



System identification for achieving robust performance[☆]

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ABSTRACT

The performance of robust controllers hinges on the underlying model set. The aim of the present paper is to develop a system identification procedure that enables the design of a controller that achieves optimal robust performance. Hereto, the complex interrelation between system identification and robust control is thoroughly analyzed and novel connections are established between (i) control-relevant and coprime factor identification and (ii) model uncertainty size and the control criterion. The key technical results include new robust-control-relevant and (W_u, W_y) -normalized coprime factorizations. The results enable the identification of multivariable model sets that achieve high robust performance in a subsequent robust control synthesis. Superiority of the proposed results compared to existing approaches is shown by means of an example.

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1. Introduction

The purpose of any model should be accounted for when evaluating model quality. In Schrama (1992), Gevers (1993) and Albertos and Sala (2002), control-relevant system identification techniques are presented that match the identification criterion to the control criterion, aiming to deliver models that are especially suitable for subsequent control design. These approaches are typically iterative and alternate between closed-loop system identification and control design. These approaches have met with different outcomes due to the lack of a convergence proof. Indeed, model errors are not explicitly accounted for during controller synthesis, hence there are no stability and performance guarantees for a newly designed controller.

Robust control (Zhou, Doyle, & Glover, 1996) explicitly deals with model errors by designing a controller that stabilizes and achieves a certain guaranteed performance for a model set. Identification for robust control involves the problem of identifying the required model set. The robust control goal introduces two requirements on the identified model set:

(R1) the model set should encompass the true system behavior to guarantee robust stability, and

(R2) the model set should enable the synthesis of a controller that achieves high robust performance.

In general, these two requirements can be conflicting, since a straightforward approach to satisfy Requirement R1 is to select a large model set. However, if the model set contains candidate models that are difficult to control, then the guaranteed performance of the robust controller is poor, violating Requirement R2. The aim of this paper is to quantify and optimize the relation between these two requirements. In contrast, several approaches have already been developed that mainly address the robust stability Requirement R1 in system identification for robust control. These approaches are investigated next.

On the one hand, system identification techniques based on stochastic assumptions have been extended in, e.g., Hakvoort and Van den Hof (1997) and Ljung (1999), to deliver model sets for subsequent robust control. Although these methods have been further refined in Gevers, Bombois, Codrons, Scorletti, and Anderson (2003) and Gevers (2005) towards a certain notion of control-relevance, the adopted approach focusses on robust stability considerations and consequently does not guarantee that it is possible or straightforward to determine a high performance robust controller. Furthermore, although stochastic embedding techniques (Goodwin, Braslavsky, & Seron, 2002) have extended the traditional stochastic identification methodologies by considering the model uncertainty as a realization of a stochastic process, at present robust control techniques that can deal with such uncertainty descriptions are not available.

On the other hand, system identification techniques have been developed that consider deterministic assumptions on the involved signals instead of the traditional stochastic ones. Deterministic time domain approaches are reported, e.g., in Milanese and

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Vicino (1991), and generally lead to polytopic uncertain model descriptions. Due to a high computational demand, the application of these polytopic model structures is limited to small data sets. Robust identification techniques, see Helmicki, Jacobson, and Nett (1991) and Chen and Gu (2000), and references therein, have been developed that directly deliver a nominal model and a bound on the model uncertainty that are compatible with \mathcal{H}_∞ -optimization based robust control. However, these techniques lead to overly large model sets and consequently conservative control designs, as is also discussed in Vinnicombe (2001, Section 9.5.2). A further overview of approaches that address aspects of system identification for robust control can be found in, e.g., Ninness and Goodwin (1995), Hjalmarsson (2005) and Reinelt, Garulli, and Ljung (2002).

Besides the development of alternative system identification procedures, model uncertainty structures for robust control have been further refined and employed in system identification to extend control-relevant models with a description of model uncertainty. Specifically, Requirement R1 precludes the use of (inverse) additive and multiplicative uncertainty since these control-relevant models are generally identified in *closed-loop*, see Schrama (1992), Gevers (1993) and Albertos and Sala (2002), hence these models are generally uncertain with respect to both the *open-loop* unstable poles and zeros. To satisfy Requirement R1, perturbations on coprime factors of the system model can be considered to represent uncertainty with respect to both the *open-loop* unstable poles and zeros. Commonly, *normalized* coprime factorizations (McFarlane & Glover, 1990) are considered, since these have a close connection to robustness properties in the graph and (ν -) gap metric; see Georgiou and Smith (1990), de Callafon, Van den Hof and Bongers (1996) and Vinnicombe (2001). Specifically, related results that connect control performance and model validation based on these normalized coprime factorizations are reported in Steele and Vinnicombe (2001) and Date and Cantoni (2005). The dual-Youla–Kučera structure (Anderson, 1998; de Callafon and Van den Hof, 1997; Douma & Van den Hof, 2005) further refines these coprime factorization-based model uncertainty structures by excluding candidate models that are not stabilized by the controller that is used during the identification experiment. In this respect, the dual-Youla–Kučera model uncertainty structure constitutes a necessary step to satisfy Requirement R2 by excluding falsified models based on the actual experimental conditions.

Although the exclusion of models that are not stabilized during the identification experiment is a necessary step to identify models that enable high performance control design, it is not sufficient to satisfy Requirement R2. Indeed, since the dual-Youla–Kučera parameterization represents all candidate systems that are stabilized by a certain controller, in general it may contain candidate models for which high performance cannot be achieved. This is further supported by the results in de Callafon and Van den Hof (2001), where the desire of achieving a certain level of robust performance using the dual-Youla–Kučera model uncertainty structure necessitates the use of highly structured model uncertainty blocks that essentially reduce the uncertainty to a large set of scalar blocks with a frequency-dependent scaling. However, such an approach leads to unnecessary conservatism and computational burden in certain uncertainty quantification procedures, see Toker and Chen (1998), and subsequent robust control synthesis (Doyle, 1982).

In this paper, a coprime-factorization solution to the system identification for the robust control problem is presented by further generalizing and extending the existing theory of coprime factorizations. The proposed solution that satisfies both Requirement R1 and Requirement R2 constitutes the main contribution of the paper and enables the design of a controller that achieves the limit of robust performance. Two main steps are taken to solve the system identification for the robust control problem. In the first step,

a new connection between control-relevant identification and the identification of coprime factorizations is established through a new technical result called a robust-control-relevant coprime factorization. Compared to the identification of *normalized* coprime factorizations, see Gu (1999), Date and Vinnicombe (2004), Van den Hof, Schrama, de Callafon and Bosgra (1995) and Zhou and Xing (2004), that are useful in certain robust control methodologies, including (McFarlane & Glover, 1990), the proposed robust-control-relevant coprime factorization has distinct advantages from a system identification perspective. Interestingly, the additional freedom associated with non-normalized coprime factorizations is also employed in the general distance measure framework in Lanzon and Papageorgiou (2009) and Lanzon, Engelken, Patra, and Papageorgiou (2012), however, in contrast to the present paper, the connection to system identification is not considered. Specifically, in this paper the following aspects are developed that extend existing results in this area.

- (C1) Existing coprime factor identification techniques are mainly aimed at identifying normalized coprime factorizations. Here, the identification of coprime factorizations is directly connected to control-relevant identification, leading to a nominal model that is especially suitable for control.
 - (C2) The control-relevant system identification problem, which is generally a four-block problem, is recast as an equivalent two block problem, enabling a significant reduction of the computational burden.
 - (C3) The identification of normalized coprime factorizations typically requires an iterative procedure; see, e.g., Van den Hof et al. (1995). The approach taken in this paper enables the direct identification of a certain coprime factorization from data, enabling a significant reduction of the computational burden.
 - (C4) The robust-control-relevant coprime factorization can be efficiently identified through a frequency response-based algorithm.
 - (C5) The identified robust-control-relevant coprime factorization is an essential ingredient for constructing a novel model uncertainty structure, as is discussed in the second step, and state-space results are presented for the actual computational procedure.
- In the second step, a new model uncertainty structure is established that extends earlier model uncertainty structures for robust control and enables the identification of multivariable model sets for achieving robust performance. The following specific aspects are developed in this paper that extend the existing results.
- (C6) State-space results are presented to compute a new (W_u, W_y)-normalized coprime factorization of the controller, which is used during the identification experiment, that in conjunction with the identified robust-control-relevant coprime factorization of the nominal model enables the direct construction of the new model uncertainty structure.
 - (C7) The proposed model uncertainty structure transparently connects the size of model uncertainty and the control criterion, which is a fundamental step forward in the nonconservative identification of model sets.

The use of existing uncertainty structures for multivariable systems necessitates the use of structured and frequency dependent uncertainty models, as in Van de Wal, van Baars, Sperling, and Bosgra (2002) and de Callafon and Van den Hof (2001). The result C7 essentially provides an appropriate scaling for the uncertainty channels, thereby enabling the use of unstructured uncertainty models that have important advantages from a computational perspective.

To facilitate the exposition, an explicit distinction is made between the identification of a nominal model and the quantification

of model uncertainty. Such a distinction is also made in, e.g., de Callafon and Van den Hof (1997), and is commonly done in applications, including (de Callafon and Van den Hof, 2001; Van de Wal et al., 2002). This distinction in the present paper is made for two reasons. First, this explicit separation sheds light on prior results in control-relevant system identification. Second, it results in a general framework that can straightforwardly be used in conjunction with a wide variety of model uncertainty quantification techniques. This separation is nonrestrictive and several approaches can be pursued to intertwine these steps, e.g., as in Hindi, Seong, and Boyd (2002). Systematic modeling errors are emphasized during the nominal model identification. The underlying assumption for the pursued frequency response-based approach is that the available measured data is sufficiently large, in which case the considered identification criterion is independent of the noise that contaminates the data.

This paper is organized as follows. In Section 2, the robust-control-relevant system identification problem is defined. As a first step, a novel connection between control-relevant system identification and the identification of coprime factorizations is presented in Section 3. The key technical result is a new robust-control-relevant coprime factorization and, in addition to Contribution C1–C5, several properties of this coprime factorization are established. As a second step, a new model uncertainty structure is presented in Section 4 that enables the identification of a model set in view of achieving robust performance, constituting Contribution C6–C7. In Section 5, the results of this paper are illustrated by means of an example that confirms improved and less conservative results compared to existing approaches based on commonly used normalized coprime factorizations. Finally, conclusions are presented in Section 6. Proofs and auxiliary are provided in the Appendix.

Notation. Throughout, a unified treatment for continuous time and discrete time systems is presented. Hereto, the indeterminate ξ is used that can be substituted for either s , z , or δ for the continuous time, discrete time, and δ -operator case, respectively. The notation \mathbb{S} and \mathbb{U} denote the strictly stable and unstable regions in the complex plane, respectively. Also, \mathbb{B} denotes the corresponding stability boundary, i.e., $\mathbb{B} = j\omega$, $\omega \in \mathbb{R} \cup \infty$ and $\mathbb{B} = e^{j\omega}$ for the continuous and discrete time, respectively. For a signal $x(t)$, $\dot{x}(t)$ denotes either $\frac{dx(t)}{dt}$ or $x(t+1)$ in the continuous and discrete time case, respectively. The argument ξ is often omitted when clear from the context. In certain cases, the results are specialized to either the continuous time or the discrete time case. These results can be related to each other by employing the Möbius transformation; see also Oomen, Van de Wal, and Bosgra (2007).

