $\omega_i \neq k\pi, k \in \mathbb{Z}$. In addition, $V_{s,N}(e^{j\omega_i})$ and $V_{s,N}(e^{j\omega_j})$ are asymptotically independent for $i \neq j$, $\omega_i, \omega_j \in [0, \pi]$.

For a proof of Theorem 8, see, e.g., Pintelon and Schoukens [2001]. For a definition of the circular complex normal distribution $\mathcal{N}_c$, see, e.g., Miller [1974]. A useful property of this distribution in the derivation of deterministic disturbance models is that for $z \in \mathcal{N}_c$, $\Re(z)$ and $\Im(z)$ are independent. Hence, lines of constant probability are circles in the complex plane.

Using the result of Theorem 8, the following result is obtained.

*Proposition 9.* Let $V_{s,N}(\omega_i)$ be a circularly complex normally distributed random variable, i.e., $V_{s,N} \in \mathcal{N}_c(0, C_{v_s}(\omega_i))$, and $\alpha \in [0, 1)$. Then,

$$\mathcal{P}(|V_{s,N}(\omega_i)| < \sqrt{\tfrac{1}{2}C_{v_s}(\omega_i)c_\chi}) = \alpha, \tag{13}$$

where $c_\chi$ denotes the $\alpha$-probability level of the $\chi^2(2)$-distribution.

Proposition 9 follows from standard results of $\chi^2$-distributions and enables the conversion of a stochastic disturbance model to a deterministic one. Specifically, let

$$\bar{\bar{V}}(\omega_i) = \sqrt{\tfrac{1}{2}C_{v_s}(\omega_i)c_\chi}, \tag{14}$$

then with a probability $\alpha$,

$$V_{s,N} \in \delta_v \bar{\bar{V}}(\omega_i), \quad |\delta_v| < 1. \tag{15}$$

In (15), $\delta_v \bar{\bar{V}}(\omega_i)$ with constraint $|\delta_v| < 1$ constitutes a deterministic disturbance model. Note that the model is frequency dependent, resulting in a nonparametric disturbance model for all $\omega_i \in \Omega^{\text{val}}$.

The variance $C_{v_s}(\omega_i)$ can be estimated from data in case a periodic input signal is used. In particular, in Oomen and Bosgra [2008], an estimator is proposed for finite time experiments, resulting in estimated covariance matrices for $\omega_i \in \Omega^{\text{val}}$. This result can be further refined to include the distribution of the estimated covariance matrix.

In Oomen and Bosgra [2008], the property that a circular complex normal distribution of random vectors can be diagonalized is exploited to generalize the approach to the multi-variate case. Specifically, for a system with $n_z$ outputs, the deterministic disturbance model

$$\mathbf{V}(\omega_i) = \left\{ T_V(\omega_i)\Delta_v \bar{\bar{V}}(\omega_i) | \Delta_v \in \mathcal{B}\mathbf{\Delta}_v \right\} \tag{16}$$

$$\mathbf{\Delta}_v = \left\{ \text{diag}(\delta_{v,1}, \delta_{v,2}, \ldots, \delta_{v,n_z}) | \delta_{v,q} \in \mathbb{C}, q = 1, \ldots, n_z \right\}. \tag{17}$$

is obtained, where $\bar{\bar{V}}(\omega_i) \in \mathbb{R}^{n_z \times n_z}$ is a diagonal matrix and $T_V(\omega_i) \in \mathbb{C}^{n_z \times n_z}$ is a coordinate transformation matrix. In this case,

$$\mathbf{V}(\omega_i)\mathbf{1}, V(\omega_i) \in \mathbf{V}(\omega_i) \tag{18}$$

constitutes a multivariable deterministic disturbance model.

## 4. AVERAGING IN A DETERMINISTIC FRAMEWORK

Model validation in a deterministic framework requires precautions to enforce independence properties of the disturbance and the input. Specifically, irrespective whether a deterministic or stochastic approach is pursued, the concept of a disturbance prohibits a dependence of the disturbance on the input. Optimal solutions to deterministic problems, however, typically correspond to situations where disturbances perfectly correlate with the input, see, e.g., Hjalmarsson [1993]. Building on the developments in Section 3, a frequency response-based approach enables a distinction between disturbances and the deterministic response due to input signals. In particular, analysis of the disturbance model (16) reveals that $v$ cannot contain periodic components. This is a direct consequence of the stochastic model (7), where $H_o \in \mathcal{RH}_\infty^{n_z \times n_z}$, and is preserved in the transformation to a deterministic model. Hence, since $v$ is nonperiodic, a distinction between $v$ and $u$ is obtained when $u$ is periodic. This idea is developed further in this section.

