Iterative Data-Driven $\mathcal{H}_\infty$ Norm Estimation of Multivariable Systems with Application to Robust Active Vibration Isolation

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Abstract—This paper aims to develop a new data-driven $\mathcal{H}_\infty$ norm estimation algorithm for model-error modeling of multivariable systems. An iterative approach is presented that requires significantly fewer prior assumptions on the true system, hence it provides stronger guarantees in a robust control design. The iterative estimation algorithm is embedded in a robust control design framework with a judiciously selected uncertainty structure to facilitate high control performance. The approach is experimentally implemented on an industrial active vibration isolation system (AVIS).

I. INTRODUCTION

Robustness is of key importance in any feedback controlled system. An important example of such a feedback controlled system is an active vibration isolation system (AVIS) [1], [2], which is used to isolate highly accurate motion systems from external disturbances in multiple degrees-of-freedom. The underlying idea of vibration isolation is based on the concept of Skyhook damping, see [3]. Model-based control designs based on $\mathcal{H}_\infty$-optimization are considered in [4] and [5]. In [4], model uncertainty is explicitly taken into account. However, the uncertainty is based on inaccurate prior assumptions, leading to potentially conservative results.

Robustness analysis and robust control design hinge on an accurate quantification of model quality. On the one hand, several model-error modeling techniques have been developed that are based on existing system identification methods [6]. For instance, in [7], the model error is quantified through the evaluation of the bias and variance errors of an estimated parametric model of the model residual. Such an approach often requires user intervention in the model parameterization step and relies on assumptions that are typically asymptotic in the data length. Alternatively, model-error modeling techniques have been proposed in [8], [9] that are based on non-parametric frequency domain models. In [9], it is acknowledged that the use of a discrete frequency grid, which is unavoidable in any finite time experiment, leads to intergrid errors. In [8], these errors are ignored by the assumption that the frequency grid is sufficiently dense. In [9], these intergrid errors are bounded by using prior assumptions, see also [10] for related approaches. However, bounding these errors generally leads to overly large estimates of the model error, as is also argued in [11, Sec. 9.5.2]. Finally, model validation techniques have been proposed, see [12], [10] for frequency domain results and [13] for time domain results. However, such approaches often lead to overly optimistic estimates of the model errors unless appropriate validation experiments are designed [14].

On the other hand, recently in [15, Sec. 12.2] a data-driven iterative procedure is proposed that direct estimates the $\mathcal{H}_\infty$ norm of single-input single-output (SISO) systems. This enables its use in model-error modeling, since reliable robust control design methodologies are available that consider model errors as $\mathcal{H}_\infty$ norm bounded perturbations. In [16], the data-driven iterative $\mathcal{H}_\infty$ norm estimation is further extended, followed by a thorough stochastic analysis in [17]. In [18], the procedure is successfully applied for robust stability analysis of a SISO experimental setup.

Although the iterative data-driven $\mathcal{H}_\infty$ norm estimation algorithm has been thoroughly analyzed and successfully used for robustness stability analysis, it is not directly applicable to model-error modeling for achieving robust performance of general multivariable systems. First, the algorithms in [15, Sec. 12.2], [16], and [17] are only suitable for SISO systems. Second, besides the size of the uncertainty in terms of the $\mathcal{H}_\infty$ norm, the uncertain model set highly depends on the model uncertainty structure and frequency weighting functions. Indeed, typical model-error modeling procedures involve the selection of a frequency dependent weighting function to shape the uncertainty, see [19, Sec. 7.4.3] for details in this direction. However, iterative data-driven $\mathcal{H}_\infty$ norm estimation algorithms do not provide the required frequency-dependent shape of the model error and cannot be used as a basis for designing frequency dependent weighting functions. Indeed, in [18], the algorithms are only used as an a posteriori stability verification. The present paper aims to develop a data-driven iterative $\mathcal{H}_\infty$ norm estimation procedure for model-error modeling of multivariable systems for robust control.

The contributions of this paper are threefold.

C1) A novel multivariable data-driven $\mathcal{H}_\infty$ norm estimation procedure is presented that applies to general multi-input multi-output systems. In addition to iteratively determining the worst-case frequency as in [15, Sec. 12.2], [16], and [17], the proposed algorithm in this paper is also able to determine the worst-case input and output direction.

C2) The proposed multivariable data-driven $\mathcal{H}_\infty$ norm estimation procedure is used in conjunction with a new model uncertainty structure that has recently been developed in [20]. This new model uncertainty structure renders the
use of weighting functions superfluous in model-error modeling by directly connecting the size of model uncertainty and the control criterion. This enables the use of the proposed data-driven $H_\infty$ norm estimation procedure for model-error modeling of multivariable systems for high performance robust control.

C3) Application of the procedure to a multivariable industrial AVIS system, which

i) confirms that the multivariable data-driven $H_\infty$ norm estimation procedure converges and performs reliably when implemented on an industrial system,

ii) reveals higher accuracy of the estimated model error compared to an alternative model-error modeling procedure, and

iii) confirms that the resulting model set leads to enhanced vibration isolation.

Preliminary research results related to C1 appear as [21], the present paper extends this with extensive theoretical and experimental results. In addition, the present paper contains the new contributions C2 and C3.

The main results of this paper are presented in the noise-free situation to facilitate the exposition. An analysis of the situation of additive stochastic noise is presented in Sec. III-C3.

The paper is organized as follows. In Sec. II, the AVIS and the control goal are presented. Then, in Sec. III, the multivariable data-driven $H_\infty$ norm estimation procedure is presented, leading to Contribution C1. Then, in Sec. IV, the proposed $H_\infty$ norm estimation procedure is used in conjunction with the new model uncertainty structure in [20], constituting Contribution C2. In Sec. V experimental results of the proposed uncertainty modeling procedure are presented. The resulting model set is then used in Sec. VI to design and implement a robust controller on the AVIS system. Hence, Sec. V and Sec. VI contain Contribution C3. Finally, a discussion and a conclusion are presented in Sec. VII.

II. DESCRIPTION OF AVIS AND CONTROL GOAL

A. AVIS

The AVIS in Fig. 1 is considered in this paper. The system consists of two main parts, i.e., a movable payload and a chassis that is connected to the floor. The payload and chassis are connected by four isolator modules. On the one hand, these isolator modules provide passive damping through a pneumatic airmount. On the other hand, the isolator modules are equipped with Lorentz motors and geophones that enable active vibration isolation. Specifically, the isolation modules are each equipped with two motors, leading to eight actuators in total. The currents applied to the motors are denoted $a = [a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8]$. In addition, three out of four modules are equipped with two geophones each that construct measurements of the velocity, leading to six sensors, which are denoted $s = [s_1, s_2, s_3, s_4, s_5, s_6]$. The inputs and outputs of the system are rigid-body decoupled with respect to the Cartesian coordinate frame depicted in Fig. 1. The manipulated input and output are denoted

$$u = T_u a \quad (1)$$
$$y = T_y s \quad (2)$$

respectively, where $T_u \in \mathbb{R}^{6 \times 8}$ and $T_y \in \mathbb{R}^{6 \times 6}$. Finally, the true system $P_o$ is given by $P_o : u \mapsto y$.