Throughout, transfer function matrices are assumed to be proper. The pair $\{N, D\}$ denotes a Right Coprime Factorization (RCF) of P if D is invertible, $N, D \in \mathcal{RH}_\infty$, $P = ND^{-1}$, and $\exists X_r, Y_r \in \mathcal{RH}_\infty$ such that the Bézout identity $X_r D + Y_r N = I$ holds. The pair $\{\bar{N}, \bar{D}\}$ is said to be a Normalized RCF (NRCF) if it is an RCF and in addition $\bar{D}^* \bar{D} + \bar{N}^* \bar{N} = I$. Many results in this paper are stated in terms of RCFs, dual definitions hold and dual results can be derived for Left Coprime Factorizations (LCFs). Throughout, N and D are used exclusively to denote rational coprime factorizations over \mathcal{RH}_∞ . Occasionally, P is represented by the polynomial Right Matrix Fraction Description (RMFD) $P = BA^{-1}$, $B, A \in \mathbb{R}[\xi]$. Here, $\mathbb{R}[\xi]^{p \times q}$ denotes a polynomial matrix of dimension $p \times q$ with real coefficients. The upper linear fractional transformation (LFT) is given by $\mathcal{F}_u(M, \Delta_u) = M_{22} + M_{21} \Delta_u (I - M_{11} \Delta_u)^{-1} M_{12}$.

2. Problem formulation

2.1. Control goal definition

Throughout, the feedback interconnection in Fig. 1 is considered, where P denotes the system and C denotes the controller. It

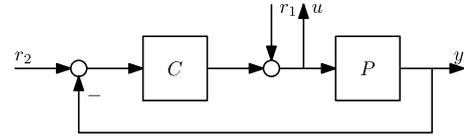


Fig. 1. Feedback interconnection.

is emphasized that for all results in the present paper, both P and C can be multivariable with appropriate dimensions. The control goal is to minimize a control criterion, denoted by $\mathcal{J}(P, C)$, for the true system P_o , i.e., determine

$$C^{\text{opt}} = \arg \min_C \mathcal{J}(P_o, C). \quad (1)$$

A norm-based criterion is considered, i.e.,

$$\mathcal{J}(P, C) = \|WT(P, C)V\|_\infty, \quad (2)$$

where

$$T(P, C) : \begin{bmatrix} r_2 \\ r_1 \end{bmatrix} \mapsto \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} P \\ I \end{bmatrix} (I + CP)^{-1} [C \ I] \quad (3)$$

is the closed-loop transfer function matrix corresponding to Fig. 1 and $W = \text{diag}(W_y, W_u)$, $V = \text{diag}(V_2, V_1)$ are weighting filters that satisfy $W, W^{-1}, V, V^{-1} \in \mathcal{RH}_\infty$. The requirement $W, W^{-1}, V, V^{-1} \in \mathcal{RH}_\infty$ is imposed to simplify the notation and is nonrestrictive, since it can be relaxed by absorbing W and V in the feedback control loop. The criterion (2)–(3) encompasses many performance goals that are based on the \mathcal{H}_∞ -norm, since essentially all closed-loop transfer function matrices are weighted in (2). The motivation for using the \mathcal{H}_∞ -norm in (2) is threefold: (i) it satisfies the multiplicativity property that facilitates the incorporation of model uncertainty; (ii) in conjunction with the four-block criterion in (2) and (3), it guarantees well-posedness and internal stability; and (iii) it facilitates the selection of W and V based on loop-shaping design techniques; see, e.g., McFarlane and Glover (1990).

2.2. Robust control

Since the true system P_o is unknown, a model-based control design approach is pursued to perform the actual optimization in (2). In general, uncertainty with respect to the dynamical behavior is always present when modeling physical systems. Thereto, a model set \mathcal{P} of candidate models is considered that presumably encompasses the true system behavior, i.e.,

$$P_o \in \mathcal{P}. \quad (4)$$

Associated with \mathcal{P} is the worst-case criterion

$$\mathcal{J}_{\text{wc}}(\mathcal{P}, C) = \sup_{P \in \mathcal{P}} \mathcal{J}(P, C)$$

that in conjunction with (4) immediately reveals that

$$\mathcal{J}(P_o, C) \leq \mathcal{J}_{\text{wc}}(\mathcal{P}, C). \quad (5)$$

Hence, if the controller satisfies a certain worst-case performance bound, then the controller is guaranteed to yield at least the same level of performance when it is implemented on the true system P_o . This motivates the robust control design

$$C^{\text{RP}} = \arg \min_C \mathcal{J}_{\text{wc}}(\mathcal{P}, C). \quad (6)$$

2.3. Robust-control-relevant system identification

In system identification for robust control, the model set \mathcal{P} is determined from measured data. As discussed in Section 1, the model set should enable a high performance control design. In terms of the criterion (2), this implies that the right hand side of (5) should be small when evaluated for C^{RP} in (6). In general,

the achievable worst-case performance in (6) is a complicated function of \mathcal{P} , both in terms of its shape and size. To ensure that the identified model set indeed delivers a high performance robust controller in (6), the identification criterion is connected to the control criterion. Conceptually, the dual problem to (6) is considered during system identification by using $\min_{\mathcal{P}} \mathcal{J}_{WC}(\mathcal{P}, C)$, as identification criterion. A crucial aspect here is that the latter criterion is a function of the variable C . In the ideal case, C should be selected as in (6), i.e., $C^{RP}(\mathcal{P})$. However, the mapping from \mathcal{P} to C^{RP} in general is complicated and no analytic results exist, thereby excluding $C^{RP}(\mathcal{P})$ as a candidate controller for use in the identification criterion. The alternative C^{opt} , see (1), is unknown and cannot be used for system identification purposes. To formulate a solvable identification criterion, a stabilizing non-optimal controller C^{exp} is employed and implemented during the identification experiment.

The choice of the controller C^{exp} that is used during the identification experiment crucially affects the robust-control-relevant identification criterion; see Definition 1. In fact, this is a well-known aspect in control-relevant system identification. Specifically, in relation to the proposed procedure in this paper, several aspects are important. (i) If the distance between C^{exp} and the designed $C^{RP}(\mathcal{P})$ or C^{opt} is too large, then $\min_{\mathcal{P}} \mathcal{J}_{WC}(\mathcal{P}, C)$ and (6) may be solved iteratively. In contrast to iterative identification and control design schemes that are solely based on a nominal model, see Gevers (1993); Schrama (1992), and Albertos and Sala (2002), an iterative system identification and control design approach based on model sets, as is considered in this paper, leads to a monotonous performance improvement; see de Callafon and Van den Hof (1997), and Garatti, Campi, and Bittanti (2010). As a result, the achievable performance after several iterations is invariant under the choice of the initial controller C^{exp} . (ii) The approximation that is introduced by replacing $C^{RP}(\mathcal{P})$ with C^{exp} can be mitigated using dynamic weighting filters. (iii) If P_o is stable, then $C^{exp} = 0$ is a stabilizing controller, hence the procedure also applies to open-loop stable systems. (iv) In the case of systems that are not strongly stabilizable, the controller C^{exp} has to be unstable to stabilize P_o in closed-loop. Such a choice is also admissible for the proposed procedure in this paper.

Prior to formally defining robust-control-relevant system identification, the model set \mathcal{P} has to be defined more precisely. To enable a direct connection with \mathcal{H}_∞ -optimization based robust control, a \mathcal{H}_∞ -norm-bounded perturbation is connected to the nominal model through an LFT-based interconnection structure, i.e.,

$$\mathcal{P} = \{P | P = \mathcal{F}_u(\hat{H}, \Delta_u), \Delta_u \in \mathbf{A}_u\}, \quad (7)$$

where $\mathbf{A}_u \subseteq \mathcal{H}_\infty$ is a norm-bounded set that is possibly subject to additional structural constraints. In addition, \hat{H} contains \hat{P} , the uncertainty structure, and possibly weighting filters to shape the uncertainty over frequency.

The involved robust-control-relevant model set system identification problem is formalized next.

Definition 1. The robust-control-relevant model set identification with respect to an existing controller C^{exp} is the result of the optimization problem

$$\mathcal{P}^{RCR} = \arg \min_{\mathcal{P}} \mathcal{J}_{WC}(\mathcal{P}, C^{exp}), \quad (8)$$

subject to the constraint in (4), where \mathcal{P} is defined in (7).

To perform the actual optimization in Definition 1, the model set \mathcal{P} has to be the result of data. Clearly, it is desired that $\hat{P} = P_o$ and $\mathbf{A}_u = \emptyset$. However, the nominal model \hat{P} is necessarily an approximation for any physical system. Hence, the use of a model set \mathcal{P} is crucial.

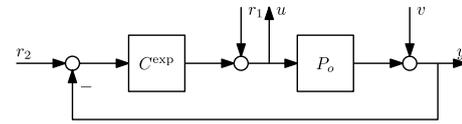


Fig. 2. System identification setup.

2.4. Identification setup and approach

The identification setup in Fig. 2 is considered, where r_1, r_2 are manipulated signals and u, y are measured signals. In addition, C^{exp} is a known, stabilizing controller. The superscript *exp* is occasionally omitted to simplify the notation. In addition, v is an additive signal that represents measurement disturbances and is defined precisely in Section 3.4.

As is motivated in Section 1, the model set \mathcal{P} is identified by subsequently (i) identifying a nominal model, followed by (ii) quantifying model uncertainty. As a result, both the nominal model (step i) and model uncertainty (step ii) should be aimed at identifying robust-control-relevant model sets in the sense of Definition 1. In the next section, a new connection between control-relevant identification of a nominal model and coprime factor identification is presented, leading to a new coprime factor realization of the nominal model. Then, in Section 4, it is shown that when the identified robust-control-relevant coprime factorization is used to construct a new model uncertainty structure, then the nominal model identification and the model uncertainty jointly aim at identifying a robust-control-relevant model set in the sense of Definition 1.

3. Robust-control-relevant coprime factor identification

3.1. Control-relevant identification of a nominal model

The quality of approximate models depends on the goal. When restricting attention to a nominal model \hat{P} , then the performance of any candidate system P can be bounded through application of the triangle inequality (Schrama, 1992), i.e.,

$$\mathcal{J}(P, C) \leq \mathcal{J}(\hat{P}, C) + \|W(T(P, C) - T(\hat{P}, C))V\|_\infty, \quad (9)$$

where $\mathcal{J}(P, C)$ is defined in (2). Evaluating (9) for P_o reveals that if the metric

$$\|W(T(P_o, C) - T(\hat{P}, C))V\|_\infty \quad (10)$$

is small, then $\mathcal{J}(\hat{P}, C) \approx \mathcal{J}(P_o, C)$ and the distance between P_o and \hat{P} is small from a closed-loop perspective for an arbitrary controller C . When evaluating (10) for the controller C^{exp} , see Section 2.4, the following control-relevant identification criterion is obtained.