Using (8) and (2), the uncertain model residual is given by

$$E(\omega_i) = Z_m(\omega_i) - \left( \mathcal{F}_u(\hat{M}, \Delta_u)W(\omega_i) + V(\omega_i)\mathbf{1} \right), \tag{19}$$

$$V(\omega_i) \in \mathbf{V}(\omega_i). \tag{20}$$

In the FDMVOP, the goal is to determine the minimum value of $\gamma$ such that the uncertain model is consistent with the data, i.e.,

$$E(\omega_i) = 0 \; \forall \omega_i \in \Omega^{\text{val}}. \tag{21}$$

The argument $\omega_i$ is omitted in further derivations for notational convenience, hence all equations involve constant matrices. Assuming consistency and using (21) in (19) and rearranging yields

$$E^{\text{nom}} = \hat{M}_{21}\Delta_u(I - \hat{M}_{11}\Delta_u)^{-1}W + T_V \Delta_v \bar{\bar{V}}\mathbf{1}. \tag{22}$$

A key issue is that the nominal model error $E^{\text{nom}}$ depends on both the size and the input direction of $W$. For a constant direction of $W$, a normalization is useful to assess the model error for different input signals. In particular,

$$\bar{E}^{\text{nom}} := \frac{E^{\text{nom}}}{\|W\|_2} = \frac{M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12}W}{\|W\|_2} + \frac{T_V \Delta_v \bar{\bar{V}}\mathbf{1}}{\|W\|_2}. \tag{23}$$

The interpretation of (23) is that for each set of validation data, the normalized nominal model residual $\bar{E}^{\text{nom}}$ has to be explained by the model uncertainty term $\frac{M_{21}\Delta_u(I-M_{11}\Delta_u)^{-1}M_{12}W}{\|W\|_2}$ and the disturbance term $\frac{T_V \Delta_v \bar{\bar{V}}\mathbf{1}}{\|W\|_2}$. Although the disturbance contribution is essential since it accounts for the fact that the measurement $Z_m$ is contaminated by disturbances, it may result in an overly optimistic model validation result. Specifically, due to the deterministic, set-based disturbance description (20), the disturbance can help explain a part of $\bar{E}^{\text{nom}}$ that is actually caused by a systematic model error.

The following proposition is the main result of this section and reveals that, in contrast to the model uncertainty contribution, the disturbance contribution in (23) averages out for a suitably chosen input signal.

*Proposition 10.* If the input $W$ increases by a factor $\alpha$, then the part of the normalized nominal model residual $\bar{E}^{\text{nom}}$ that can be attributed to disturbances decreases by $\alpha$, whereas the contribution of the model uncertainty is invariant under the increase of the $W$.

**Proof.** Only a sketch of the proof for the SISO case is provided. Observe that in this case, the two quantities on the right hand side of (23) are scalar. Next, by increasing the input with $\alpha$, and employing the homogeneity property of norms,

$$\bar{E}^{\text{nom}} := \frac{M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12}W}{\|W\|_2} + \frac{\delta_v \bar{\bar{V}}_q}{\alpha\|W\|_2}, \tag{24}$$

Next, from (24) and using the fact that $\delta_v$ is norm-bounded, i.e., $\bar{\sigma}(\delta_v) < 1$, the part of $\bar{E}^{\text{nom}}$ that can be

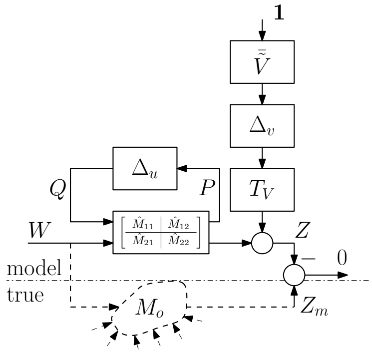


Fig. 3. Consistency in the FDMVDP.

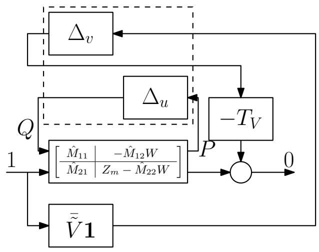


Fig. 4. Recasting Figure 3 as an implicit LFT.

attributed to disturbances decreases with a factor $\alpha$ if the input is increased with a factor $\alpha$, completing the proof.