The goal of the AVIS is to isolate the payload with respect to exogenous disturbances. The AVIS is schematically illustrated in Fig. 2. The signal $d_1$ refers to force disturbances that directly affect the payload, and $d_2$ refers to force disturbances that are induced by floor vibrations. The goal is to keep the absolute velocity of the payload equal to zero by manipulating the input $u$ through the velocity measurement $y$, see (2). The key advantage of active vibration control compared to the passive isolation by the pneumatic airmounts is the availability of absolute velocity measurements. This enables skyhook damping [3], which connects the fictive damper (implemented as a control algorithm) to the fixed world. As a result, skyhook damping implies that $y$ should be kept as small as possible for both disturbances $d_1$ and $d_2$. This is in sharp contrast to the situation where velocity measurements relative to the floor are available, in which case $y$ should be kept as small as possible only if $d_2 = 0$.

The resulting feedback interconnection is depicted in Fig. 3, where

$$d = d_1 + H_d d_2$$
and $H_d$ is a causal stable transfer function matrix that characterizes the transfer of the flow vibrations through the pneumatic airmount to the payload.

To show the potential performance improvement by skyhook damping, an initial controller $C_{\text{exp}}$ is designed. This controller consists of a multi-loop SISO controller, where each diagonal element is a gain with high-frequency roll-off, see Fig. 14 in Sec. VI-A for the diagonal elements corresponding to the $z$ and $\phi$ directions. The resulting closed-loop transfer function

$$P_{cl,\text{exp}}: d \mapsto y = P_o(I + C_{\text{exp}}P_o)$$

characterizes the closed-loop disturbance attenuation properties. In Fig. 4, frequency response function measurements of the open-loop system $P_o$ and the closed-loop system $P_{cl,\text{exp}}$ are depicted in the vertical translational direction $z$. The actual identification of these frequency response functions is described in Sec. V. Clearly, Fig. 4, reveals that the controller $C_{\text{exp}}$ damps the resonance phenomenon at approximately 3.2 Hz by 10 dB in the translational $z$ direction.

Although $C_{\text{exp}}$ increases the damping properties of the AVIS, see Fig. 4, the increased damping is insufficient for satisfactory control performance. This will be confirmed by the experimental results in Sec. VI-B, specifically Fig. 16 and Fig. 17. Analysis of the power spectral densities in Fig. 17 reveals a dominant frequency content between 1 and 10 Hz.

**B. Control goal and approach**

The goal in this paper is to improve vibration isolation properties of the AVIS through enhanced controller design. The key performance limiting factor in increasing the gain of the controller $C_{\text{exp}}$, and hence increasing the skyhook damping, involves the high-frequency flexible dynamics beyond 100 Hz, as can be observed in Fig. 4. As a result, performance and robustness objectives have to be appropriately specified. The pursued approach to achieve high performance active vibration isolation is a multivariable robust controller design.

The performance objectives are specified using a criterion $\mathcal{J}(P, C)$, where the goal is to compute the optimal controller

$$C_{\text{opt}} = \arg\min C(P_o, C).$$

Before specifying $\mathcal{J}(P, C)$ in more detail, see Sec. II-C, the connection between the control objective and the true unknown system $P_o$ is established in further detail.

To address robustness objectives, note that the minimization of $\mathcal{J}(P_o, C)$ requires knowledge of $P_o$. This knowledge is reflected by a model set $P$. This model set $P$ will be defined more precisely in Sec. II-D. The key property of this model set $P$ is that it will be chosen such that it encompasses the true AVIS dynamical behavior, i.e., including the high-frequency flexible dynamics. Hence, the assumption

$$P_o \in P$$

is assumed to hold throughout the paper. The key motivation for considering a model set is the fact that any model is uncertain and cannot represent the true system behavior exactly. In view of (4), given the model set $P$, the robust controller synthesis

$$C_{\text{RP}} = \arg\min_C \mathcal{J}_{\text{WC}}(P, C),$$

is considered, where $\mathcal{J}_{\text{WC}}(P, C) = \sup_{P_o \in P} \mathcal{J}(P, C)$. This provides a performance guarantee when implementing $C_{\text{RP}}$ on the true system, since by (4), the bound

$$\mathcal{J}(P_o, C_{\text{RP}}) \leq \mathcal{J}_{\text{WC}}(P, C_{\text{RP}})$$

holds.

To facilitate the exposition, a two-input two-output robust feedback controller is designed for the $z$ and $\phi$ direction, i.e.,

$$\begin{bmatrix} u_z \\ u_\phi \end{bmatrix} = C_{\text{RP}} \begin{bmatrix} y_z \\ y_\phi \end{bmatrix}.$$  

The presented approach applies equally well to the full multivariable situation, i.e., three rotations and three translations. In this paper, the selection of a rotational and translational degree-of-freedom is made to show that the presented approach automatically scales the uncertainty channels, see (28). Hence, it automatically deals with the various units of the measurements. Note that all physical actuators $s$ and sensors $s$ are typically used due to $T_u$ and $T_y$ in (1) and (2), respectively.

**C. Control criterion**

The control goal in this paper is specified by

$$\mathcal{J} = \|WT(P, C)V\|_\infty,$$

where

$$T(P, C) = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix},$$

$$\begin{bmatrix} P \\ I \end{bmatrix} (I + CP)^{-1} \begin{bmatrix} C \\ I \end{bmatrix}.$$  

The transfer function matrix $T(P, C)$ maps $\begin{bmatrix} r_2 \\ r_1 \end{bmatrix}$ onto $\begin{bmatrix} y \\ u \end{bmatrix}$, where $r_1$ corresponds to $d$ in Fig. 3 and $r_2$ is an additional signal, see Fig. 7 in Sec. IV-C for a block diagram. In addition, $W$ and $V$ in (7) are stable and minimum-phase weighting filters. The motivation for considering the $H_\infty$ norm in (7) is to enable a systematic robust controller synthesis.

A four-block problem is considered, i.e., $T(P, C)$ in (8) contains four blocks. The four-block problem guarantees internal stability of the resulting optimal controller. This has important implications from a theoretical perspective, since it
will enable the construction of a specific coprime factorization that leads to (28). In addition, the four-block problem enables the use of the systematic loop-shaping approach in [22] to select $W$ and $V$ such that it enhances active vibration isolation performance. Note that the relevant closed-loop transfer function for vibration isolation is essentially given by (3). Interestingly, the loop-shaping approach in [22] is essentially based on the fact that the loop-gain $|CP|$ is the only degree of freedom in shaping the four-block problem (8). In this paper, the transparent connection between shaping the loop-gain $|CP|$ and the relevant transfer function (3) is exploited to design the weighting functions.

In the considered design framework, so-called loop-shaping weighting filters $W_1$ and $W_2$ are adopted that shape the desired open-loop gain $W_2P(W_1$. Next, observe that the initial controller leads to a certain loop-gain $C\hat{P}$. The desired loop-gain typically has a smaller amplitude in the low frequency range compared to $W_2PW_1$. Hence, the rationale in this paper to design the weighting filter $W_1$ and $W_2$ is to increase the gain of $C\exp$, i.e.,

$$W_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 19.9 & 0 \\ 0 & 11.75 \end{bmatrix} C\exp.$$

Although the design procedure to select $W_2$ resembles the controller $C\exp$, a robust controller synthesis procedure is required to deliver a robustly stabilizing feedback controller. These weighting functions $W_2$ and $W_1$ directly fit in criterion (7) through $W = \begin{bmatrix} W_2 & 0 \\ 0 & W_1^{-1} \end{bmatrix}$, $V = \begin{bmatrix} W_2^{-1} & 0 \\ 0 & W_1 \end{bmatrix}$.