Definition 2. The control-relevant identification of a nominal model \hat{P} with respect to an existing controller C^{exp} is the result of the optimization problem

$$\min_{\hat{P}} \|W(T(P_o, C^{exp}) - T(\hat{P}, C^{exp}))V\|_\infty. \quad (11)$$

Criterion (2) guarantees internal stability of the feedback loops corresponding to $T(P_o, C^{exp})$ and $T(\hat{P}, C^{exp})$, respectively. Indeed, since $T(P_o, C^{exp}) \in \mathcal{H}_\infty$ by assumption, potential unstable modes in $T(\hat{P}, C^{exp})$ cannot cancel out in the difference in (11).

The criterion (11) is at the heart of common iterative identification and control approaches, including (Albertos & Sala, 2002; Gevers, 1993; Schrama, 1992). These approaches are solely based on nominal models and alternate between minimizing the two terms on the right hand side of (9). A severe criticism of these

approaches is that these are not guaranteed to converge. Indeed, due to the iterative procedure, the controller C is updated, whereas the bound in (9) holds for a single C . In contrast, in this paper it is shown that the nominal model resulting from the criterion (11) can effectively be used to construct a robust-control-relevant model set, which can possibly be used in an iterative scheme with guaranteed convergence.

3.2. Towards coprime factor identification

To anticipate on robust control design, the constraint (4) needs to be satisfied when extending the control-relevant nominal model \hat{P} , see (11), with model uncertainty $\Delta_u \in \mathbf{A}_u$. This imposes severe constraints on the model uncertainty structure, see \hat{H} in (7), since analysis of (11) reveals that boundedness of the norm implies that the interconnection of (i) P_o and C^{exp} and (ii) \hat{P} and C^{exp} are stable. However, no statements can be made with respect to open-loop stability or the minimum phase character of both P_o and \hat{P} . As a result, standard (inverse) additive or multiplicative model uncertainty structures cannot be used to guarantee (4), as is exemplified next.

Example 1. Let P_o and \hat{P} be simultaneously stabilized by C^{exp} and suppose that $P_o \notin \mathcal{RH}_\infty$ and $\hat{P} \in \mathcal{RH}_\infty$. Then, an additive uncertainty model $\mathcal{P}_A = \{P|P = \hat{P} + \Delta_u, \|\Delta_u\|_\infty \leq \gamma\}$, leads to $P_o \notin \mathcal{P}_A \forall \gamma \in \mathbb{R}_+$.

To guarantee that the constraint (4) holds for a certain \mathcal{H}_∞ -norm-bounded perturbation in the case of both uncertain open-loop unstable poles and unstable zeros, a coprime factorization-based model uncertainty structure is considered. In this case, let

$$\Delta_u = [\Delta_N^T \Delta_D^T]^T, \quad (12)$$

let $\{\hat{N}, \hat{D}\}$ be an RCF of \hat{P} , and consider the model set

$$\{P|P = (\hat{N} + \Delta_N)(\hat{D} + \Delta_D)^{-1}, \|\Delta_u\|_\infty \leq \gamma\}. \quad (13)$$

In this case, it is straightforward to verify that (4) holds for a certain $\gamma \in \mathbb{R}_+$.

The preceding discussion reveals that the internal structure of \hat{P} based on coprime factorizations is essential to guarantee that the constraint (4) holds for a certain norm-bound. Although this fact has been observed in the system identification for control literature, as is evidenced by the results in Gu (1999), Date and Vinnicombe (2004), Van den Hof et al. (1995) and Zhou and Xing (2004), this is not taken into account in the control-relevant criterion (11) that only involves the input–output behavior of the model. To show this, let $\{\hat{N}, \hat{D}\}$ be an RCF of \hat{P} . Then, the set of all RCFs of \hat{P} is generated by

$$\{\{\hat{N}Q, \hat{D}Q\}|Q, Q^{-1} \in \mathcal{RH}_\infty\}, \quad (14)$$

hence there is freedom in the choice of coprime factorization of \hat{P} . In fact, (14) reveals that infinitely many RCFs of \hat{P} exist. Next, observe that the control-relevant identification criterion (11) for a coprime factorization in the set (14) is given by

$$\|W(T(P_o, C^{\text{exp}}) - T(\hat{N}Q(\hat{D}Q)^{-1}, C^{\text{exp}}))V\|_\infty. \quad (15)$$

The result (15) directly reveals that the criterion in (11) is invariant under a change of coprime factorization. In the next section, a novel connection between control-relevant identification and coprime factor identification is presented. The key technical result is a new robust-control-relevant coprime factorization that leads to a new model uncertainty structure in Section 4.

3.3. Robust-control-relevant coprime factor identification

In Section 3.1, a motivation for identification of control-relevant nominal models is given, whereas the use of coprime factorization-based model uncertainty structures is advocated in Section 3.2.

However, (15) revealed that criterion (11) is invariant under a change of coprime factorization. Hence, from a control-relevant perspective of a nominal model, see Definition 2, there is no preference for a specific realization. In, e.g., Van den Hof et al. (1995), an approach is presented that exploits the nonuniqueness of coprime factorizations to identify normalized coprime factors. Such normalized coprime factors can directly be used in the robust control procedure in McFarlane and Glover (1990). In addition, these normalized coprime factorizations have a McMillan degree that is upper bounded by the McMillan degree of the underlying model \hat{P} . In contrast to the identification of normalized coprime factors, the aim of this paper is to exploit the freedom in coprime factorizations to establish a further connection with system identification for robust control. The following definition is essential in the subsequent derivations.

Definition 3. The pair $\{\tilde{N}, \tilde{D}\}$ is called an LCF of $[CV_2 V_1]$ with co-inner numerator if it is an LCF of $[CV_2 V_1]$ and, in addition, $\tilde{N}\tilde{N}^* = I$.

An LCF with co-inner numerator of $[CV_2 V_1]$ exists if and only if $[CV_2 V_1]$ has no zeros on \mathbb{B} , see, e.g., Chu (1988), which is satisfied under the assumptions that are posed in this paper; see Remark 2. The following theorem, in conjunction with Theorem 2, is the main result of this section. This result provides a novel connection between control-relevant identification of a nominal model, see Definition 2, and coprime factor identification through the introduction of a new coprime factorization.

Theorem 1. Let $\{\tilde{N}_e, \tilde{D}_e\}$ be an LCF of $[CV_2 V_1]$ with co-inner numerator, where $\tilde{N}_e = [\tilde{N}_{e,2} \tilde{N}_{e,1}]$ is partitioned accordingly. Then, (11) is equivalent to

$$\min_{\hat{N}, \hat{D}} \|W \left(\begin{bmatrix} N_o \\ D_o \end{bmatrix} - \begin{bmatrix} \hat{N} \\ \hat{D} \end{bmatrix} \right)\|_\infty, \quad (16)$$

where

$$\begin{bmatrix} N_o \\ D_o \end{bmatrix} = \begin{bmatrix} P_o \\ I \end{bmatrix} (\tilde{D}_e + \tilde{N}_{e,2} V_2^{-1} P_o)^{-1} \quad (17)$$

$$\begin{bmatrix} \hat{N} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} \hat{P} \\ I \end{bmatrix} (\tilde{D}_e + \tilde{N}_{e,2} V_2^{-1} \hat{P})^{-1}. \quad (18)$$

Proof. Let $\{\tilde{N}_e, \tilde{D}_e\}$ be any LCF of $[CV_2 V_1]$. Then,

$$\begin{aligned} WT(P, C)V &= W \begin{bmatrix} P \\ I \end{bmatrix} (I + CP)^{-1} \tilde{D}_e^{-1} \tilde{N}_e \\ &= W \begin{bmatrix} P \\ I \end{bmatrix} (\tilde{D}_e + \tilde{D}_e CV_2 V_2^{-1} P)^{-1} \tilde{N}_e. \end{aligned} \quad (19)$$

Substitution of (19) into (11), using $CV_2 = \tilde{D}_e^{-1} \tilde{N}_{e,2}$, and rearranging yields: $\min_{\hat{N}, \hat{D}} \|W \left(\begin{bmatrix} N_o \\ D_o \end{bmatrix} - \begin{bmatrix} \hat{N} \\ \hat{D} \end{bmatrix} \right)\tilde{N}_e\|_\infty$. Next, $\|XY\|_\infty = \|X\|_\infty$ when Y is co-inner. By considering an LCF with co-inner numerator $\{\tilde{N}_e, \tilde{D}_e\}$ of $[CV_2 V_1]$ immediately yields the desired result in (16). \square

Theorem 1 reveals that the control-relevant identification criterion (11) can be reduced from a four-block problem to a two-block problem over stable factors $\{N_o, D_o\}$ and $\{\hat{N}, \hat{D}\}$. The following theorem reveals that $\{N_o, D_o\}$ and $\{\hat{N}, \hat{D}\}$ are in fact coprime factorizations of P_o and \hat{P} , respectively, if a certain stability condition is satisfied.

Theorem 2. Let $T(P, C) \in \mathcal{RH}_\infty$ and let $\{\tilde{N}_e, \tilde{D}_e\}$ be an LCF of $[CV_2 V_1]$, with $\tilde{N}_e = [\tilde{N}_{e,2} \tilde{N}_{e,1}]$. Then,

$$\{P(\tilde{D}_e + \tilde{N}_{e,2}V_2^{-1}P)^{-1}, (\tilde{D}_e + \tilde{N}_{e,2}V_2^{-1}P)^{-1}\} \quad (20)$$

is an RCF of P .

Summarizing, the results of Theorems 1 and 2 constitute the first step in solving the system identification for the robust control problem and provide a new connection between control-relevant system identification and the identification of coprime factorizations. The results of Theorems 1 and 2 correspond to Contribution C1, C2, and C3 in Section 1. The specific coprime factorization in (17) and (18) is called a robust-control-relevant coprime factorization for reasons that will become clear in Section 4.

In the remainder of this section, properties of the robust-control-relevant coprime factorizations (17) and (18) are investigated. First, uniqueness is investigated.

