 Suppressing intersample behavior in iterative learning control

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Abstract

Iterative Learning Control (ILC) is a control strategy to improve the performance of digital batch repetitive processes. Due to its digital implementation, discrete time ILC approaches do not guarantee good intersample behavior. In fact, common discrete time ILC approaches may deteriorate the intersample behavior, thereby reducing the performance of the sampled-data system. In this paper, a generally applicable multirate ILC approach is presented that enables to balance the at-sample performance and the intersample behavior. Furthermore, key theoretical issues regarding multirate systems are addressed, including the time-varying nature of the multirate ILC setup. The proposed multirate ILC approach is shown to outperform discrete time ILC in realistic simulation examples.

1. Introduction

Good at-sample performance is a necessary, yet not sufficient condition for good continuous time performance in sampled-data systems. Sampled-data systems include many physical systems that evolve in continuous time and are controlled by a digital controller (Chen & Francis, 1995). Since the plant evolves in continuous time, it is most natural to evaluate performance in continuous time, i.e., to analyze both the at-sample response and the intersample behavior. In fact, achieving a good at-sample performance can go at the expense of a poor intersample behavior (Oomen, van de Wal, & Bosgra, 2007b). In this perspective, any high performance digital control design approach for a sampled-data system should be accompanied by a thorough intersample behavior consideration.

Iterative Learning Control (ILC) (Bien & Xu, 1998; Bristow, Tharayil, & Alleyne, 2006; Gorinevsky, 2002; Moore, 1999) is a high performance digital control strategy used to improve the performance of batch repetitive processes, by iteratively updating the command signal from one experiment (trial) to the next. Basically, ILC results in a command signal that can compensate for trial-invariant deterministic components in the discretized error signal, even if imperfect plant knowledge is available.

Although ILC for discrete time LTI systems based on discretized error signals has been well developed, a systematic ILC approach for sampled-data systems is lacking. In Chien (1998) and Sun and Wang (2001), an ILC approach is presented that is suitable for nonlinear sampled-data systems. The approach essentially deals with the fact that nonlinear systems do not have equivalent discrete time representations as is the case for LTI systems (Åström & Wittenmark, 1990; Chen & Francis, 1995), and does not address the intersample behavior. In contrast, the purpose of this paper is to develop a systematic ILC approach for sampled-data systems that extends discrete time ILC approaches by addressing both the at-sample performance and intersample behavior.

In LeVoci and Longman (2004) and Longman and Lo (1997), it is shown that sampling zeros (Åström, Hagander, & Sterlynb, 1984) can indeed deteriorate the intersample response in ILC and ad hoc solutions are provided for performance improvement. In Hara, Tetsuka, and Kondo (1990), repetitive control, which is closely related to ILC, for sampled-data systems is considered. The proposed solutions to handle the intersample behavior require modifications to the sampler and hold function. However, these functions are commonly unalterable in practice. In Ishii and Yamamoto (1998) and Langari and Francis (1994) repetitive control of sampled-data systems is considered by employing lifted system descriptions. Although intersample behavior is addressed in the approach, these methods do not directly extrapolate to ILC due to its batch repetitive behavior.

Sampled-data systems can be considered as the limiting case of multirate systems (Chen & Francis, 1995; Vaidyanathan, 1993). In Zhang, Wang, Ye, Wang, and Zhou (2007) and Zhang, Wang, Zhou, Ye, and Wang (2008), multirate ILC is employed to ensure that an Arimoto-type ILC controller has desired properties,
including an exponential decay rate of the tracking error and effective handling of initial state errors. Though multirate aspects are used in Zhang et al. (2007) and Zhang, Wang, Zhou et al. (2008), the signals are sampled before learning. Hence, the intersample behavior in the signals is essentially discarded and instead a downsampled, discrete time ILC problem is obtained. In Moore (1993, Section 5.3), a multirate ILC approach is presented that uses a faster input rate to avoid unbounded input signals in case of nonminimum phase systems. However, an analysis of the intersample behavior is not provided. In Xu, Lee, and Zhang (2005), a multirate ILC approach with a faster input rate in conjunction with an estimator is presented to enable the ILC algorithm to exploit the fast input rate. However, the approach does not guarantee satisfactory intersample behavior.