### D. Identification of $P$ using $\mathcal{H}_\infty$ norm bounded perturbations

The uncertain model set $P$ that is introduced in Sec. II-B is constructed as an $\mathcal{H}_\infty$ norm bounded perturbation around a nominal model $\hat{P}$, i.e.,

$$P = \{P | P = F_u(\hat{H}(\hat{P}), \Delta_u), \Delta_u \in \Delta_u \},$$

where $\hat{H}(\hat{P})$ represents the nominal model $\hat{P}$ and uncertainty structure. Also, the upper linear fractional transformation (LFT) is given by

$$F_u(\hat{H}, \Delta_u) = \hat{H}_{22} + \hat{H}_{21} \Delta_u (I - \hat{H}_{11} \Delta_u)^{-1} \hat{H}_{12}.$$ 

As an example of (9), note that $\hat{H}(\hat{P})$ in the case of additive uncertainty $\hat{P} + \Delta_u$ is given by $H_{11} = 0$, $H_{12} = H_{21} = I$, and $H_{22} = \hat{P}$. In addition, unstructured model uncertainty is considered, i.e.,

$$\Delta_u := \{\Delta_u ||\Delta_u||_\infty \leq \gamma \}$$

To actually identify $P$, three aspects are of importance:

1) identifying a nominal model $\hat{P}$,
2) constructing a model uncertainty structure, see $\hat{H}(\hat{P})$, and
3) determining the size of model uncertainty $\gamma$ in (10).

In the next section, a new approach for the latter aspect, i.e., determining $\gamma$, is presented. The former two aspects are subsequently addressed in Sec. IV, leading to a complete procedure that encompasses 1)-3).

### III. A DATA-DRIVEN APPROACH TO MULTIVARIABLE UNCERTAINTY MODELING

In this section, a novel procedure is presented to estimate the size $\gamma$ of $\Delta_u$, given a nominal model $\hat{P}$ and uncertainty structure $\hat{H}(\hat{P})$. As a result, $\gamma$ depends on the choice of $\hat{P}$ and the uncertainty structure, where a suitable structure will be presented in Sec. IV. In addition, in view of (4), $\gamma$ depends on the true system $P_o$. To show this, let $\Delta_u$ uniquely generate $P_o$ in (9), i.e.,

$$P_o = F_u(\hat{H}(\hat{P}), \Delta_o).$$

In this case,

$$\gamma = ||\Delta_o||_\infty$$

is the minimum-norm bound such that the model set defined by (9) and (10) satisfies (4). Thus, the goal in model-error modeling essentially is to determine the $\mathcal{H}_\infty$ norm of the model error $\Delta_u$. Note that $\Delta_u$ generally is a dynamic system, e.g., in the case of additive uncertainty, $\Delta_o = \hat{P} - P_o$.

The pursued approach to estimate $\gamma$ is to perform experiments on $\Delta_u$. To achieve this, the input $u_\Delta$ is applied to $\Delta_o$ and the output $y_\Delta = \Delta_o u_\Delta$ is measured. Note that in general experimentation on $\Delta_u$ involves both an experiment on the true system $P_o$ and a simulation with the model $\hat{P}$. For instance, in the case of additive uncertainty, $y_\Delta = \Delta_o u_\Delta = \hat{P} u_\Delta - P_o u_\Delta$.

The key idea in this paper is to directly estimate the size $\gamma$ of $\Delta_o$ based on measured data, hence without the need for identification of an intermediate model. In contrast, the approach in [7] involves the estimation of an auxiliary model of the model error $\Delta_o$, whereas [8] and [9] employ a nonparametric model, i.e., an identified frequency response function, of $\Delta_o$. In these approaches, the identified model is subsequently used to compute the $\mathcal{H}_\infty$ norm.

Next, the basic procedure is presented, revealing that the auxiliary model estimation step can be rendered superfluous.

#### A. Iterative data-driven $\mathcal{H}_\infty$ norm estimation: SISO systems

In this section, the basic principle for data-driven $\mathcal{H}_\infty$ norm estimation is presented. To introduce the mechanism, first assume that $\Delta$ is SISO and linear time invariant (LTI). Hence,

$$\Delta(z) = \sum_{k=0}^\infty \delta_k z^{-k} \quad \text{with} \quad \delta_k, k = 0, \ldots, \infty \quad \text{the Markov parameters of the system.}$$

Next, recall that the $\mathcal{H}_\infty$ norm is an induced norm, i.e., $||\Delta||_\infty = ||\Delta||_{2}$ see [19, Appendix A.5.7]
for a proof. Assume that the signals have finite length \( N \in \mathbb{N} \), i.e., \( u_A, y_A \in \mathbb{R}^{N \times 1} \), hence
\[
\|\Delta\|_2 = \sup_{u_A \in \mathbb{R}^{N \times 1}} \frac{\|y_A\|_2}{\|u_A\|_2}.
\]
Note that \( \|\Delta\|_2 \to \|\Delta\|_\infty \) for \( N \to \infty \), see [17, Theorem 3] for a proof. Next, on the finite time interval of length \( N \),
\[
y_A = \Delta u_A,
\]
where no measurement noise is assumed and
\[
\Delta = \begin{bmatrix}
\delta_0 & 0 & 0 & \cdots & 0 \\
\delta_1 & \delta_0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\delta_{N-1} & \delta_{N-2} & \delta_{N-3} & \cdots & \delta_0
\end{bmatrix}.
\]
As a result, (12) is equivalent to
\[
\|\Delta\|_2 = \sup_{u_A \in \mathbb{R}^{N \times 1}} \sqrt{\frac{u_A^T \Delta^T \Delta u_A}{u_A^T u_A}} = \sqrt{\lambda_{\text{max}}(\Delta^T \Delta)}.
\]
Now, observe that
\[
\Delta^T = T_N \Delta T_N,
\]
where
\[
T_N = \begin{bmatrix}
0 & \cdots & 0 & 1 \\
0 & \cdots & 1 & 0 \\
\vdots & \cdots & \vdots & \vdots \\
1 & \cdots & 0 & 0
\end{bmatrix},
\]
i.e., \( T_N \) is an involutory permutation matrix of size \( N \times N \). Here, \( T_N \) has the interpretation of a time-reversal operator. Next, (15) reveals that \( T_N \Delta \) is symmetric, hence
\[
\|\Delta\|_2 = \lambda_{\text{max}}(T_N \Delta).
\]
As a result, \( \|\Delta\|_2 \) equals the largest eigenvalue of the matrix \( T_N \Delta \). Hence, given the impulse response \( \delta_i \), \( i = 1, \ldots, N \), \( \|\Delta\|_2 \) can directly be computed through an eigenvalue analysis. In contrast, in this paper another approach is pursued to determine \( \|\Delta\|_2 \) that does not require knowledge of \( \delta_i \), \( i = 1, \ldots, N \). Such an approach is given next.