The signal $W$ can be increased by a factor $\alpha$ by amplifying the time domain signals by a factor $\alpha$. In addition, repeating the experiment has a similar result in the frequency domain case. In particular, suppose that $w(t)$ is periodic with period $N$. Then, in case the total measurement time is equal to an integer number of periods $n_{\text{per}}$, i.e., $n_{\text{per}}N$, then for $\omega_i = \frac{2\pi p}{N}$, $p = 0, 1, \ldots, N-1$,

$$W_{n_{\text{per}}N}(\omega_i) = \sqrt{n_{\text{per}}} W_N(\omega_i). \tag{25}$$

The admissible contribution of the disturbance term $v$ in the normalized nominal model error $\bar{E}^{\text{nom}}$ thus decreases with a factor $\sqrt{n_{\text{per}}}$ if $n_{\text{per}}$ periods of the periodic signal $w$ are applied. In this sense, the deterministic disturbance averages out with a factor $\sqrt{n_{\text{per}}}$ if the measurement time is increased with a factor $n_{\text{per}}$.

## 5. SOLUTION TO THE VALIDATION PROBLEM

In this section, a solution to the FDMVDP is provided. The FDMVDP amounts to verifying consistency of the model with the data at each frequency $\omega_i \in \Omega^{\text{val}}$. The consistency equation, i.e., $E = 0$, using (8), (2), and (16), becomes

$$0 = Z_m - \mathcal{F}_u(M, \Delta_u)W - T_v \Delta_v \bar{\bar{V}}\mathbf{1}, \tag{26}$$

where $\Delta_u \in \mathbf{\Delta}_u$, $\Delta_v \in \mathcal{B}\mathbf{\Delta}_v$, see Figure 3. Next, selecting a scalar input equal to one and rearranging yields the implicit LFT of Figure 4, i.e., an LFT with an input and an output equal to zero.

By linearity, Figure 4 corresponds to the implicit LFT

$$\mathcal{F}_u(\bar{M}, \bar{\Delta})\alpha = 0, \tag{27}$$

where

$$\bar{M} = \left[\begin{array}{cc|c} 0 & 0 & \bar{\bar{V}}\mathbf{1} \\ 0 & \gamma M_{11} & -M_{12}W \\ \hline -T_V & \gamma M_{21} & Z_m - M_{22}W \end{array}\right] \tag{28}$$

$$\bar{\Delta} \in \mathcal{B}\bar{\mathbf{\Delta}},\ \bar{\mathbf{\Delta}} = \left\{ \begin{bmatrix} \Delta_v & 0 \\ 0 & \Delta_u \end{bmatrix} | \Delta_v \in \mathbf{\Delta}_v, \Delta_u \in \mathbf{\Delta}_u \right\}, \tag{29}$$

and $\alpha \in \mathbb{C}\backslash 0$. In (28) and (29), the norm of $\Delta_u$, i.e., $\gamma$, has been absorbed in $\bar{M}$. By manipulating the consistency equation (26) to the block diagram of Figure 4, the FDMVDP is recast as the existence of signals that satisfy the implicit LFT (27), which is related to the Structured Singular Value (SSV) for autonomous LFTs. In fact, the following definition provides an extension of the SSV that is useful in model validation.

*Definition 11.* (Generalization of the SSV). [Paganini and Doyle, 1994] Given complex matrices $M, N$ of appropriate sizes, $\bar{\mu}_{\bar{\mathbf{\Delta}}}(M, N)$ is defined as

$$\bar{\mu}_{\bar{\mathbf{\Delta}}}(M, N) := \left( \min_{\bar{\Delta}} \{ \bar{\sigma}(\bar{\Delta}) | \bar{\Delta} \in \bar{\mathbf{\Delta}}, \operatorname{Ker}\left( \begin{bmatrix} I - \bar{\Delta}M \\ N \end{bmatrix} \right) \neq 0 \} \right)^{-1},$$

unless $\operatorname{Ker}\left( \begin{bmatrix} I - \bar{\Delta}M \\ N \end{bmatrix} \right) \neq 0$, $\forall \bar{\Delta} \in \bar{\mathbf{\Delta}}$, in which case $\bar{\mu}_{\bar{\mathbf{\Delta}}}(M, N) := 0$.

The definition of $\bar{\mu}_{\bar{\mathbf{\Delta}}}(M, N)$ leads to the main result of this section, which is a necessary and sufficient test for the FDMVDP.

*Proposition 12.* In the FDMVDP, the model is not invalidated if and only if $\bar{\mu}_{\bar{\mathbf{\Delta}}}(\bar{M}_{11} - \bar{M}_{12}\bar{X}_{22}\bar{M}_{21}, \bar{M}_{21} - \bar{M}_{22}\bar{X}_{22}\bar{M}_{21}) > 1$, where $\bar{X}_{22}$ is a matrix that satisfies $\bar{X}_{22}\bar{M}_{22} = I$.