The main contribution of this paper is a systematic procedure that extends existing ILC approaches for sampled-data systems by explicitly dealing with the intersample behavior. By employing optimal ILC (van de Wijdeven & Bosgra, 2008), the intersample response is explicitly quantified in the optimization problem. In addition, a multirate ILC approach is pursued, where the response is explicitly quantified in the optimization problem. By employing the fast input rate. However, the approach does not suffer from aliasing problems as in Zhang, Wang, Ye, and Zhou (2008), however, the resulting multirate ILC problem is time-varying. Time-variance is caused by the fact that the input and output signals of the resulting ILC controller have different sampling frequencies, which is appropriately dealt with in this paper.

The paper is organized as follows. In Section 2, the sampled-data ILC problem is formulated. In Section 3, the multirate ILC setup is defined and the main theoretical issues regarding multirate systems are presented. In Section 4, the main results regarding the ILC controller design for multirate systems are presented. In Section 5, simulation examples, which address sampling zeros (Åström et al., 1984) and aliased disturbances (Oomen et al., 2007b), illustrate the necessity of dealing with intersample behavior in ILC. Finally, in Section 6, concluding remarks are given.

\textbf{Notation.} Throughout, \( t \subseteq \mathbb{Z} \) and \( t_c \subseteq \mathbb{R} \) denote discrete time and continuous time, respectively. In block diagrams, continuous time signals are represented by solid lines, and fast sampled discrete time signals are represented by dashed lines, and fast sampled discrete time signals are indicated by dotted lines. All systems are assumed to be single-input single-output, finite dimensional, and linear time invariant (LTI). Generalization to the multivariable case is conceptually straightforward. The delay operator \( D \) is defined by \( (D_s f)(t) = f(t - \tau) \), where \( \tau \in \mathbb{R} \).

### 2. Problem definition

In this section, the ILC problem for closed-loop sampled-data systems is defined. The considered setup is depicted in Fig. 1. Here, \( y = P u \), where \( P \) denotes the continuous time plant. The plant input is given by

\[
\begin{align*}
\dot{u} &= h^l(u^l + C^{dl}S^l e) \\
e &= r - y,
\end{align*}
\]

where \( r \) is the reference signal and \( C^{dl} \) is a discrete time controller operating at a sampling frequency \( f^l \). In (1), the ideal sampler and zero-order-hold are defined by

\[S^l: e(t_c) \to e^l(t) \], \hspace{1cm} e^l(t_c) = e(t_c h^l)
\]

\[h^l: u^l(t) \to u(t_c) \], \hspace{1cm} u(t_c h^l + \tau) = u^l(t_c), \hspace{1cm} \tau \in [0, h^l),
\]

respectively, where \( t_c \in \mathbb{R} \), sampling frequency \( f^l = \frac{1}{h^l} \), and \( h^l \) denotes the sampling time. The variable \( q \) represents a low or high sampling frequency. Specifically, the superscript \( l \) refers to the low sampling frequency \( f^l \), hence \( q = 1 \to f^q = f^l \). Similarly, the superscript \( h \) refers to the high sampling frequency \( f^h \).

Typically, the sampling time \( h^l \) of the feedback controller is lower bounded, since a new control signal has to be computed in real-time. The command signal \( u^l \) is generated by the ILC algorithm. It is assumed that \( u^l \) operates at the same sampling frequency as the feedback controller, since this is commonly encountered in digital computer implementations. Finally, it is remarked that in (1) and (2), sampled values of the reference signal \( r \) could be used, i.e., \( S^l r \), since by linearity, see Proposition 7, \( S^l e = S^l \tau - S^l y \). However, for the forthcoming sampled-data analysis, it is instructive to consider \( r \) as a continuous time signal.