**Procedure 1 (SISO \( \|\Delta\|_2 \) estimation):** Perform the following sequence of steps.

1) set \( n = 1 \) and initialize with arbitrary \( u_A^{(1)} \in \mathbb{R}^{N \times 1} \backslash 0 \).
2) determine \( y_A^{(n)} = \Delta u_A^{(n)} \) by performing an experiment on the system.
3) time-reverse: \( x_A^{(n)} = T_N y_A^{(n)} \).
4) set \( u_A^{(n+1)} = x_A^{(n)} \).
5) set \( n \mapsto n + 1 \) and repeat from Step 2 until a stopping criterion is met.

Procedure 1 coincides with the power method [23, Sec. 7.3.1], which is an iterative algorithm to determine the maximal eigenvalue of a matrix. It is well-known that this estimator
\[
\gamma_1^{(n)} = \frac{(u_A^{(n)})^T x_A^{(n)}}{(u_A^{(n)})^T u_A^{(n)}},
\]
converges under mild conditions to \( \|\Delta\|_2 \) for \( n \to \infty \).

Essentially, two approaches can be pursued to complete Step 2 in Procedure 1. On the one hand, \( \Delta^{(n)} u_A^{(n)} \) can be evaluated on a model using impulse response coefficients \( \delta_i \), \( i = 1, \ldots, N \). On the other hand, the approach taken in this paper is to perform an experiment of length \( N \) on the true system \( \Delta_o \) to evaluate \( \Delta^{(n)} u_A^{(n)} \). As a result, Procedure 1 does not need any structural knowledge of \( \Delta_o \).

Summarizing, the iterative procedure 1 leads to an estimate \( \hat{\gamma}_1^{(n)} \) of \( \|\Delta_o\|_\infty \) for a sufficiently large number of iterations \( n \) and sufficiently long experiment length \( N \). Each iteration requires one experiment of length \( N \) on the true system \( \Delta_o \), provided that \( \Delta_o \) is SISO. Since the approach is data-driven, the only computational step involves (17), which has linear complexity in \( N \) and thus is not restrictive for large \( N \).

In the next section, \( H_\infty \) norm estimation of multi-input multi-output (MIMO) systems is investigated.

**B. Data-driven \( H_\infty \) norm estimation of MIMO systems**

In this section, a novel procedure for estimating the \( H_\infty \) norm of MIMO systems is presented. It turns out that Procedure 1 does not directly apply to MIMO systems, since (15) is not valid for general MIMO systems. Hence, MIMO systems introduce a significant complication when compared to the SISO case in Sec. III-A.

To develop a data-driven \( H_\infty \) norm estimator for MIMO systems, consider the transfer function matrix of the MIMO system \( \Delta \) with \( p \) outputs and \( q \) inputs:
\[
\Delta = \begin{bmatrix}
\Delta_{11} & \cdots & \Delta_{1q} \\
\vdots & \ddots & \vdots \\
\Delta_{p1} & \cdots & \Delta_{pq}
\end{bmatrix} \in \mathcal{RH}_{\infty}^{p \times q}
\]
with finite time representation for experiment length \( N \)
\[
\begin{bmatrix}
\Delta_{11} & \cdots & \Delta_{1q} \\
\vdots & \ddots & \vdots \\
\Delta_{p1} & \cdots & \Delta_{pq}
\end{bmatrix} = \begin{bmatrix}
\gamma_1^{(n)} & \cdots & \gamma_q^{(n)} \\
\vdots & \ddots & \vdots \\
\gamma_{pq}^{(n)} & \cdots & \gamma_{pq}^{(n)}
\end{bmatrix} = \begin{bmatrix}
u_A^{(n)} \\
\vdots \\
u_A^{(n)}
\end{bmatrix},
\]
where \( u_A^{(n)} \in \mathbb{R}^{N \times 1} \), \( i = 1, \ldots, q \) and \( y_A^{(n)} \in \mathbb{R}^{N \times 1} \), \( i = 1, \ldots, p \). In addition, \( \Delta \in \mathbb{R}^{pN \times qN} \).

To proceed, observe that (14) is also valid for the multivariable system (18). Next, note that
\[
\Delta^T = \begin{bmatrix}
T_N & 0 \\
0 & T_N
\end{bmatrix},
\]
i.e., \( \Delta^T \) is the finite time representation of the transfer function matrix
\[
\begin{bmatrix}
\Delta_{11} & \cdots & \Delta_{1q} \\
\vdots & \ddots & \vdots \\
\Delta_{p1} & \cdots & \Delta_{pq}
\end{bmatrix}.
\]

In general, \( \tilde{\Delta} \neq \Delta \). As a result, an analogous step from (14) to (16) can only be made in the restrictive case where \( \tilde{\Delta} = \Delta \), i.e., if the system is symmetric. As a result, the case \( \tilde{\Delta} \neq \Delta \) requires a different approach. The key step in the forthcoming developments is the observation that
\[
\tilde{\Delta} = \sum_{i=1}^{q} \sum_{j=1}^{p} \mathcal{I}_{ij} \Delta \mathcal{I}_{ij},
\]
where \( \mathcal{I}_{ij} \) is the \( ij \)-th identity matrix.
where
\[ I_{ij} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & I_N & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{q \times p} \]

and the 1 is on the \((i, j)^{th}\) location. To derive the finite time equivalent of (19), note that the finite time representation of \(I_{ij}\) is given by
\[ \tilde{I}_{ij} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & I_N & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{qN \times pN}, \tag{20} \]

where \(\tilde{I}_{ij}\) is a block-diagonal matrix of \(q \times p\) blocks of dimension \(N \times N\). Also, the \((i, j)^{th}\) block-element is equal to an identity matrix of size \(N\). Note that the system \(I_{ij}\) can be interpreted as a static system, i.e., containing only a direct feedthrough term, leading to a diagonal matrix in (20). Next,
\[ \tilde{A} = \sum_{i=1}^{q} \sum_{j=1}^{p} \tilde{I}_{ij} \Delta \tilde{I}_{ij}. \tag{21} \]

The result (21) is the basis for the following procedure, which constitutes Contribution C1 of this paper.

Procedure 2 (MIMO \(\|\Delta\|_{2}\) estimation): Perform the following sequence of steps.

1) set \(n = 1\) and initialize with arbitrary \(u_{\Delta}^{(1)} \in \mathbb{R}^{qN \times 1}\), \(u_{\Delta}^{(1)} \neq 0\).
2) determine \(y_{\Delta}^{(n)} = \Delta u_{\Delta}^{(n)}\) by performing an experiment on the system.
3) time-reverse: \(x_{\Delta}^{(n)} = T_{pN} y_{\Delta}^{(n)}\).
4) set \(z_{\Delta}^{(n)} = 0\) and perform \(pq\) experiments:
   for \(i = 1, \ldots, q\)
   for \(j = 1, \ldots, p\)
   \[ z_{\Delta}^{(n)} \mapsto z_{\Delta}^{(n)} + \tilde{I}_{ij} \Delta \tilde{I}_{ij} x_{\Delta}^{(n)} \]
   \end
5) time reverse: \(w_{\Delta}^{(n)} = T_{qN} z_{\Delta}^{(n)}\).
6) set \(w_{\Delta}^{(n+1)} = w_{\Delta}^{(n)}\).
7) set \(n \mapsto n + 1\) and repeat from Step 2 until a stopping criterion is met.

The multivariable \(H_\infty\) norm can then be estimated as
\[ \gamma_2^{(n)} = \sqrt{\frac{(u_{\Delta}^{(n)})^T w_{\Delta}^{(n)}}{(u_{\Delta}^{(n)})^T u_{\Delta}^{(n)}}}, \tag{22} \]

where in contrast to Procedure 1, a square root appears due to the fact that the reduction from two experiments to a single experiment per step in the SISO procedure (based on (14) to (16)) is not made in the derivation of Procedure 2.