A proof can be obtained by employing the theory for implicit LFTs that is presented in Paganini and Doyle [1994], where also computable upper and lower bounds of $\bar{\mu}_{\bar{\mathbf{\Delta}}}$ are provided. Numerical algorithms are available in Balas et al. [2007]. The result of Proposition 12 can directly be used to solve the FDMVOP via bisection.

## 6. EXAMPLE

In this section, the results of the preceding sections are illustrated by means of an example. Specifically, the nonparametric estimation of the deterministic disturbance model of Section 3 is illustrated. Subsequently, it is shown that disturbances average out if the measurement time is increased.

Consider the true ARX system

$$\left(1 - 0.9z^{-1} + 0.7z^{-2}\right) y = 0.3z^{-2}u + e, \tag{30}$$

where $e(t)$ is zero mean white noise with unit variance. The goal is to verify the validity of the model

$$\hat{P} = \frac{0.3z^{-2}}{1 - z^{-1} + .7z^{-2}}, \tag{31}$$

i.e., to determine the minimum-norm $\Delta_u$ such that the model is consistent with the data. An additive perturbation model is considered, i.e., $\hat{P} + \Delta_u$. The input signal is chosen as

$$u(t) = \sum_{\omega_i = 1}^{n_\omega} \sin(\omega_i t),\ \omega_i \in \Omega^{\text{val}}, \tag{32}$$

where
$\Omega^{\text{val}} = \left\{ \frac{2\pi}{N}, \frac{2\pi 5}{N}, \frac{2\pi 10}{N}, \frac{2\pi 15}{N}, \frac{2\pi 20}{N}, \frac{2\pi 25}{N}, \frac{2\pi 30}{N}, \frac{2\pi 35}{N}, \frac{2\pi 40}{N}, \frac{2\pi 45}{N} \right\}$,
with $N = 100$.

The first step in the model validation procedure is the estimation of a nonparametric disturbance model, where

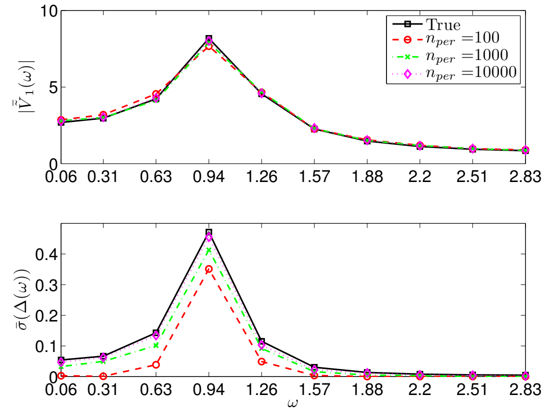


Fig. 5. Results of model validation test for varying $n_{\text{per}}$. Estimated deterministic disturbance model (top), estimated model uncertainty (bottom).

$\alpha = 0.99$. The estimated disturbance models for $n_{\text{per}} = \{100, 1000, 10000\}$ are depicted in Figure 5 (top). For increasing measurement times, the estimate converges rapidly to the true value.

In Figure 5 (bottom), the results of the FDMVOP are depicted for different values of $n_{\text{per}}$, together with the true systematic model error. For $n_{\text{per}} = 100$, it is observed that the minimum-norm validating $\Delta_u$ is significantly smaller than the true systematic model error. In fact, for certain frequencies $\omega_i \in \Omega^{\text{val}}$, the minimum-norm validating $\Delta_u$ is zero. Indeed, in this case the nominal model residual can be attributed entirely to disturbances.

For increasing $n_{\text{per}}$, it is observed that the FDMVOP results in an increased norm bound that converges from below to the true value. This is indeed the desired averaging of disturbances in a moderately optimistic model validation framework, leading to an asymptotically correctly estimated model uncertainty.

## 7. CONCLUSIONS

In this paper, a two-stage approach that remedies the ill-posedness in the deterministic model validation problem formulation is presented. A thorough analysis reveals that the notion of disturbance should be properly defined to separate model uncertainty and disturbances in a systematic manner. This is achieved by first estimating realistic, nonparametric, deterministic, and frequency dependent disturbance models. Subsequently, averaging properties of deterministic disturbances are considered, enabling a separation between disturbances and model uncertainty. The key idea is to select an appropriate periodic input signal. In fact, experiment design seems to be an unexplored area in model validation for robust control.

Experimental results confirm that the proposed approach enables an accurate estimation of model uncertainty for sufficiently long measurement times. Concluding, a computationally tractable model validation has been presented that separates disturbances and model uncertainty, resulting in asymptotically correctly estimated uncertainty and disturbance models. In this perspective, the proposed framework provides a further connection in the estimation of model sets for robust control.

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