On the Properties of Iterative Schemes

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Background
The idea of using iterative experiments appears in many system identification approaches. In this research, the value of iterations and the limits of accuracy are investigated.

\( \ell_2 \)-Induced Gain Estimation
Case study: gain estimation [1]
- perform iterative experiments on \( G \), see Fig. 1 (left)
- corresponding transfer function: Fig. 1 (right)
- resulting \( u_k \) for \( k \rightarrow \infty \):
  - sinusoid with frequency \( \omega^* \)
  - result: \( \lim_{k \rightarrow \infty} \gamma_k = \lim_{k \rightarrow \infty} \| u_k \|_2 = \| G \|_\infty \)
  - resembles power iterations method

Extended setup
A more realistic setup is considered in Fig. 2, including
- noise: \( \varepsilon_k \) is assumed ZMWN with variance \( \lambda_e \)
- normalization: \( \alpha_k \) due to bound on \( \| u_k \|_2 \)

Analysis
Resulting Spectrum
The extended setup in Fig. 2 is investigated through a spectral analysis. Assuming convergence for \( k \rightarrow \infty \), then

\[
\Phi_{u_k}(\omega) = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |G(\omega)|^2 \Phi_{u_k}(\omega)d\omega + \lambda_e \right) - |G(\omega)|^2
\]

Observations:
- \( \Phi_{u_k}(\omega) \) has maximum at \( \omega^* \)
- \( \Phi_{u_k}(\omega) \geq \lambda_e \)

Convergence Analysis
Extended system satisfying eigenvalue equation
\[
\frac{1}{\alpha_\infty^2} \left[ \Phi_{u_k}(\omega) \right] = \left[ \frac{1}{\pi} \int_{-\pi}^{\pi} |G(\omega)|^2 \Phi_{u_k}(\omega)d\omega + \lambda_e \right] \left[ \Phi_{u_k}(\omega) \right]
\]
- convergence proof via Hilbert projective metric [2]
- \( \omega \)-discretization: computation of \( \Phi_{u_k}(\omega) \) for given \( G \)

Example
Given \( G \) in Fig. 1, (1) gives \( \Phi_{u_k}(\omega) \) in Fig. 3.

Implications
Bias Analysis for Non-Parametric Estimation
The nonparametric estimator in [1] can be written as

\[
\hat{\gamma}_k = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} |G(\omega)|^2 \Phi_{u_k-1}(\omega)d\omega}
\]

Combining this with the limit spectrum \( \Phi_{u_\infty} \) reveals
- \( \hat{\gamma}_\infty = \| G \|_\infty \) for \( \lambda_e = 0 \)
- \( \hat{\gamma}_\infty < \| G \|_\infty \) for \( \lambda_e > 0 \) (biased)

Limits of Accuracy
Fisher information matrix
\[
I_\theta = \sum_{i=1}^{k} \frac{1}{2\pi \lambda_e} \int_{-\pi}^{\pi} G'(e^{i\omega}, \theta) \Phi_{u_i}(\omega) \left( G'(e^{-j\omega}, \theta) \right)^T d\omega
\]
- additivity property
  - only increase of information for increasing \( k \)
  - if \( \psi_{u_i}(\omega) = \delta(\omega - \omega^*) \)
  - optimal accuracy for \( G(\omega^*) \) for FIR model [3]

Final Remarks
Analysis of iterative experiments in identification
- case study: non-parametric \( \ell_2 \)-gain estimation

Present extensions
- finite time implementation: time reversal
- only one experiment per iteration
- MIMO: multiple experiments per iteration
- nonparametric Hankel-norm estimation

Future work: analysis of the value of iterations in
- iterative learning control
- iterative identification and control
- iterative feedback tuning

References