Theorem 3. *The coprime factorizations $\{N_o, D_o\}$ and $\{\hat{N}, \hat{D}\}$ in Theorem 1 are unique modulo a right multiplication by a constant unitary matrix Q .*

Theorem 3 reveals that the freedom in robust-control-relevant coprime factorizations is limited to the class of constant unitary matrices. This has important consequences when comparing the new robust-control-relevant coprime factorization to the well-known normalized case; see Vidyasagar (1985, Section 7.3) for a definition. These normalized coprime factorizations have a dominant role in certain robust control design methodologies; see McFarlane and Glover (1990) and Vinnicombe (2001). In fact, if the robust-control-relevant coprime factorizations in Theorem 1 are not normalized, then these cannot be normalized by exploiting the freedom in coprime factorizations in Theorem 3. Indeed, let \hat{N}, \hat{D} be as defined in Theorem 1, then $\hat{N}^*\hat{N} + \hat{D}^*\hat{D} = Z$, and assume that $\{\hat{N}, \hat{D}\}$ are not normalized, i.e., $Z \neq I$. Since Q in Theorem 3 is a constant unitary matrix, it is clearly not a stable and minimum phase spectral factor of Z . As a result, the freedom in Q cannot be exploited to normalize $\{\hat{N}, \hat{D}\}$.

Remark 1. A similar dimension reduction as in Theorem 1 can be achieved for different coprime factorizations, including NR-CFs. Hereto, observe that if the pair $\{\hat{N}, \hat{D}\}$ is a robust-control-relevant coprime factorization of \hat{P} , then $\exists \bar{Q}, \bar{Q}^{-1} \in \mathcal{RH}_\infty$ such that $\{\bar{N}, \bar{D}\} = \{\hat{N}, \hat{D}\}\bar{Q}$ is a NR-CF of \hat{P} , where \bar{Q} is unique modulo a constant unitary matrix. Then, (16) is equivalent to $\min_{\bar{N}, \bar{D}} \|W(\begin{bmatrix} N_o \\ D_o \end{bmatrix} \bar{Q} - \begin{bmatrix} \bar{N} \\ \bar{D} \end{bmatrix})\bar{Q}^{-1}\|_\infty$, where \bar{Q}^{-1} has the role of weighting filter. Although the complexity also reduces, it is emphasized that NR-CFs lack the advantages of robust-control-relevant coprime factors in Contribution C3 and C5.

3.4. Towards a frequency response-based algorithm

The control-relevant identification criterion (11) and consequently also (16) are not straightforward to solve due to the presence of the \mathcal{H}_∞ -norm. The pursued strategy to obtain a tractable identification problem is to exploit the frequency domain interpretation of the \mathcal{H}_∞ -norm, i.e., the maximum modulus theorem states that if $G(\xi) \in \mathcal{RH}_\infty$, then

$$\|G(\xi)\|_\infty = \sup_{\omega \in \mathbb{B}} \bar{\sigma}(G(\omega)). \quad (21)$$

The result (21) motivates the intermediate step of nonparametric frequency response function identification. In this case, the frequency response function of the true system is generally obtained on a discrete frequency grid $\Omega \subset \mathbb{B}$, which is an immediate consequence of the use of finite time data sets. In contrast, the \mathcal{H}_∞ -norm in (21) is defined on a continuous

frequency grid. Although identification methods, e.g., Chen and Gu (2000), are available that can deal with a discrete frequency grid by imposing certain assumptions regarding the true system, these methods typically lead to conservative results as is discussed in Vinnicombe (2001, Section 9.5.2).

In this section, an alternative procedure is pursued that is based on the following lemma.

Lemma 1. *The optimization problem*

$$\min_{\hat{N}, \hat{D}} \sup_{\omega_i \in \Omega} \bar{\sigma} \left(W \left(\begin{bmatrix} N_o \\ D_o \end{bmatrix} - \begin{bmatrix} \hat{N} \\ \hat{D} \end{bmatrix} \right) \right), \quad (22)$$

$$\text{subject to } T(\hat{P}, C) \in \mathcal{RH}_\infty. \quad (23)$$

minimizes a lower bound of the \mathcal{H}_∞ -norm in (16).

The proof of Lemma 1 follows directly from (16) and (21). It is emphasized that the discrete frequency grid Ω should be chosen sufficiently dense to guarantee that the lower bound (22) of (16) is tight.

To solve the optimization problem (22)–(23), the frequency response of the coprime factorization $\{N_o, D_o\}$ for $\omega_i \in \Omega$ is required. This frequency response function can be identified through the following procedure.

Procedure 1. Perform the following sequence of steps:

- (1) Identify $T(P_o, C^{\text{exp}})$ for $\omega_i \in \Omega$.
- (2) Append $T(P_o, C^{\text{exp}})$ with weighting filters V , as a result $T(P_o, C^{\text{exp}})V$ is obtained for $\omega_i \in \Omega$.
- (3) Right multiply $T(P_o, C^{\text{exp}})V$ by $[\tilde{N}_{e,2}(\omega_i) \tilde{N}_{e,1}(\omega_i)]^*$, yielding $[N_o^T(\omega_i) D_o^T(\omega_i)]^T$ for $\omega_i \in \Omega$.

To reduce the effects of finite time noisy observations, a suitable identification experiment should be designed to obtain a good estimate in Step 1. For many systems, a realistic assumption on v is given by $v = He$, where $H, H^{-1} \in \mathcal{RH}_\infty$ and e is a sequence of independent identically distributed random vectors. Consequently, the error in estimating $T(P_o, C^{\text{exp}})$ is circular complex normally distributed and the estimation error can be quantified solely through its variance; see Pintelon and Schoukens (2001, Theorem 14.25). On the one hand, the resulting variance error may be made negligibly small through an appropriate experiment design using periodic input signals r_2 and r_1 and collecting sufficiently large data sets. On the other hand, in the case where variance error is significant, then these can be taken into account, e.g., as in Pintelon and Schoukens (2001, Chapter 8).

A remaining aspect in the optimization problem (22)–(23) is the parameterization of $\{\hat{N}, \hat{D}\}$ that is investigated in detail in the next section.

3.5. Parametric coprime factor identification

The coprime factorization $\{\hat{N}, \hat{D}\}$ needs to be appropriately parameterized to perform the actual optimization in (22)–(23). During the optimization, two constraints should be satisfied: (i) $\hat{N}, \hat{D} \in \mathcal{RH}_\infty$ for the pair $\{\hat{N}, \hat{D}\}$ to be an RCF over \mathcal{RH}_∞ , and (ii) $T(\hat{P}, C) \in \mathcal{RH}_\infty$; see (23). The main contribution of this section is a novel parameterization that connects the two stability constraints. Hereto, let

$$\hat{P}(\theta) = B(\theta)A(\theta)^{-1}, \quad (24)$$

where $A(\theta) \in \mathbb{R}[\xi]^{n_u \times n_u}$, $B(\theta) \in \mathbb{R}[\xi]^{n_y \times n_u}$, and θ is a real-valued parameter vector. If $A(\theta)$ and $B(\theta)$ are coprime polynomial matrices, then the McMillan degree of \hat{P} equals $\text{deg det } A$. An example of a suitable parameterization is the full polynomial matrix fraction description that can directly be used to represent strictly proper

systems and enables the straightforward formulation of polynomial matrices that are linear in its coefficients.

Next, the coprime factorization $\{\hat{N}, \hat{D}\}$ is parameterized by substituting (24) into (18), leading to

$$\begin{bmatrix} \hat{N}(\theta) \\ \hat{D}(\theta) \end{bmatrix} = \begin{bmatrix} B(\theta) \\ A(\theta) \end{bmatrix} (\tilde{D}_e A(\theta) + \tilde{N}_{e,2} V_2^{-1} B(\theta))^{-1}. \quad (25)$$

The main advantage of explicitly introducing the knowledge of the weighting filters V_1, V_2 and the controller C^{exp} , which is defined in Section 2.4, in the parameterization in (25) is that it enables a connection between the two stability conditions, above, as is formalized next.

Theorem 4. Consider parameterization (25). Then, $T(\hat{P}(\theta), C) \in \mathcal{RH}_\infty$ if and only if $\hat{N}(\theta), \hat{D}(\theta) \in \mathcal{RH}_\infty$.

Theorem 4 facilitates the design of an identification algorithm, see, e.g., Oomen and Bosgra (2008) for a closed-loop generalization of Lawson, Sanathanan–Koerner, and Gauss–Newton iterations.

Summarizing, the nonparametric and parametric identification results in Sections 3.4 and 3.5, respectively, jointly constitute Contribution C4.

3.6. State-space results

In this section, essential results are presented that enable the analysis of the McMillan degree of the robust-control-relevant coprime factorization and the construction of a model uncertainty structure in Section 4. These state-space results constitute Contribution C5. The following theorem is the main result of this section.

Theorem 5. Given minimal state-space realizations (A_c, B_c, C_c, D_c) of C , (A_2, B_2, C_2, D_2) of V_2 , (A_1, B_1, C_1, D_1) of V_1 , let L_e denote the output injection and M_e output transformation matrix corresponding to the LCF with co-inner numerator of $[CV_2 \ V_1]$, and let (A_p, B_p, C_p, D_p) be a state-space realization of \hat{P} . Then, a state-space realization of $[\hat{N}^T \ \hat{D}^T]^T$ is given by $(A_{ND}, B_{ND}, C_{ND}, D_{ND})$, where

$$A_{ND} = \begin{bmatrix} A_p + B_p(I - D_{z,2}D_p)^{-1}D_{z,2}C_p & B_p(I - D_{z,2}D_p)^{-1}C_z \\ B_{z,2}(I - D_pD_{z,2})^{-1}C_p & A_z + B_{z,2}(I - D_pD_{z,2})^{-1}D_pC_z \end{bmatrix} \quad (26)$$

$$B_{ND} = \begin{bmatrix} B_p(I - D_{z,2}D_p)^{-1}D_{z,1} \\ B_{z,1} + B_{z,2}(I - D_pD_{z,2})^{-1}D_pD_{z,1} \end{bmatrix}$$

$$C_{ND} = \begin{bmatrix} (I - D_pD_{z,2})^{-1}C_p & (I - D_pD_{z,2})^{-1}D_pC_z \\ (I - D_{z,2}D_p)^{-1}D_{z,2}C_p & (I - D_{z,2}D_p)^{-1}C_z \end{bmatrix}$$

$$D_{ND} = \begin{bmatrix} (I - D_pD_{z,2})^{-1}D_pD_{z,1} \\ (I - D_{z,2}D_p)^{-1}D_{z,1} \end{bmatrix}, \quad (27)$$

and

$$\begin{bmatrix} A_z & B_{z,1} & B_{z,2} \\ C_z & D_{z,1} & D_{z,2} \end{bmatrix} = \left[\begin{array}{ccc|cc} \begin{bmatrix} A_c & B_c C_2 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_1 \end{bmatrix} & -L_e M_e^{-1} & - \begin{bmatrix} B_c \\ 0 \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} C_c & D_c C_2 & C_1 \end{bmatrix} & M_e^{-1} & -D_c \end{array} \right],$$

and M_e and L_e are defined as in Lemma 4 in Appendix C.

Remark 2. The output injection matrix and output transformation matrix corresponding to the LCF $\{\tilde{N}_e, \tilde{D}_e\}$ of $[CV_2 \ V_1]$, see also Lemma 4 in Appendix C, can be computed by solving an algebraic Riccati equation and can be determined by considering the dual problem to the results in Chu (1988) and references therein. The assumption $V_1, V_1^{-1} \in \mathcal{RH}_\infty$ guarantees the existence of an LCF $\{\tilde{N}_e, \tilde{D}_e\}$ with $\tilde{N}_e \tilde{N}_e^* = I$.