Basically, (19) and Procedure 2 recast the evaluation of \(\Delta^T\) as \(p \cdot q\) experiments on \(\Delta\), for which the true system \(\Delta_0\) is available for evaluation. Hence, the approach does not require explicit knowledge of \(\Delta^T\) and \(\Delta\). The underlying system property that is exploited here is linearity. A single iteration of the proposed procedure 2 applied to a two-input two-output MIMO system is illustrated in Fig. 6.

C. Convergence and implementation aspects

1) Convergence aspects: Procedure 1 and Procedure 2 are globally convergent. The proof follows directly from the power iterations method for computing the largest eigenvalue of a matrix, see [23, Sec. 8.2]. In addition, the finite time induced 2 norm in (12) converges to the \(H_\infty\) norm for \(N \to \infty\), see [17, Theorem 3]. Hence, \(N\) should be chosen sufficiently large to avoid transient effects. Such transient effects are well-known in system identification, e.g., in frequency domain system identification these are directly related to leakage errors, see, e.g., [24]. Also, note that this result differs from time-domain model validation, including [13]. In particular, such model validation techniques also exploit the result (14) but are exact for both the infinite time and finite time case. The key difference lies in the fact that the power iteration approach in this paper involves an estimation problem instead of a model validation problem. The key advantage of the power iterations method is that very effectively deals with noise, as is described in Sec. III-C3. In fact, in [17, Section 4.2], it is shown that the power iterations method can be interpreted as an iterative experiment design approach for estimating the \(H_\infty\) norm.

2) Input constraints: The actual implementation on physical systems of the data-driven \(H_\infty\) norm estimation procedures 1 and 2 for SISO and MIMO systems, respectively, requires some additional attention. First, the input \(u_{\Delta}^{(n)}\) to the system \(\Delta_0\) is usually subject to constraints, including energy, power, and amplitude constraints. These constraints can directly be dealt with. In particular, let \(\mu^{(n)}\) be a normalization constant such that the input
\[ \frac{1}{\mu^{(n)}} u_{\Delta}^{(n)} \]
satisfies the input constraints. Next, apply the scaled input in (23) to the system \(\Delta_0\). Then, by linearity,
\[ \mu^{(n)} \Delta_0 \frac{1}{\mu^{(n)}} u_{\Delta}^{(n)} = \Delta_0 u_{\Delta}^{(n)}. \]

Hence, either the output of \(\Delta_0\) should be scaled by \(\mu^{(n)}\) or the estimators \(\hat{\gamma}_1\) and \(\hat{\gamma}_2\) should be appropriately extended with the reciprocal of the normalization constant \(1/\mu^{(n)}\), i.e., \(\mu^{(n)}\).

3) Noise aspects: In Sec. III-A and Sec. III-B, an iterative estimation approach is presented that applies to the noise-free situation. A key aspect in any uncertainty modelling
procedure for robust control involves the distinction between exogenous noise and unmodelled dynamics. For instance, in model-validation approaches [12], [10], [13], it is tested whether there exists an element in the model uncertainty class and disturbance signal that explain the measured data. In contrast, the proposed power iteration method in Sec. III-A and Sec. III-B involve an estimation problem. In the case where additive noise affects the measurements, i.e., (13) is extended to $y_f = Δu + e$, where $e$ is zero-mean white noise with variance $λ_e$, then two aspects are relevant.

- if $u_f$ is assumed to converge to the eigenvector corresponding to the largest eigenvalue $λ_{max}(TNΔ)$, see (16) for the SISO case, then the estimator (17) is unbiased and its variance decreases with $N$ and is proportional to $λ_e$, see [16, Section 3.1] for a proof.
- in [17], the convergence of the algorithm has been established in the situation where noise is present. In this case, the signal $u_f$ may not converge to the desired eigenvector. In fact, the noise affecting the output can be interpreted as a re-initialization of the algorithm. To avoid bias of the estimator $γ_f$, it has been argued in [17] that the iterative procedure should be extended to ensure convergence.

Summarizing, the aspect of additive noise of the power iteration algorithm has been extensively addressed in [16, Section 3.1],[17]. For the considered mechatronic application in this paper, the signal-to-noise ratio can be made sufficiently large by scaling the input appropriately using the results in Sec. III-C2. As a result, the input $u_f$ converges to the eigenvector corresponding to the largest eigenvalue and the estimators (17) and (22) are unbiased. In addition, due to a good signal to noise ratio and relatively long experiment, the variance of the estimator is small.

IV. TOWARDS A MULTIVARIABLE MODELING PROCEDURE FOR ACHIEVING ROBUST PERFORMANCE

In the previous section, a novel approach has been presented to estimate the $H_∞$ norm $γ$ of the model error in (10). As is argued in Sec. II-D, two aspects are remaining that determine the shape of the model set $P$, i.e., 1) the nominal model $P^*$, and 2) the uncertainty structure leading to $H(P)$ in (9).

The shape of the model set $P$ clearly determines the worst-case performance bound in (6). In this section, a three-step procedure is proposed such that the nominal model $P^*$, the uncertainty structure, i.e., $H(P)$, and the approach in Sec. III-B to estimate $γ$ jointly aim at achieving a small worst-case performance bound in (6). This is of crucial importance in view of the approach in Sec. III-B that only deliver a norm-bound. In contrast, alternative uncertainty modeling procedures often adopt weighting functions to reduce conservatism, e.g., [19, Sec. 7.4.3].

First, the general objective for modeling $P$ is defined, after which the three steps are described in detail. These three steps constitute Contribution C2 of the paper and are summarized as follows.

Procedure 3: Perform the following three steps for identifying $P$:

1) Identify $P$ in (25), see Sec. IV-B.
2) Construct uncertainty model (27), leading to (28), see Sec. IV-C.
3) Apply Procedures 2 and 3, leading to $γ$ and hence $P$ in (9) and (10).

A. Modeling goal

The function $J_{WC}(P,C)$ is a complex function of both $P$ and $C$. By noting that $C^{RP}$ in (5) depends on $P$, i.e., $C^{RP}(P)$, it is desired to determine $P$ such that it minimizes $J_{WC}(P,C^{RP}(P))$, subject to (4). However, this is in general difficult to solve.

The key step in this section is to exploit knowledge of $C^{exp}$, see Sec. II-A, to obtain a tractable approach that is aimed at achieving high performance in (5). Note that $J_{WC}(P,C^{RP}) ≤ J_{WC}(P,C^{exp})$. Hence, $C^{exp}$ provides an upper bound for the guaranteed performance in (5). Hence, the aim is to determine

$$P = \arg\min_{P} J_{WC}(P,C^{exp})$$
subject to $P_0 ∈ P$. (24)

B. Step I: nominal modeling

In the first step, a nominal model $P$ is identified. In view of (24), the control objective is also adopted in control-relevant identification. In particular, the control-relevant identification criterion in [25] is adopted, i.e., $P$ is minimized according to

$$\min_{P} \left\| WT(P_o,C^{exp})V - WT(P,C^{exp})V \right\|_∞.$$  (25)

The actual minimization in (25) is performed using the approach in [20] and is described in Sec. V-A. First, it is shown in Sec. IV-C that this criterion is useful in view of (24).