The main problem considered in this paper is given by the optimal sampled-data problem.

\textbf{Definition 1 (Optimal Sampled-Data ILC).} Given the norm-based criterion \( J_{SD}(u^l, e) \), the optimal sampled-data ILC problem amounts to determining

\[w^l_{SD} = \arg\min_{u^l} J_{SD}(u^l, e).\]  

In the optimal sampled-data ILC problem, an optimal discrete time command signal \( u^l \) is determined that achieves good continuous time performance \( e \), see Fig. 1. This implies that the problem involves both continuous time and discrete time signals. This definition of sampled-data systems is consistent with the literature on sampled-data systems, including Chen and Francis (1995). In contrast, standard ILC algorithms (Bristow et al., 2006; Gorinevsky, 2002; Moore, 1999) employ discrete time measurements of the error \( e \). In particular, the optimal discrete time ILC problem is given by the following definition.

\textbf{Definition 2 (Optimal Discrete Time ILC).} Given the norm-based criterion \( J_{DT}(u^l, e) \), the optimal discrete ILC problem amounts to determining

\[w^l_{DT} = \arg\min_{u^l} J_{DT}(u^l, e^l).\]  

In the discrete time ILC criterion \( J_{DT}(u^l, e^l) \), only the at-sample response is minimized, whereas the sampled-data criterion \( J_{SD}(u^l, e) \) includes the intersample response. This implies that discrete time ILC approaches may result in poor intersample behavior, which is quantified by

\[J_{SD}(w^l_{DT}, e) \geq J_{SD}(w^l_{SD}, e).\]  

The gap in (7) depends on the particular system and exogenous signals and can become arbitrarily large, as is illustrated in Section 5.

In this paper, the sampled-data ILC problem in Definition 1 is addressed. The sampled-data setup can theoretically be handled using lifted system descriptions, e.g., Bamieh, Pearson, Francis, and Tannenbaum (1991), Chen and Francis (1995) and Yamamoto (1994). However, actual implementation of the resulting ILC controller requires a continuous time measurement of \( e \) that is unavailable in a digital computer environment. In the next section, a multirate approximation to the sampled-data ILC problem in Definition 1 is presented to enable digital computer implementation.
3. Multirate setup

3.1. Multirate ILC setup

To enable digital computer implementation of ILC controllers that explicitly address intersample behavior, a multirate approximation of the sampled-data ILC problem, see Definition 1 and Fig. 1, is presented. The key idea is that in many applications, it is possible to measure error signals at a higher sampling frequency \( f^h \) than the frequency \( f^l \) at which \( C^d, l \) operates. Indeed, the bound on \( f^l \) is caused by the fact that in feedback control the new control input has to be computed in real-time. In contrast, although ILC is implemented in real-time, it can exploit the time in between trials for the actual computation of the command signal.

To enable usage of standard tools from multirate signal processing (Vaidyanathan, 1993), the following assumption is imposed.

**Assumption 3.** Let the sampling frequencies \( f^l \) and \( f^h \) be related by
\[
f^h = H f^l, \quad 1 < F \in \mathbb{Z}. \tag{8}
\]

The multirate ILC setup is depicted in Fig. 2. Herein, \( \delta^h \) and \( \mathcal{H}^h \) are defined in (3) and (4), respectively, where \( q = h \), i.e., a high sampling frequency \( f^h \) is assumed. The downsampling operator \( \delta_d \) is defined by
\[
\delta_d : e^h(t) \mapsto e^l(t), \quad e^l(t) = e^h(F t), \quad t \in \mathbb{R}. \tag{9}
\]
In addition, the multirate zero-order-hold \( \mathcal{H}_u \) is defined as Oomen et al. (2007b)
\[
\mathcal{H}_u = t^F(z) \delta_u, \tag{10}
\]
where the upsampler \( \delta_u \) and zero-order-hold interpolator \( t^F(z) \) are defined as
\[
\delta_u : u^h(t) \mapsto \tilde{u}^h(t), \quad \tilde{u}^h(t) = \begin{cases} u^h \left( \frac{t}{F} \right) & \text{for } t \in \mathbb{R}, \frac{t}{F} \in \mathbb{Z} \\ 0 & \text{for } t \in \mathbb{R}, \frac{t}{F} \notin \mathbb{Z}. \end{cases} \tag{11}
\]
\[
t^F(z) = \sum_{f=0}^{F-1} z^{-f}. \tag{12}
\]