Theorem 5 reveals that the McMillan degree of $[\hat{N}^T \ \hat{D}^T]^T$ is bounded by the sum of the McMillan degrees of \hat{P}, C, V_2 , and V_1 . Although coprime factorizations exist that have a McMillan degree that is bounded by the McMillan degree of \hat{P} , see Vinnicombe (2001), the incorporation of the dynamics in C, V_2 , and V_1 leads to a specific internal structure of the model that enables the construction of a new model uncertainty structure that has distinct advantages in view of Definition 1, as is shown in the next section. It is emphasized that due to the specific parameterization that is employed in (25), the additional dynamics that are introduced by C, V_2 , and V_1 cancel out exactly when computing $\hat{P} = \hat{N}\hat{D}^{-1}$. Consequently, the internal structure of the model does not affect the input–output behavior of \hat{P} .

4. Robust-control-relevant model uncertainty structure

In the previous section, the first step towards solving the system identification for the robust control problem was established by connecting control-relevant system identification and the identification of coprime factorizations through a new robust-control-relevant coprime factorization. In this section, the second and final step towards solving the system identification for the robust control problem is taken by constructing a new model uncertainty structure. A key technical result to enable this is first established in Section 4.1. Then, limitations of existing model uncertainty structures are presented in Section 4.2. The new model uncertainty structure that, in conjunction with the control-relevant nominal model, solves the system identification for the robust control problem is presented in Section 4.3.

4.1. (W_u, W_y) -normalized coprime factorizations

In this section, a new coprime factorization is introduced that, in conjunction with the coprime factorization in Section 3.3, is used to construct a model uncertainty structure that directly connects to the control criterion. This factorization is referred to as a (W_u, W_y) -normalized coprime factorization and state-space results are presented for the actual computation, thereby constituting Contribution C6 of this paper.

Definition 4. The pair $\{N_c, D_c\}$ is a (W_u, W_y) -normalized RCF of C if it is an RCF of C and, in addition,

$$\begin{bmatrix} W_u N_c \\ W_y D_c \end{bmatrix}^* \begin{bmatrix} W_u N_c \\ W_y D_c \end{bmatrix} = I. \quad (28)$$

The following result addresses uniqueness of (W_u, W_y) -normalized RCFs.

Theorem 6. A (W_u, W_y) -normalized RCF of C is unique modulo a right multiplication by a constant unitary matrix Q .

The following results enable the computation of such a (W_u, W_y) -normalized coprime factorization.

Theorem 7. Given the discrete time systems W_u, W_y , and C , where $W_u, W_u^{-1}, W_y, W_y^{-1} \in \mathcal{RH}_\infty$ and minimal state-space realizations (A_u, B_u, C_u, D_u) of W_u , $(A_{yi}, B_{yi}, C_{yi}, D_{yi})$ of W_y^{-1} , and (A_c, B_c, C_c, D_c) of C , then a (W_u, W_y) -normalized RCF $\{N_c, D_c\}$ of C is given by the state-space realization

$$\begin{bmatrix} D_c \\ N_c \end{bmatrix} = \left[\begin{array}{cc|c} \mathcal{A} + \mathcal{B}F & \mathcal{B}G \\ \hline \mathcal{C} + \mathcal{D}F & \mathcal{D}G \end{array} \right],$$

where

$$\begin{aligned} G &= Z^{-\frac{1}{2}}, \quad F = -Z^{-1}(\mathcal{B}^*X\mathcal{A} + \mathcal{D}^*c), \\ Z &= R + \mathcal{B}^*X\mathcal{B}. \end{aligned} \quad (29)$$

X is the unique, positive semi-definite solution to the algebraic Riccati equation

$$\tilde{\mathcal{A}}^*X\tilde{\mathcal{A}} - X - \tilde{\mathcal{A}}^*X\mathcal{B}(\mathcal{B}^*X\mathcal{B} + R)^{-1}\mathcal{B}^*X\tilde{\mathcal{A}} + Q = 0, \quad (30)$$

and

$$\begin{aligned} \tilde{\mathcal{A}} &= \mathcal{A} - \mathcal{B}R^{-1}\mathcal{D}^*c & Q &= c^*\tilde{R}^{-1}c \\ R &= I + \mathcal{D}^*\mathcal{D} > 0 & \tilde{R} &= I + \mathcal{D}\mathcal{D}^* > 0 \end{aligned}$$

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline c & \mathcal{D} \end{array} \right] = \left[\begin{array}{c|c} \left[\begin{array}{ccc} A_u & B_u C_c & B_u D_c C_{yi} \\ 0 & A_c & B_c C_{yi} \\ 0 & 0 & A_{yi} \end{array} \right] & \left[\begin{array}{c} B_u D_c D_{yi} \\ B_c D_{yi} \\ B_{yi} \end{array} \right] \\ \hline \left[\begin{array}{ccc} 0 & 0 & C_{yi} \\ 0 & C_c & D_c C_{yi} \end{array} \right] & \left[\begin{array}{c} D_{yi} \\ D_c D_{yi} \end{array} \right] \end{array} \right].$$

Corollary 1. Given the continuous time systems W_u, W_y , and C , where $W_u, W_u^{-1}, W_y, W_y^{-1} \in \mathcal{RH}_\infty$ and minimal state-space realizations (A_u, B_u, C_u, D_u) of W_u , $(A_{yi}, B_{yi}, C_{yi}, D_{yi})$ of W_y^{-1} , and (A_c, B_c, C_c, D_c) of C , then a (W_u, W_y) -normalized RCF $\{N_c, D_c\}$ of C is given by the state-space realization

$$\left[\begin{array}{c} D_c \\ N_c \end{array} \right] = \left[\begin{array}{c|c} \mathcal{A} + \mathcal{B}F & \mathcal{B}G \\ \hline c + \mathcal{D}F & \mathcal{D}G \end{array} \right],$$

where $G = R^{-\frac{1}{2}}$ and $F = -R^{-1}(\mathcal{B}^*X + \mathcal{D}^*c)$, and X is the unique, symmetric, positive semi-definite solution to the algebraic Riccati equation

$$\tilde{\mathcal{A}}^*X + X\tilde{\mathcal{A}} - X\mathcal{B}R^{-1}\mathcal{B}^*X - Q = 0,$$

where $\tilde{\mathcal{A}}, Q, R, \tilde{R}, \mathcal{A}, \mathcal{B}, c, \mathcal{D}$ are defined as in Theorem 7.

In the next section, it is shown that the specific (W_u, W_y) -normalized RCF of C in conjunction with the identified coprime factorization $\{\hat{N}, \hat{D}\}$ of \hat{P} enable the construction of a structure for model uncertainty that transparently connects to the control criterion.

4.2. Limitations of existing model uncertainty structures

In Section 1, two Requirement R1 en R2 are stated with respect to system identification for robust control procedures. These requirements are defined mathematically in (4) and (8), respectively. In Section 3.2, it is argued that the use of coprime factorizations guarantees that the constraint (4) and hence Requirement R1 is satisfied for a certain perturbation in \mathcal{H}_∞ .

To investigate the connection between the coprime factor perturbation model, see (13), and the worst-case performance criterion associated with the model set, see (8), observe that the worst-case performance for general model uncertainty structures that can be recast as an LFT, see (7), including (inverse) additive and multiplicative uncertainty, is given by

$$\mathcal{J}_{\text{WC}}(\mathcal{P}, C) = \sup_{\Delta_u \in \mathcal{A}_u} \|\hat{M}_{22} + \hat{M}_{21}\Delta_u(I - \hat{M}_{11}\Delta_u)^{-1}\hat{M}_{12}\|_\infty. \quad (31)$$

The following result is obtained for the coprime factor-based uncertainty structure (13).

Lemma 2. Consider the uncertainty structure (13) with associated perturbation set given by $\mathcal{A}_u = \{\Delta_u | \Delta_u \text{ satisfies (12), } \|\Delta_u\|_\infty \leq \gamma\}$. Then, $\mathcal{J}_{\text{WC}}(\mathcal{P}, C^{\text{exp}})$ is bounded iff $\|[-\hat{D}^{-1}C^{\text{exp}}\hat{D}^{-1}]\|_\infty < \frac{1}{\gamma}$.

Proof. Observe that the LFT (31) is well-posed iff $\|\hat{M}_{11}\|_\infty < \frac{1}{\gamma}$. Next, (13) leads to $\hat{M}_{11} = [-\hat{D}^{-1}C^{\text{exp}}\hat{D}^{-1}]$ directly yields the desired result. \square

The additional constraint in Lemma 2 may be conflicting with (4). Indeed, if (4) is to be satisfied, then this may lead to an unbounded \mathcal{J}_{WC} . The interpretation is that if a certain candidate model $P \in \mathcal{P}$ is not stabilized by C^{exp} , this leads to a model set that is clearly not robust-control-relevant in the sense of Definition 1.

To resolve this deficiency of coprime factorization-based model uncertainty structures, the set of candidate models is restricted to the class of models that is stabilized by C^{exp} , leading to a dual version of the Youla–Kučera-parameterization; see Anderson (1998) and Douma and Van den Hof (2005).

Theorem 8. Let \hat{P} be internally stabilized by C^{exp} and let $\{\hat{N}, \hat{D}\}$ and $\{N_c, D_c\}$ be an RCF of \hat{P} and C^{exp} , respectively. Then, the class of all systems P that are internally stabilized by C^{exp} is given by

$$\mathcal{P}^{\text{DY}} = \{P | P = (\hat{N} + D_c\Delta_u)(\hat{D} - N_c\Delta_u)^{-1}, \Delta_u \in \mathcal{H}_\infty\}. \quad (32)$$

The proof of Theorem 8 is dual to the proof of the Youla–Kučera parameterization of all controllers that stabilize a certain system by interchanging P and C . When comparing (13) and (32), the refinement involves a connection between the perturbations Δ_N and Δ_D through a coprime factorization of the stabilizing controller $C = N_c D_c^{-1}$. Typically, the model uncertainty structure (32) is used in conjunction with a certain norm-bound γ to satisfy (4), i.e.,

$$\mathcal{A}_u = \{\Delta_u | \|\Delta_u\|_\infty \leq \gamma\}. \quad (33)$$

When evaluating the worst-case performance for the model set defined by (32) and (33), this leads to

$$\mathcal{J}_{\text{WC}}(\mathcal{P}, C^{\text{exp}}) = \sup_{\Delta_u \in \mathcal{A}_u} \|\hat{M}_{22} + \hat{M}_{21}\Delta_u\hat{M}_{12}\|_\infty.$$

The affine dependence of \mathcal{J}_{WC} on Δ_u guarantees that the LFT is well-posed for all $\Delta_u \in \mathcal{A}_u$, and consequently \mathcal{J}_{WC} remains bounded. This advantage from a robust stability perspective is well-established. However, the analysis from a performance perspective is less obvious, since in general \hat{M}_{21} and \hat{M}_{12} are multivariable and frequency dependent. This is resolved in Section 4.3 by introducing a new uncertainty structure that connects the size of model uncertainty and the control criterion.