C. Step II: uncertainty model structure selection

Having identified a nominal model that minimizes (25), the next step is the selection of an uncertainty model structure that extends $P$ to $H(P)$. As in Step 1, the essence lies in selecting the uncertainty structure such that it facilitates solving (24).

To establish the connection between $H(P)$ and the criterion (24), note that the latter can be expressed as an LFT. This involves the construction of a generalized plant as is explained in [19, Sec. 3.8]. In particular, the uncertain model $P$ in (9) is appended with weighting filters and interconnected with $C^{exp}$, leading to the setup in Fig. 7. As a result, $J_{WC}(P,C^{exp}) = \sup_{Δ_u ∈ Δ_u} \left\| Mf2 + Mf1Δ_u(1 - Mf1Δ_u)^{-1}Mf1 \right\|_∞$  (26)

Since (26) depends on a complicated manner on $Δ_u$ and hence $γ$, see (10), a specific approach that relies on the results in [20] is adopted. In particular, in [20], it is suggested to

i) adopt the dual-Youla uncertainty structure [26], [27]

$P^{DY} = \left\{ P | P = (\hat{N} + DcΔ_u)(\hat{D} - NcΔ_u)^{-1}, Δ_u ∈ Δ_u \right\}$,  (27)

where $\{ \hat{N}, \hat{D} \}$ is a robust-control-relevant coprime factorization of $P$ as defined in [20, Sec. 3.3], and
To gain access to \( u_\Delta \) and \( y_\Delta \), note that (27) in closed-loop with \( C^{\exp} \) implemented can be represented as in Fig. 8. Inspection reveals

\[
u_\Delta = \hat{D}^{-1}(r - C^{\exp} \hat{N} u_\Delta - C^{\exp} D_c y_\Delta + N_c y_\Delta),
\]

implying that the reference signal

\[
r = (\hat{D} + C^{\exp} \hat{N}) u_\Delta.
\]

should be applied. Next, observe that

\[
y_\Delta = D_c^{-1}(y - \hat{P}(I + C^{\exp} \hat{P})^{-1} r).
\]

Equations (30) and (31) reveal how experiments can be performed on \( \Delta_o \). This is summarized in the following procedure.

Procedure 4 (Performing experiments on \( \Delta_o \)): Let \( u_\Delta \) be given and perform the following sequence of steps.

1) Compute \( r = (\hat{D} + C^{\exp} \hat{N}) u_\Delta \).
2) Perform a closed-loop experiment on \( P_o \) with \( C^{\exp} \) implemented, i.e., \( y = P_o(I + C^{\exp} P_o)^{-1} r \).
3) Compute \( y_\Delta = D_c^{-1}(y - \hat{P}(I + C^{\exp} \hat{P})^{-1} r) \).

Procedure 4 can directly be implemented in Procedure 2, enabling the data-driven estimation for multivariable uncertainty structures given by (27).

Remark 1: Besides the fact that the specific coprime factor representation used in (27) leads to the result (28), the general representation of (27) has important consequences for the analysis of the algorithm in the presence of noise. Indeed, by the results presented in [28], the impact of noise that enters the feedback loop in fact is equivalent to an open-loop identification problem. Hence, the analysis of noise applied to open-loop systems in [17] directly applies to the closed-loop results presented in this paper.

V. EXPERIMENTAL RESULTS UNCERTAINTY MODELING

In this section, the multivariable modeling procedure presented in Sec. IV is applied to the AVIS in Sec. II-A. Together with Sec. VI, this constitutes Contribution C3 of the paper.

A. Step I. Nominal modeling

In the first step, the nominal model \( \hat{P} \) in (25) is identified. A frequency response function of the closed-loop system \( T(P_o, C^{\exp}) \) is identified using the approach in [24]. The key reason for the intermediate step of frequency response function identification is that it enables the solution of (25) by exploiting the frequency domain interpretation of the \( H_\infty \) norm. By exploiting a multisine experiment design, the approach in [24] enables the accurate identification of frequency response functions by effectively reducing the variance error without introducing bias. By virtue of (8), an estimate of \( P \) can be obtained from the relation \( P = T_{12} T_{22}^{-1} \). The identified frequency response function of \( P_o \) is depicted in Fig. 9.

Next, a model parameterization for \( P \) is used. An 8th order model is used. The reason for this low order is that the criterion (25) essentially shapes the bias of the parametric model. Hence, a low-order model suffices for control purposes. The actual optimization is performed using the algorithm in [20, Section 3.4-3.5]. The resulting parametric model \( \hat{P} \) is

\[
\Delta_o = D_c^{-1}(I + P_o C)^{-1}(P_o - \hat{P}) \hat{D}.
\]

The result (29) reveals that \( \Delta_o \) depends on the model \( \hat{P} = N \hat{D} \), the true system \( P_o \), and \( C^{\exp} = N_c D_c^{-1} \). Hence, it is expected that performing an experiment on \( \Delta_o \) both involves computations using the model \( \hat{P} \), as well as performing experiments with \( P_o \).
depicted in Fig. 9. In addition, the nonparametric frequency response function of $P_o$ is depicted. Inspection reveals that the suspended rigid-body mode at 4 [Hz] and the first resonance at 135 [Hz] are accurately modeled.

B. Step II. Uncertainty model structure

Given the identified nominal model $\hat{P}$ in Step 1, the model uncertainty structure in (27) is constructed. Importantly, the required coprime factorizations in (27) follow directly from the pursued approach in [20], leading to the result (28).

To obtain an idea of the shape of $\Delta_o$, the frequency response function of $\Delta_o$ is identified. To achieve this, Procedure 4 is employed in conjunction with the approach in [24, Chapter 2]. It is emphasized that the resulting frequency response function of $\Delta_o$ is not required for using the power iterations algorithm in Sec. III. However, it does provide insight into the shape of $\Delta_o$ and a reference value for $\gamma$. In particular, the peak value of $\sigma(\Delta_o)$ over frequency equals 1.95, which is attained at a frequency of 296 [Hz]. Note that a rather dense frequency grid has been employed to determine the frequency response function $\Delta_o$, leading to a fairly long experimentation time.

In the next section, Procedure 2 is applied to estimate $\gamma$.

C. Step III. Data-driven estimation of $\gamma$

In this section, the key experimental results of this paper are presented, which involve the application of Procedure 2 in view of uncertainty modeling of the AVIS.

The application of Procedure 2 leads to the following observations.

1) The iteration is initialized with $u(1)$ being zero mean white noise, see Fig. 11 (Iteration 1) and Fig. 12 (Iteration 1) for the corresponding power spectral density $\Psi_{u(1)}$.
2) The output $w(1)$ of $\Delta_o^T \Delta_o$ in Fig. 11 (Iteration 1) is colored noise, since it contains several dominant frequency components.
3) The resulting input $u$ and output $w$ in iteration $n = 3$ are depicted in Fig. 11 (Iteration 3) and Fig. 12 (Iteration 3). Both $u$ and $w$ contain several dominant frequency components. In addition, note that the inputs during the different iterations are normalized with respect to the 2 norm. Interestingly, the output in iteration $n = 3$ already has a significantly larger amplitude compared to the output in iteration $n = 1$, see Fig. 11.
4) After 40 iterations, the input is mostly sinusoidal except for truncation effects, see Fig. 11 and Fig. 12.
5) After convergence, $\gamma^{(40)}$ is nonzero in both $z$-direction and $\phi$-direction, see Fig. 11. Hence, Procedure 2 also determines the worst-case input direction.