**Remark 4.** An ideal sampler is considered in (3) and (9). This is not restrictive, since sensor dynamics and anti-aliasing filters can be incorporated in \( P \). Signal quantization is not addressed, see, e.g., Bamieh (2003) for appropriate extensions. In addition, the multirate zero-order-hold can be defined directly instead of (10). However, the present definition via the upsampler, which inserts zero values, is consistent with standard definition (Vaidyanathan, 1993), and enables a straightforward incorporation of general interpolators.

The multirate setup in Fig. 2 leads to the following optimal multirate ILC problem.

**Definition 5 (Optimal Multirate ILC).** Given the norm-based criterion \( J_{MR}(w^l, e^h) \), the optimal multirate ILC problem amounts to determining
\[
\begin{align*}
\hat{w}^l_{MR} &= \arg \min_{w^l} J_{MR}(w^l, e^h).
\end{align*} \tag{13}
\]

The optimal multirate ILC problem is a sensible approximation of the sampled-data ILC problem, since under appropriate technical conditions (Kannai & Weiss, 1993)
\[
J_{SD}(w^l, e) - J_{MR}(w^l, e^h) \to 0 \quad \text{for} \quad h^h \to 0. \tag{14}
\]

In practice, \( F \) and hence \( h^h \), see (8), are upper bounded. In the zero-order-hold case, a low \( F \), e.g., \( 2 \leq F \leq 5 \) typically suffices due to the low-pass character of the interpolator (Oomen, van de Wal, & Bosgra, 2007a; Oomen et al., 2007b). In the next section, the multirate setup in Fig. 2 is analyzed in detail.

3.2. Multirate analysis

In this section, the multirate setup in Fig. 2 is analyzed to enable proper formulation of the multirate system description for ILC. Initially, admissible feedback controllers and sampling frequencies are discussed. Subsequently, linearity and time-variance of the multirate ILC setup are investigated.

Throughout, it is assumed that the feedback system in Fig. 1 is well-posed and internally stable (Zhou, Doyle, & Glover, 1996). A nonpathological sampling frequency is assumed, which is formalized in the following assumption (Kalman, Ho, & Narendra, 1963).

**Assumption 6.** Let \( A, B, C, D \) be a minimal state space realization of \( P \). Then, it is assumed that \( A \) does not have two eigenvalues \( \lambda_p(A) = \sigma_p + j \omega_p \) and \( \lambda_q(A) = \sigma_q + j \omega_q \), \( p \neq q \) that satisfy
\[
(\lambda_p - \lambda_q)\sigma_p = \sigma_q \quad \text{and} \quad \omega_p = \omega_q + 2 \pi f^l, \quad r \in \mathbb{Z} \setminus \{0\}. \tag{15}
\]

**Assumption 6** ensures sampling preserves observability and controllability. Hence, there cannot be any unstable modes in the intersample behavior.

Next, linearity and time-variance of the multirate system are investigated. The following proposition reveals that sampling and hold operators are linear.

**Proposition 7.** The operators \( \delta^h, \mathcal{H}^h, \delta_u, \) and \( \mathcal{H}_u \) in (3), (4), (9), and (10), respectively, are linear.

The proof of Proposition 7 follows directly from the definition of linearity (Zhou et al., 1996).

To analyze time-variance, the notion of periodically time-varying operators is useful.