4.3. Enabling system identification for robust control through a new model uncertainty structure

In this section, the second and final step towards solving the system identification for the robust control problem is taken by establishing a new model uncertainty structure that transparently connects to the control criterion. The key idea is to employ the robust-control-relevant coprime factorization of \hat{P} in (18) in conjunction with the (W_u, W_y) -normalized coprime factorization of C^{exp} in (28). The following theorem constitutes Contribution C7 of the paper and reveals that the procedure presented in this paper provides a solution to the system identification for the robust control problem.

Theorem 9. Consider the model set in (32) and (33), where $\{\hat{N}, \hat{D}\}$ satisfies (18), and $\{N_c, D_c\}$ satisfies (28). Then, the worst-case performance is bounded by

$$\mathcal{J}_{\text{WC}}(\mathcal{P}, C^{\text{exp}}) \leq \mathcal{J}(\hat{P}, C^{\text{exp}}) + \gamma. \quad (34)$$

Proof. The interconnection of the model set (32) and C as in Fig. 1 leads to the LFT (31) with

$$\hat{M} = \begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{bmatrix} = \left[\begin{array}{c|c} 0 & [\tilde{N}_{e,2} \ \tilde{N}_{e,1}] \\ \hline \begin{bmatrix} W_y D_c \\ -W_u N_c \end{bmatrix} & W \begin{bmatrix} \hat{P} \\ I \end{bmatrix} (I + C\hat{P})^{-1} [C \ I] V \end{array} \right]. \quad (35)$$

With respect to \hat{M}_{12} , (18) leads to

$$\hat{M}_{12} = (\hat{D} + C\hat{N})^{-1} [C \ I] V = [\tilde{N}_{e,2} \ \tilde{N}_{e,1}].$$

Next, substitution of (35) into the LFT (31) and application of the triangle inequality yields

$$\|\mathcal{F}_u(\hat{M}, \Delta_u)\|_\infty \leq \|\hat{M}_{22}\|_\infty + \|\hat{M}_{21}\Delta_u\hat{M}_{12}\|_\infty. \quad (36)$$

Observing that \hat{M}_{21} and \hat{M}_{12} are inner and co-inner, respectively, implies that $\exists \hat{M}_{21}^\perp, \hat{M}_{12}^\perp$ such that $[\hat{M}_{21} \ \hat{M}_{21}^\perp], [\hat{M}_{12}^\perp \ \hat{M}_{12}] \in \mathcal{RH}_\infty$ are all-pass transfer function matrices. Next, observing that (36) equals

$$\|\hat{M}_{22}\|_\infty + \left\| [\hat{M}_{21} \ \hat{M}_{21}^\perp] \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{M}_{12} \\ \hat{M}_{12}^\perp \end{bmatrix} \right\|_\infty,$$

and employing the norm-preserving property of all-pass matrices and noting that $\|\Delta_u\|_\infty \leq \gamma$ for all $\Delta_u \in \Delta_u$ immediately reveals the desired result in (34). \square

The result of Theorem 9 provides a transparent connection between the size of model uncertainty γ on the one hand and the worst-case performance criterion $\mathcal{J}_{WC}(\mathcal{P}, C^{\text{exp}})$ on the other hand. Intuitively, this result reveals that the control-relevant nominal model \hat{P} , see Theorem 1, and the model uncertainty structure in Theorem 9 jointly aim at identifying a robust-control-relevant model set in view of Definition 1.

An important aspect in Theorem 9 is that the considered model uncertainty Δ_u , see (33), is an unstructured set. Hence, the proposed approach enables the use of unstructured model uncertainty, see Contribution C7. This is an important extension of dual-Youla–Kučera-based model uncertainty, since prior applications, including (de Callafon and Van den Hof, 2001), resorted to a structured model uncertainty formulation in attempt to reduce conservatism. The use of unstructured uncertainty has significant advantages for the complexity and conservatism of certain uncertainty quantification procedures, see Toker and Chen (1998), and robust control; see Doyle (1982). In this respect, the result of Theorem 9 can be interpreted as a scaling of the model uncertainty channels with respect to the control criterion.

5. Example

Consider the true system P_o with McMillan degree 6 in Fig. 3. Closed-loop frequency response functions $T(P_o, C^{\text{exp}})$ are identified with a stabilizing C^{exp} implemented on the system; see Fig. 4. In addition, W and V in (2) according to the procedure in McFarlane and Glover (1990).

In view of Procedure 1, $T(P_o, C^{\text{exp}})$ is appended with the weighting filters W and V and $\{N_o, D_o\}$ is computed for $\omega_i \in \Omega$; see Fig. 5. Next, a parametric coprime factorization $\{\hat{N}, \hat{D}\}$ is determined that is control-relevant in view of Theorem 1 and Lemma 1; see Fig. 5. The parametric model $\hat{P} = \hat{N}\hat{D}^{-1}$ is restricted to have a McMillan degree equal to 4, leading to undermodeling; see also Fig. 3. The resulting performance when the experimental controller C^{exp} is evaluated on the model \hat{P} equals $\mathcal{J}(\hat{P}, C^{\text{exp}}) = 19.9$.

Next, a model set \mathcal{P}^{RCR} is constructed using the results in Section 4. Hereto, a (W_u, W_y) -normalized coprime factorization of

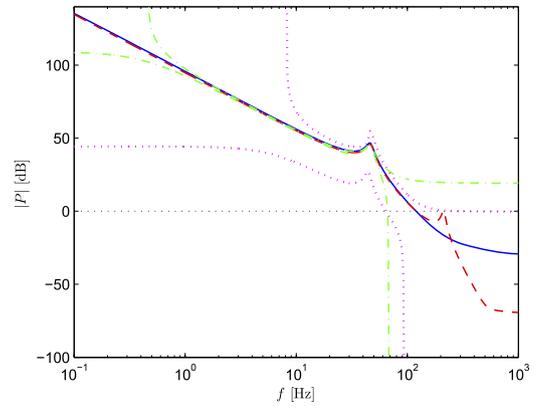


Fig. 3. Model \hat{P} (solid), true system P_o (dashed), bounds on candidate model set \mathcal{P}^{RCR} (dash-dotted), bounds on candidate model set $\tilde{\mathcal{P}}$ (dotted).

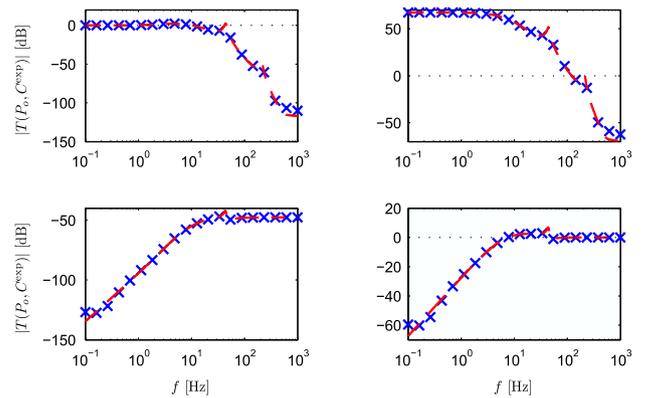


Fig. 4. Identified $T(P_o, C^{\text{exp}})$ (\times), true closed-loop system $T(P_o, C^{\text{exp}})$ (dashed).

C^{exp} is computed. This leads to the model uncertainty structure in Section 4. Hence, it remains to estimate the size of Δ_u to guarantee that (4) holds. For the sake of exposition, the dual-Youla–Kučera parameter corresponding to the true system, i.e., the $\Delta_{u,o}$ that induces P_o in (32) is computed, leading to the bound $\gamma^{\text{RCR}} = \|\Delta_{u,o}\|_\infty = 0.2$. Next, a skewed structured singular value analysis reveals that $\mathcal{J}_{WC}(\mathcal{P}^{\text{RCR}}, C^{\text{exp}}) = 20.1$. Clearly, the obtained values for $\mathcal{J}(\hat{P}, C^{\text{exp}})$, γ^{RCR} , and $\mathcal{J}_{WC}(\mathcal{P}^{\text{RCR}}, C^{\text{exp}})$ reveal that the bound in (34) holds and is tight. Hence, the model uncertainty indeed is estimated in a robust-control-relevant structure and consequently also a robust-control-relevant model set \mathcal{P}^{RCR} is obtained.

To confirm that \mathcal{P}^{RCR} enables the design of a high performance robust controller, (6) is computed, leading to $\mathcal{J}_{WC}(\mathcal{P}^{\text{RCR}}, C^{\text{RP,RCR}}) = 2.7$. Since the worst-case performance has been reduced by an order of magnitude. The question is whether C^{exp} was sufficiently close to $C^{\text{RP}}(\mathcal{P}^{\text{RCR}})$. Hereto, the true optimal performance is given by (1) and given by $\mathcal{J}(P_o, C^{\text{opt}}) = 2.5$. Since $\mathcal{J}(P_o, C^{\text{opt}}) \approx \mathcal{J}_{WC}(\mathcal{P}^{\text{RCR}}, C^{\text{RP,RCR}})$, the definition of robust-control-relevance in Definition 1 indeed is useful. Finally, a comparison is made with commonly used NRCFs (McFarlane & Glover, 1990; Vidyasagar, 1985; Vinnicombe, 2001). Hereto, $\{\tilde{N}, \tilde{D}\} = \{\hat{N}Q^{-1}, \hat{D}Q^{-1}\}$ and $\{\tilde{N}_c, \tilde{D}_c\} = \{N_c Q_c, D_c Q_c\}$ of \hat{P} and C^{exp} are computed, with $Q, Q^{-1}, Q_c, Q_c^{-1} \in \mathcal{RH}_\infty$; see Fig. 5. This leads to a dual-Youla–Kučera model uncertainty structure based on normalized coprime factorizations; see (32). The norm bound on the dual-Youla–Kučera parameter to guarantee that (4) holds is given by $\tilde{\gamma} = 0.9$. The resulting model set $\tilde{\mathcal{P}}$ achieves a worst-case performance $\mathcal{J}_{WC}(\tilde{\mathcal{P}}, C^{\text{exp}}) = 38.6$, for which the bound (34) clearly does not hold. To corroborate the fact that $\tilde{\mathcal{P}}$ is not robust-control-relevant in view of Definition 1, the model sets \mathcal{P}^{RCR} and

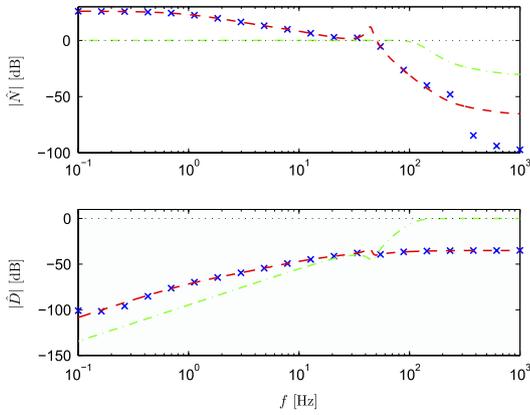


Fig. 5. Coprime factors: N_o, D_o (\times), \hat{N}, \hat{D} (dashed), \tilde{N}, \tilde{D} (dash-dotted).