Next, estimator (22) is evaluated for all 40 iterations, see Fig. 13. This leads to the following observations. First, the estimate $\hat{\gamma}^{(n)}$ converges for increasing $n$. This leads to

$$\hat{\gamma}^{(40)} = 1.997.$$

Second, when comparing the estimated $\hat{\gamma}^{(40)} = 1.997$ with the result in Sec. V-B, which is based on the frequency response function estimate in Fig. 10, it is observed that Procedure 2 leads to a higher value $\gamma^{(40)}$ compared to $\sup_o \sigma(\Delta_o)$. Interestingly, inspection of Fig. 12 (Iteration 40) reveals that the dominant frequency component is 121 [Hz], which differs significantly from the worst-case frequency of 296 [Hz] that was obtained in Sec. V-B. Hence, due to its guaranteed global convergence, Procedure 2 provides an effective means for determining the worst-case frequency in view of $\mathcal{H}_\infty$ norm estimation. This is in sharp contrast to the situation where a fixed pre-chosen frequency grid is used, as is confirmed by the considered application. In this respect, Procedure 2 is considered as a useful adaptive experiment design approach.

D. Summary

The obtained nominal model $\hat{P}$ in Step I, together with the uncertainty structure in Step II, and the size $\hat{\gamma}^{(40)} = 1.997$ in Step III, lead to a model set $\mathcal{P}$, see (9). In the next section, the model set $\mathcal{P}$ is used to design a robust controller.
VI. CONTROLLER SYNTHESIS AND EXPERIMENTAL IMPLEMENTATION

A. Robust controller synthesis

The model set $\mathcal{P}$ is used to analyze and synthesize several controllers. In particular, the following controllers are considered:

- $C_{\text{exp}}$: initial controller that is described in Sec. II-A.
- $C_{\text{NP}}$: nominal controller $C_{\text{NP}} = \arg\min_{C} J(\hat{P}, C)$
- $C_{\text{RP}}$: robust controller for model set $\mathcal{P}$ in (5).

The controllers $C_{\text{NP}}$ and $C_{\text{RP}}$ are computed using $H_{\infty}$-optimization and skewed-$\mu$-synthesis, see [19] for details.

The controllers are depicted in Fig. 14, whereas the closed-loop process sensitivity functions $\hat{P}(I + C\hat{P})^{-1}$ are depicted in Fig. 15. In addition, the achieved performance for both the model $\hat{P}$ and the model set $\mathcal{P}$ in terms of the control criterion are presented in Table I.

The following observations are made.

1) When comparing $J(\hat{P}, C_{\text{exp}})$ and $J_{\text{WC}}(\mathcal{P}, C_{\text{exp}})$, it is observed that the bound (28) indeed holds and is tight.

2) From Table I, $C_{\text{NP}}$ indeed achieves optimal performance for the nominal model $\hat{P}$. However, $J_{\text{WC}}(\mathcal{P}, C_{\text{NP}})$ is unbounded. Hence, performance and stability cannot be guaranteed when implementing $C_{\text{NP}}$ on $\hat{P}$.

3) The controller $C_{\text{RP}}$ achieves the smallest worst-case performance, i.e., $J_{\text{WC}}(\mathcal{P}, C_{\text{RP}})$. In addition, the performance for the nominal model, i.e., $J(\hat{P}, C_{\text{RP}})$ is also significantly improved when compared to $J(\hat{P}, C_{\text{exp}})$.

4) The improved performance in terms of the criterion coincides with improved disturbance attenuation properties at low frequencies in Fig. 15, see also Sec. II-A.

B. Controller implementation

The synthesized controllers in Sec. VI-A are now implemented for validation. Measured time domain responses are depicted in Fig. 16, whereas the corresponding cumulative power spectral densities are depicted in Fig. 17.

The following observations are made. First, the controller $C_{\text{NP}}$ does not stabilize the actual system. This is observed from the response in the $\phi$-direction in Fig. 16, where the

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**TABLE I**

<table>
<thead>
<tr>
<th>Controller</th>
<th>Minimized criterion</th>
<th>$J(\hat{P}, C)$</th>
<th>$J_{\text{WC}}(\mathcal{P}, C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{\text{NP}}$</td>
<td>None</td>
<td>14.30</td>
<td>16.30</td>
</tr>
<tr>
<td>$C_{\text{RP}}$</td>
<td>$J(\hat{P}, C)$</td>
<td>1.87</td>
<td>$\infty$</td>
</tr>
<tr>
<td></td>
<td>$J_{\text{WC}}(\mathcal{P}, C)$</td>
<td>4.80</td>
<td>5.02</td>
</tr>
</tbody>
</table>

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Fig. 11. Power iterations: iterations 1 (top), 3 (middle), 40 (bottom) of Procedure 2 - measured inputs $u_{\Delta}$ and outputs $w_{\Delta}$.

Fig. 12. Power iterations: iterations 1 (top), 3 (middle), 40 (bottom) of Procedure 2 - power spectral density of the input $u_{\Delta}$.

Fig. 13. Estimated norm $\hat{\gamma}_{2}^{(n)}$.

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**A. Robust controller synthesis**

The model set $\mathcal{P}$ is used to analyze and synthesize several controllers. In particular, the following controllers are considered:

- $C_{\text{exp}}$: initial controller that is described in Sec. II-A.
- $C_{\text{NP}}$: nominal controller $C_{\text{NP}} = \arg\min_{C} J(\hat{P}, C)$
- $C_{\text{RP}}$: robust controller for model set $\mathcal{P}$ in (5).

The controllers $C_{\text{NP}}$ and $C_{\text{RP}}$ are computed using $H_{\infty}$-optimization and skewed-$\mu$-synthesis, see [19] for details.

The controllers are depicted in Fig. 14, whereas the closed-loop process sensitivity functions $\hat{P}(I + C\hat{P})^{-1}$ are depicted in Fig. 15. In addition, the achieved performance for both the model $\hat{P}$ and the model set $\mathcal{P}$ in terms of the control criterion are presented in Table I.

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2) From Table I, $C_{\text{NP}}$ indeed achieves optimal performance for the nominal model $\hat{P}$. However, $J_{\text{WC}}(\mathcal{P}, C_{\text{NP}})$ is unbounded. Hence, performance and stability cannot be guaranteed when implementing $C_{\text{NP}}$ on $\hat{P}$.

3) The controller $C_{\text{RP}}$ achieves the smallest worst-case performance, i.e., $J_{\text{WC}}(\mathcal{P}, C_{\text{RP}})$. In addition, the performance for the nominal model, i.e., $J(\hat{P}, C_{\text{RP}})$ is also significantly improved when compared to $J(\hat{P}, C_{\text{exp}})$.

4) The improved performance in terms of the criterion coincides with improved disturbance attenuation properties at low frequencies in Fig. 15, see also Sec. II-A.

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The synthesized controllers in Sec. VI-A are now implemented for validation. Measured time domain responses are depicted in Fig. 16, whereas the corresponding cumulative power spectral densities are depicted in Fig. 17.