**Definition 8 (Bamieh et al., 1991; Chen & Francis, 1995).** An operator \( G \) is periodically time-varying with period \( h \) if it commutes with the delay operator \( D_h \), i.e., \( D_h G = G D_h \), where \( h \in \mathbb{R} \).

If Definition 8 does not apply, then the system is considered time-varying. Recall that time-invariance is a special case of periodic time variance.

**Definition 9.** An operator \( G \) is time-invariant if it commutes with \( D_h \) for all \( h \in \mathbb{R} \).

The following results reveal that linearity is preserved in multirate systems, yet time-invariance may be lost, i.e., multirate systems can become linear periodically time-varying (LPTV).

**Proposition 10.** Consider the sampler \( \delta^h \) and zero-order-hold \( \mathcal{H}^h \). Then,
\begin{itemize}
  \item[(a)] \( \delta^h \mathcal{H}^h \) is an LTI operator \\
  \item[(b)] \( \mathcal{H}^h \delta^h \) is an LPTV operator with period \( h^h \).
\end{itemize}

**Proof.** (a): (3) and (4) yield \( \delta^h \mathcal{H}^h = I \) that is LTI. (b): follows from (3) and (4) and evaluating the delay operator \( D_h \), for different values of \( r \). \( \square \)
The following results are required for the manipulation of multirate systems in the next sections.

**Proposition 11.** Consider the setup in Fig. 2 and let \((A, B, C, D)\) be defined as in Assumption 6. Then, the following properties hold:

(a) \(x^f = \delta_d x^h\)
(b) \(A^h = A^h A^f\)
(c) \(p^{d,h} = \delta_d p^h A^h\) has state-space realization \((A^h, B^h, C, D)\)
(d) \(p^{d,h} = \delta_d p^h A^h\) has state-space realization \((A^h, B^h, C, D)\)
(e) \(p^{d,h}\) and \(p^{d,h}\) are LTI and minimal,

where \(A^h = e^{\delta_d b^h}, B^h = \int_0^{\delta_d b^h} e^{\delta_d b^h} \mathrm{d}x B, A^h \) and \(B^h\) follow from \(\left[\begin{array}{cc} A^h & B^h \\ * & \end{array} \right] = \left[\begin{array}{cc} a^h & b^h \\ \cdot & \end{array} \right]\), where \(*\) is not used in further computations.

**Proof.** (a), (b): follows from (3), (9) and (4), (10), respectively.
(c): follows by integrating differential equations, (d): follows by successive substitution in the difference equations. (e): LTI follows similarly as in Proposition 10, and the following lemmas are required for the manipulation of finite time LTI systems.

**Proposition 12.** Consider the closed-loop system of Fig. 2. Then, the operators mapping \(w^f\) and \(r\) to \(e^f\) and \(e^h\) are given by

\[ 
\begin{align*}
\dot{e}^f &= (I - p^{d,h}(I + C^d p^{d,h})^{-1} C^d) \dot{e}^h - J_{DT} w^f, \\
\dot{e}^h &= (I - p^{d,h} H_{d}(I + C^d p^{d,h})^{-1} C^d) \delta \dot{e}^h - J_{MR} w^f, \\
J_{MR} &= p^{d,h} H_{d}(I + C^d p^{d,h})^{-1}.
\end{align*}
\]

In addition, the mapping \(-J_{DT}: w^f \mapsto e^h\) is LTI, whereas the mapping \(-J_{MR}: w^f \mapsto e^f\) is time-varying.

**Proof.** Eqs. (16) and (18) result from the interconnection structure in Fig. 2 and Proposition 11. The fact that \(w^f \mapsto e^f\) is LTI follows from Proposition 11(e). Time-variance of \(w^f \mapsto e^f\) is best interpreted by the fact that the delay operator applied to \(w^f\) corresponds to time steps \(h^f\), whereas the delay operator applied to \(e^h\) corresponds to time steps \(h^h\). Additionally, if a fictitious signal satisfying \(w^f = \delta_d w^h\) is introduced, then the mapping \(w^h \mapsto e^h\) is LTV, see Proposition 10(b). The main consequence of Proposition 12 is that discrete time ILC can resort to standard LTI design techniques. The mapping \(w^f \mapsto e^h\), as required in multirate ILC, is time-varying, hence transfer functions in the usual sense do not apply and standard LTI design techniques cannot be employed. In the next section, a general framework for multirate ILC is proposed that deals with the time-varying nature of the mapping \(w^f \mapsto e^h\) and provides a solution to the multirate ILC problem in Definition 5.