$\tilde{\mathcal{P}}$ are visualized. Hereto, bounds on the Bode magnitude diagram are generated; see Fig. 3 for the results. Clearly, the bounds for the model set \mathcal{P}^{RCR} are tight in the region that corresponds to the cross-over region of both the experimental controller C^{exp} of 10 [Hz] and the desired cross-over region of 25 [Hz]. These regions are known to be of high importance for control. In contrast, the model set $\tilde{\mathcal{P}}$ is only accurate at the high frequent parasitic dynamics that are not relevant for control.

6. Conclusions

This paper addresses the system identification for the robust control problem. Existing results have employed coprime factorization-based uncertainty descriptions. However, these can unintentionally lead to conservative results that have not been well understood.

To avoid the conservatism and solve the system identification for the robust control problem, two main steps are taken in this paper. First, a new connection between control-relevant identification and the identification of coprime factorizations is developed, see Theorem 1, leading to a new robust-control-relevant coprime factorization that can be identified directly from data, see Procedure 1, and that is control-relevant in the sense of Definition 2. Second, a new coprime factor-based model uncertainty structure is presented that is based on the robust-control-relevant coprime factorization of Theorem 1 in conjunction with a new (W_u, W_y) -normalized coprime factorization, see Definition 4 and Theorem 7, leading to a transparent connection in Theorem 9 between the size of model uncertainty γ and the control criterion $\mathcal{J}(P, C)$. Thus, Theorem 9 reveals that the use of the proposed coprime factorizations avoids the introduction of conservatism and provides a solution to the considered system identification for the robust control problem. The improvements of the presented results compared to existing approaches are confirmed in a simulation example.

Acknowledgments

The authors thank Prof. Paul Van den Hof for inspiring discussions regarding an early draft of the paper.

Appendix A. Proofs of Section 3.3

The following auxiliary result is required to show that both $\{N_o, D_o\}$ and $\{\hat{N}, \hat{D}\}$ are in fact right coprime factorizations of P_o and \hat{P} , respectively; see Theorem 2.

Lemma 3. Let $[CV_2 V_1]$ have full row rank and let $\{\tilde{N}_e, \tilde{D}_e\}$ be an LCF of $[CV_2 V_1]$, where $\tilde{N}_e = [\tilde{N}_{e,2} \tilde{N}_{e,1}]$. Then, $\tilde{N}_{e,2}$ and $\tilde{N}_{e,1}$ are left coprime if and only if $[CV_2 V_1]$ has no transmission zeros in \mathbb{U} .

Proof. Suppose that $z_o \in \mathbb{U}$ is a transmission zero of $[CV_2 V_1]$. Then, by virtue of Zhou et al. (1996, Lemma 3.28), $\exists u_o \neq 0$ such that

$$u_o^*([CV_2 V_1](z_o)) = 0. \quad (\text{A.1})$$

Since $\{\tilde{N}_e, \tilde{D}_e\}$ is an LCF of $[CV_2 V_1]$, $\tilde{D}_e \in \mathcal{RH}_\infty$, hence (A.1) is equivalent to $u_o^* \tilde{N}_e(z_o) = 0$. To show that $\tilde{N}_{e,2}$ and $\tilde{N}_{e,1}$ are not left coprime, recall that left coprimeness implies that $\exists Y_l, X_l \in \mathcal{RH}_\infty$ such that

$$\tilde{N}_{e,2} Y_l + \tilde{N}_{e,1} X_l = I. \quad (\text{A.2})$$

However, $Y_l, X_l \in \mathcal{RH}_\infty$ implies that $Y_l(z_o)$ and $X_l(z_o)$ are finite. As a result,

$$u_o^*(\tilde{N}_{e,2} Y_l)(z_o) + u_o^*(\tilde{N}_{e,1} X_l)(z_o) = 0.$$

Since no $Y_l, X_l \in \mathcal{RH}_\infty$ exist that satisfy the Bézout identity (A.2), $\tilde{N}_{e,2}$ and $\tilde{N}_{e,1}$ are not left coprime if $[CV_2 V_1]$ has a transmission zero in \mathbb{U} .

To show the converse, suppose that $[CV_2 V_1]$ does not have any transmission zeros in \mathbb{U} . Then, by left coprimeness of $\{\tilde{N}_e, \tilde{D}_e\}$, \tilde{N}_e does not have any transmission zeros in \mathbb{U} , i.e., there does not exist a $z_o \in \mathbb{U}$ and a vector $u_o \neq 0$ such that $u_o^*([\tilde{N}_{e,2} \tilde{N}_{e,1}](z_o)) = 0$. Since such z_o and u_o do not exist, $\exists Y_l, X_l \in \mathcal{RH}_\infty$ such that (A.2) holds, see, e.g., Vidyasagar (1985), hence $\tilde{N}_{e,2}$ and $\tilde{N}_{e,1}$ are left coprime. \square

Proof of Theorem 2. Suppose that $T(P, C) \in \mathcal{RH}_\infty$ and $V, V^{-1} \in \mathcal{RH}_\infty$. Then, $V^{-1} \in \mathcal{RH}_\infty$ implies that V is invertible and thus $[CV_2 V_1]$ has full row rank. In addition, $V^{-1} \in \mathcal{RH}_\infty$ implies that V_1 does not have any transmission zeros in \mathbb{U} , hence by virtue of Lemma 3, $\tilde{N}_{e,2}$ and $\tilde{N}_{e,1}$ are left coprime, i.e., $\exists Y_l, X_l \in \mathcal{RH}_\infty$ such that $\tilde{N}_{e,2} Y_l + \tilde{N}_{e,1} X_l = I$. Hence, $T(P, C) \in \mathcal{RH}_\infty \Leftrightarrow T(P, C)V \in \mathcal{RH}_\infty \Leftrightarrow \begin{bmatrix} P \\ I \end{bmatrix} (\tilde{D}_e + \tilde{N}_{e,2} V_2^{-1} P)^{-1} [\tilde{N}_{e,2} \tilde{N}_{e,1}] \in \mathcal{RH}_\infty \Rightarrow \begin{bmatrix} P \\ I \end{bmatrix} (\tilde{D}_e + \tilde{N}_{e,2} V_2^{-1} P)^{-1} [\tilde{N}_{e,2} \tilde{N}_{e,1}] \begin{bmatrix} Y_l \\ X_l \end{bmatrix} \in \mathcal{RH}_\infty \Leftrightarrow \begin{bmatrix} P \\ I \end{bmatrix} (\tilde{D}_e + \tilde{N}_{e,2} V_2^{-1} P)^{-1} \in \mathcal{RH}_\infty$, revealing that (20) is a stable factorization of P .

The factorization of P in (20) is an RCF if, in addition to stability of the factors, $\exists X_r, Y_r$ such that the Bézout identity holds. Let $X_r = \tilde{N}_{e,2} V_2^{-1}$ and $Y_r = \tilde{D}_e$, hence $X_r, Y_r \in \mathcal{RH}_\infty$. In addition, $[X_r Y_r] \begin{bmatrix} P \\ I \end{bmatrix} (\tilde{D}_e + \tilde{N}_{e,2} V_2^{-1} P)^{-1} = I$, completing the proof that (20) indeed is an RCF of P . \square

Proof of Theorem 3. Let $\{N_o, D_o\}$ be defined as in Theorem 1. Then, all coprime factorizations are generated by $\{N_o Q, D_o Q\}$, where $Q, Q^{-1} \in \mathcal{RH}_\infty$. Substitution of the coprime factorization $\{N_o Q, D_o Q\}$ into $WT(P, C)V$ yields $\begin{bmatrix} N_o \\ D_o \end{bmatrix} Q Q^{-1} \tilde{D}_e$. Clearly, the latter expression can only be reduced from a four-block problem into a two-block problem if $Q^{-1} \tilde{D}_e$ is co-inner. If \tilde{D}_e is co-inner, this implies that $Q^{-1} \tilde{D}_e (Q^{-1} \tilde{D}_e)^* = Q^{-1} (Q^{-1})^* = I$, hence Q is a constant unitary matrix. A proof for $\{\hat{N}, \hat{D}\}$ follows along similar lines. \square

Appendix B. Proofs of Section 3.5

Proof of Theorem 4. Let $\{\tilde{N}_e, \tilde{D}_e\}$ be an LCF of $[CV_2 V_1]$, where $\tilde{N}_e = [\tilde{N}_{e,2} \tilde{N}_{e,1}]$ and $\tilde{N}_{e,2}$ and $\tilde{N}_{e,1}$ are left coprime by virtue of Theorem 2. Hence, $\exists Y_l, X_l \in \mathcal{RH}_\infty$ such that $\tilde{N}_{e,2} Y_l + \tilde{N}_{e,1} X_l = I$. To show the if part, suppose that $T(\hat{P}(\theta), C) \in \mathcal{RH}_\infty$. Substituting (24), this is equivalent to $\begin{bmatrix} B(\theta) \\ A(\theta) \end{bmatrix} (A(\theta) + CB(\theta))^{-1} [CV_2 V_1] \in$

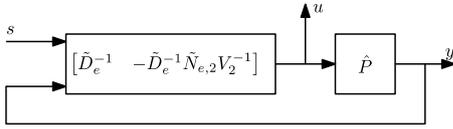


Fig. C.1. Interconnected system for the proof of Theorem 5.

$\mathcal{RH}_\infty \Rightarrow \begin{bmatrix} B(\theta) \\ A(\theta) \end{bmatrix} (\tilde{D}_e A(\theta) + \tilde{N}_{e,2} V_2^{-1} B(\theta))^{-1} [\tilde{N}_{e,2} \tilde{N}_{e,1}] \begin{bmatrix} Y_l \\ X_l \end{bmatrix} \in \mathcal{RH}_\infty \Leftrightarrow \begin{bmatrix} \hat{N}(\theta) \\ \hat{D}(\theta) \end{bmatrix} \in \mathcal{RH}_\infty$, which is the desired result. Next, to show the converse, suppose that $\begin{bmatrix} \hat{N}(\theta) \\ \hat{D}(\theta) \end{bmatrix} \in \mathcal{RH}_\infty$, implying that $\begin{bmatrix} \hat{N}(\theta) \\ \hat{D}(\theta) \end{bmatrix} [\tilde{N}_{e,2} \tilde{N}_{e,1}] \in \mathcal{RH}_\infty$, which is in fact equivalent to $T(\hat{P}(\theta), C) \in \mathcal{RH}_\infty$. Since $V, V^{-1} \in \mathcal{RH}_\infty$, the result is immediate. \square

Appendix C. Proofs of Section 3.6

Several auxiliary results are useful in the proof of Theorem 5 that are discussed first.