The following observations are made. First, the controller $C_{\text{NP}}$ does not stabilize the actual system. This is observed from the response in the $\phi$-direction in Fig. 16, where the
Fig. 14. Controllers: $C^{\exp}$ (solid blue), $C^{NP}$ (dashed red), $C^{RP}$ (dashed-dotted green).

Fig. 15. Closed-loop process sensitivities: $\hat{P}(I+C^{\exp}\hat{P})^{-1}$ (solid blue), $\hat{P}(I+C^{NP}\hat{P})^{-1}$ (dashed red), $\hat{P}(I+C^{RP}\hat{P})^{-1}$ (dashed-dotted green). In addition, the nominal model $\hat{P}$ is depicted (dotted magenta).

Fig. 16. Measured time domain responses: $C^{\exp}$ (solid blue, top), $C^{NP}$ (dashed red, middle), $C^{RP}$ (dashed-dotted green, bottom). Here, low-frequency noise (normally distributed white noise, filtered through a low-pass filter with cut-off frequency of 8 Hz) is added to the system in both the $z$-translation and $\phi$-rotation simultaneously.

Fig. 17. Cumulative power spectral density (CPS) of the measured error signals in Fig. 16: $C^{\exp}$ (solid blue, top), $C^{RP}$ (dashed-dotted green, bottom).

system hits a safety guardrail due to the unstable behavior. This corroborates the results in Table I, where $J_{WC}(P_{r}, C^{NP})$ is unbounded, and hence no performance and stability guarantees can be given when implementing $C^{NP}$ on the true system $P_{r}$.

Second, the experimental controller $C^{\exp}$ and optimal robust controller $C^{RP}$ both stabilize the true system, which is revealed by the stationary behavior in Fig. 16. In addition, the controller $C^{RP}$ leads to a significantly better performance when compared to $C^{\exp}$, which is visible from the time domain responses in Fig. 16 and especially from the cumulative power spectral density in Fig. 17. In particular, the controller $C^{RP}$ leads to a performance improvement of more than a factor 4 in $z$-translation and more than a factor 2 in $\phi$-rotation.

VII. DISCUSSION AND CONCLUSION

A. Discussion

1) A model-error modeling perspective: From the perspective of model-error modeling, the proposed approach does not require severe prior assumptions as in alternative approaches. Hence, it provides stronger guarantees in robust control design.

As is experimentally shown in Sec. V, Procedure 2 leads to a higher and hence more accurate estimation of the $\mathcal{H}_{\infty}$ norm compared to the use of an $a$ priori chosen frequency
grid, as is exemplified in Sec. V-B and also done in, e.g., [8]. From this perspective, Procedure 2 can be seen as an adaptive experiment design procedure. As a result, Procedure 2 reduces estimation errors due to the use of a discrete frequency grid when compared to the uncertainty modeling procedure in [8]. From this perspective, it is emphasized that the use of prior assumptions to bound the interpolation error, as is done in, e.g., [9], is likely to lead to conservative results, as is also argued in [11, 9.5.2].

2) Connections to related (iterative) approaches: The approach presented in Sec. III is related to several identification and iterative control algorithms. For a connection to maximum likelihood estimation, see [17, Remark 10].

The iterative algorithm in Sec. III can be related to iterative learning control (ILC) algorithms [29]. For simplicity, consider the situation of Procedure 1, in which case the input during the next iteration is given by

\[ u^{(n+1)}_\Delta = T_N w^{(n)}_\Delta, \]

which closely resembles ILC update laws that are typically also a linear function of the signals \( w^{(n)}_\Delta \) and \( u^{(n)}_\Delta \). However, there are some principal differences when comparing typical ILC algorithms and Procedure 1. To show this, assume as in Sec. III-C that an input normalization is implemented\(^1\) as in (23), with \( \mu^{(n+1)} = \|w^{(n)}_\Delta\|_2 \), i.e., the 2 norm of the input is normalized to 1, as is also done in [17]. This leads to the input to the true system

\[ u^{(n+1)}_\Delta = \frac{1}{\|w^{(n)}_\Delta\|_2} w^{(n+1)}_\Delta = \frac{1}{\|w^{(n)}_\Delta\|_2} T_N u^{(n)}_\Delta, \]

with output

\[ w^{(n+1)}_\Delta = \Delta_0 w^{(n+1)}_\Delta = \Delta_0 \frac{1}{\|w^{(n)}_\Delta\|_2} T_N w^{(n)}_\Delta, \tag{32} \]

In (32), the operator \( \Delta_0 \frac{1}{\|w^{(n)}_\Delta\|_2} T_N \) maps \( \mathbb{R}^N \) onto itself. Note that the amplification of \( \Delta_0 \) for the input \( w^{(n)}_\Delta \) equals the 2 norm of the output \( u^{(n+1)}_\Delta \). Under mild assumptions, which are related to the comment in [17, Remark 9], \( \|w^{(n+1)}_\Delta\|_2 > \|w^{(n)}_\Delta\|_2 \). Since \( T \) has induced 2 norm equal to one, the induced 2 norm of \( \Delta_0 \frac{1}{\|w^{(n)}_\Delta\|_2} T_N \) is larger than one. As a result, (32) is a Lipschitz continuous function with Lipschitz constant larger than one, hence (32) is not a contraction. In contrast, ILC algorithms are also of the form (32), but designed such that the iteration (32) is a contraction map, corresponding to a Lipschitz constant that is strictly smaller than one. This ensures convergence to a unique fixed point, generally being an error signal of the system converging to zero.

The novel multivariable result of Procedure 2 provides further extensions to the implementation of ILC algorithms. In particular, in certain ILC algorithms referred to as adjoint ILC algorithms [30], it is desired to filter through the transpose of the system. It is clear that the use of time reversal operators as in (15) enables a direct evaluation on the true system and effectively renders the need for a model in ILC algorithms superfluous. Interestingly, a slight modification of Procedure 2 directly enables its use for multivariable ILC algorithms.

\(^1\)As is argued in Sec. III-C, this requires a slight modification of the estimators (17) and (22), see also [17].

B. Conclusion

In this paper, a novel data-driven \( H_\infty \) norm estimation procedure, which relies on iterative experiments on the system, is proposed for multivariable systems. This procedure is employed for model-error modeling purposes, and embedded in a system identification and robust control design framework. The key advantage of the proposed approach is that it provides guarantees to satisfy (4) without relying on severe prior assumptions that are adopted in alternative approaches. Experimental results of an active vibration isolation system show fast convergence of the algorithm and its ability to determine both the worst-case frequency and worst-case input and output directions in \( H_\infty \) norm estimation. In addition, the experimental results confirm its ability to provide enhanced vibration isolation properties through a robust control design. Finally, the experimental results confirm that the iterative algorithm is useful for experiment design, i.e., to iteratively/adaptively determine improved inputs for \( H_\infty \) norm estimation, leading to improved results compared to existing approaches.

The proposed results can be directly extended. Besides its possible use in iterative learning control algorithms, see Sec. VII-A2, it can be directly used in any multivariable estimation problem involving eigenvalues. In addition, the proposed approach exhibits global convergence for multivariable systems (see Section III-C). This requires an additional number of experiments as is illustrated in Fig. 6. Present research focuses on reducing the number of required experiments.

ACKNOWLEDGEMENTS

The authors would like to thank Okko Bosgra, Robbert van Herpen, and Bo Wahlberg for their contribution to this work.

REFERENCES

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