Lemma 4. Consider minimal state-space realizations (A_c, B_c, C_c, D_c) of C , (A_2, B_2, C_2, D_2) of V_2 and (A_1, B_1, C_1, D_1) of V_1 , and let $\tilde{D}_e, \tilde{N}_{e,2}$ be defined as in Theorem 1, where L_e denotes the output injection and M_e the output transformation matrix corresponding to the LCF with co-inner numerator of $[CV_2 V_1]$. Then, a state-space realization of $[\tilde{D}_e \tilde{N}_{e,2} V_2^{-1}]$ is given by

$$\left[\begin{array}{cc|cc} A_e + L_e C_e & L_e & L_e D_c + \tilde{B}_c & \\ M_e C_e & I & M_e D_c & \end{array} \right], \quad (C.1)$$

where $\tilde{B}_c^T = [B_c^T \ 0 \ 0]$.

See Appendix D for a proof of Lemma 4.

Lemma 5. Let a state-space realization of $[\tilde{D}_e \tilde{N}_{e,2} V_2^{-1}]$, see Lemma 4, be given. Then, a state-space realization of $[\tilde{D}_e^{-1} \tilde{D}_e^{-1} \tilde{N}_{e,2} V_2^{-1}]$ is given by

$$\left[\begin{array}{c|cc} A_e & -L_e M_e^{-1} & -\tilde{B}_c \\ C_e & M_e^{-1} & -D_c \end{array} \right]. \quad (C.2)$$

See Appendix D for a proof of Lemma 5. The preceding lemmas enable the following proof of Theorem 5.

Proof of Theorem 5. Consider the feedback interconnection in Fig. C.1. Then, it is straightforward to verify that $[\hat{N}^T \hat{D}^T]^T$, see (18), corresponds to the system $s \mapsto [y^T \ u^T]^T$. Next, direct manipulations using the state-space realization of $\tilde{D}_e^{-1} \tilde{D}_e^{-1} \tilde{N}_{e,2} V_2^{-1}$, in (C.2) in conjunction with a state-space realization of \hat{P} leads to a state-space realization of $s \mapsto [y^T \ u^T]^T$ that is given by (26)–(27). \square

Appendix D. Proofs of Appendix C

Proof of Lemma 4. Consider the interconnected system in Fig. D.1. Then, a state-space realization of the system $\begin{bmatrix} u_2 \\ u_1 \end{bmatrix} \mapsto e$ is given by

$$\left[\begin{array}{c|c} A_e & B_e \\ C_e & D_e \end{array} \right] = \left[\begin{array}{ccc|cc} A_c & B_c C_2 & 0 & B_c D_2 & 0 \\ 0 & A_2 & 0 & B_2 & 0 \\ 0 & 0 & A_1 & 0 & B_1 \\ \hline C_c & D_c C_2 & C_1 & D_c D_2 & D_1 \end{array} \right],$$

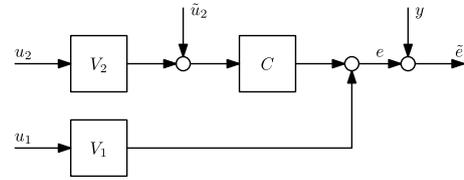


Fig. D.1. Interconnected system for the proof of Lemma 4.

with corresponding state vector $x = [x_c^T \ x_2^T \ x_1^T]^T$, where x_c, x_2 , and x_1 are the corresponding state vectors of C, V_2 , and V_1 , respectively.

Next, consider an output injection, resulting in $\dot{x} = A_e x + B_e u + L_e(e+y)$ and define the transformed output $\tilde{e} = M_e(e+y)$. Observe that by an appropriate selection of L_e and M_e ,

$$[\tilde{D}_e \tilde{N}_e] : [y^T \ u_2^T \ u_1^T]^T \mapsto \tilde{e} \quad (D.1)$$

corresponds to an LCF with co-inner numerator of $[CV_2 V_1]$, see Remark 2. A state-space realization for (D.1) is given by

$$\left[\begin{array}{cc|cc} A_e + L_e C_e & L_e & B_e + L_e D_e & \\ M_e C_e & I & M_e D_e & \end{array} \right]. \quad (D.2)$$

Considering Fig. D.1 again in conjunction with (D.1) and (D.2) reveals that $[\tilde{D}_e \tilde{N}_{e,2} V_2^{-1}]$ corresponds to the system $\begin{bmatrix} y \\ u_2 \end{bmatrix} \mapsto \tilde{e}$, see Fig. D.1, hence the desired state-space realization in (C.1) follows immediately. \square

Proof of Lemma 5. Observe from Lemma 4 that a state-space realization of $[I \ -\tilde{N}_{e,2} V_2^{-1}]$ is given by

$$\left[\begin{array}{cc|cc} A_e + L_e C_e & 0 & -L_e D_c - \tilde{B}_c & \\ M_e C_e & I & -M_e D_c & \end{array} \right].$$

By observing that a state-space realization of H^{-1} is given by $H^{-1} = (A - BD^{-1}C, -BD^{-1}, D^{-1}C, D^{-1})$, where $H = (A, B, C, D)$ with D square and full rank, which can easily be verified since $H^{-1}H = HH^{-1} = I$, a state-space realization of \tilde{D}_e^{-1} is given by $\left[\begin{array}{c|c} A_e & -L_e M_e^{-1} \\ C_e & M_e^{-1} \end{array} \right]$. Next, a state-space realization of the series connection of $[I \ -\tilde{N}_{e,2} V_2^{-1}]$ and \tilde{D}_e^{-1} is given by

$$\left[\begin{array}{cc|cc} A_e + L_e C_e & 0 & 0 & -L_e D_c - \tilde{B}_c \\ -L_e M_e^{-1} M_e C_e & A_e & -L_e M_e^{-1} & L_e M_e^{-1} M_e D_c \\ \hline M_e^{-1} M_e C_e & C_e & M_e^{-1} & -M_e^{-1} M_e D_c \end{array} \right] \quad (D.3)$$

Next, applying a state transformation $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ to the state-space realization in (D.3) and subsequently removing the unobservable states leads to the desired result in (C.2). \square

Appendix E. Proofs of Section 4.1

Proof of Theorem 6. Suppose that $\{N_c, D_c\}$ is a (W_u, W_y) -normalized RCF of C . Then, all coprime factorizations of C are generated by $\{N_c Q, D_c Q\}$, $Q, Q^{-1} \in \mathcal{RH}_\infty$. In view of (28), these coprime factorizations are (W_u, W_y) -normalized RCF iff $Q^* Q = I$. \square

Proof of Theorem 7. First, consider the interconnected system in Fig. E.1, where a state-space realization of the system $u_{yi} \mapsto \begin{bmatrix} y_{yi} \\ y_c \end{bmatrix}$ is given by $\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline C & \mathcal{D} \end{array} \right]$, with state vector $x = [x_u^T \ x_c^T \ x_{yi}^T]^T$, where x_u, x_c , and x_{yi} are the state vectors corresponding to W_u, C , and

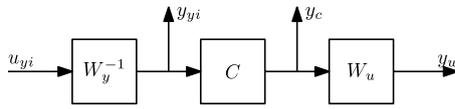


Fig. E.1. Interconnected system for the proof of Theorem 7.

W_y^{-1} , respectively. Next, consider a state feedback by defining a new input u_{ND} that satisfies

$$u_{yi} = Gu_{ND} + Fx, \quad (\text{E.1})$$

where F and G are matrices of appropriate dimensions. Then, a state-space realization of the system $\begin{bmatrix} D_c \\ N_c \end{bmatrix} : u_{ND} \mapsto \begin{bmatrix} y_{yi} \\ y_c \end{bmatrix}$ is given by

$$\begin{bmatrix} \mathcal{A} + \mathcal{B}F & \mathcal{B}G \\ \mathcal{C} + \mathcal{D}F & \mathcal{D}G \end{bmatrix}. \text{ Clearly, by the specific choice of } F \text{ in (29), } \mathcal{A} + \mathcal{B}F$$

is stable by construction, hence $N_c, D_c \in \mathcal{RH}_\infty$ and $C = N_c D_c^{-1}$. It remains to be shown that the pair $\{N_c, D_c\}$ indeed constitutes a (W_u, W_y) -normalized RCF of C . Hereto, consider the system $u_{yi} \mapsto y_u$ as defined in Fig. E.1, which has a state-space realization

$$\left[\begin{array}{ccc|c} \mathcal{A} & & & \mathcal{B} \\ \hline C_u & D_u C_c & D_u D_c C_{yi} & D_u D_c D_{yi} \end{array} \right].$$

Using (E.1), $\begin{bmatrix} D_w \\ N_w \end{bmatrix} : u_{ND} \mapsto \begin{bmatrix} u_{yi} \\ y_u \end{bmatrix}$ is given by

$$\left[\begin{array}{ccc|c} \mathcal{A} + \mathcal{B}F & & & \mathcal{B}G \\ \hline F & & & G \\ C_u & D_u C_c & D_u D_c C_{yi} + D_u D_c D_{yi} F & D_u D_c D_{yi} G \end{array} \right].$$

It can directly be verified that the pair $\{N_w, D_w\}$ constitutes a normalized RCF of $W_u C W_y^{-1}$. Indeed, observe that (30) can be written as

$$\tilde{\mathcal{A}}^* X (I + \mathcal{B}R^{-1} \mathcal{B}^* X)^{-1} \tilde{\mathcal{A}} - X + Q = 0, \quad (\text{E.2})$$

which corresponds to the algebraic Riccati equation in Zhou et al. (1996, Theorem 21.25) to compute a normalized coprime factorization. In addition, this also reveals that a unique, positive semi-definite solution to (E.2) exists. Next, to show that $\{N_c, D_c\}$ is an RCF, observe that coprimeness of N_w, D_w implies $\exists X_{r,w}, Y_{r,w} \in \mathcal{RH}_\infty$ satisfying

$$X_{r,w} N_w + Y_{r,w} D_w = I. \quad (\text{E.3})$$

Next, let $X_{r,c} = X_{r,w} W_u$ and $Y_{r,c} = Y_{r,w} W_y$, hence $X_{r,c}, Y_{r,c} \in \mathcal{RH}_\infty$. Observing that $N_c = W_u^{-1} N_w$ and $D_c = W_y^{-1} D_w$ and using (E.3) reveals that $X_{r,c} N_c + Y_{r,c} D_c = I$, hence the pair $\{N_c, D_c\}$ indeed is an RCF. To show that $\{N_c, D_c\}$ is a (W_u, W_y) -normalized RCF of C , note that $\{W_u N_c, W_y D_c\} = \{N_w, D_w\}$. Hence (28) is satisfied and $\{N_c, D_c\}$ is a (W_u, W_y) -normalized RCF of C . \square

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