### 4. Multirate ILC

In this section, the solution to the multirate ILC problem is presented. In Section 4.1, appropriate finite time system descriptions of the required time-varying operators are presented. Then, in Section 4.2, the optimal ILC controller is presented. Finally, design aspects and convergence results are discussed in Section 4.3.

#### 4.1. Finite time system descriptions

Consider the system \(P^{d,h}\) with Markov parameters \(m_i\), operating over a finite time interval \(t_i \in [0, N_i - 1] \subseteq t\), where the state of the system is reset to zero after each trial. Then, the input–output behavior is represented by its convolution matrix (Frueh & Phan, 2000; Phan & Longman, 1988):

\[ 
\begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix},
\]

For causal SISO LTI systems, \(P^{d,h} \in \mathbb{R}^{N_i \times N_i}\) is a square lower triangular Toeplitz matrix that maps input vector \(u^f \in \mathbb{R}^{N_i}\) to output vector \(y^h \in \mathbb{R}^{N_i}\). The signals \(u^f\) and \(y^h\) are obtained by stacking the corresponding time signals for finite interval \(t_i \in [0, N_i - 1] \subseteq t\), where \(N_i\) in vectors.

**Lemma 13.** Consider two operators \(A\) and \(B\) with finite time block triangular Toeplitz representation \(A_0 B_0\), respectively, and \(C = B A\). If \(A\) and \(B\) are lower triangular block Toeplitz, i.e., if \(A\) and \(B\) are linear and causal operators, then \(C = A B\).
Lemma 14. Let $A$ be a causal operator with causal inverse $A^{-1}$ and finite time lower triangular Toeplitz representation $A$. Then, $A^{-1}$ has a finite time lower triangular Toeplitz representation $A^{-1} = (A)^{-1}$.

Lemmas 13 and 14 are proved in Böttcher and Silbermann (1999, Section 6.2), Zimmerman (1989) and Kailath and Koltracht (1986), respectively.

4.2. Optimal multirate ILC

To determine the ILC controllers, the finite time mapping between the command signal and error in Proposition 12 is required, see also van de Wijdeven and Bosgra (2008).

Proposition 15. Consider the interconnection structure in Fig. 2. Then, the finite time mappings $-J_{f_{DT}} : u^l \mapsto e^l$ and $-J_{MR} : u^l \mapsto e^l$, corresponding to (17) and (19), respectively, are given by

$$J_{f_{DT}} = \mathcal{P}d^1 (\mathcal{I}_{w} + \mathcal{C}_d^1 (\mathcal{P}d^1))^{-1} \in \mathbb{R}^{N \times N}$$

and

$$J_{MR} = \mathcal{P}d^h (\mathcal{I}_{w} + \mathcal{C}_d^1 (\mathcal{P}d^1))^{-1} \in \mathbb{R}^{N \times N}.$$ (23)

In addition, $J_{f_{DT}} = J_{f_{DT}}$.

Proof. Follows by applying Lemmas 13 and 14 to the results in Proposition 12.

With the relevant systems and signals for ILC defined, the optimal ILC controller can be designed. In general, the objective in ILC control design is to minimize the error during trial $k+1$, $k \in \mathbb{Z}_+$, in an appropriate norm by determining a suitable command signal. In optimal ILC, additional criteria are included in the objective to bound the command amplitude and the change in amplitude of the command signal from trial $k$ to trial $k+1$. Specifically, the criterion for determining the command input $u^l$ in trial $k+1$ is given by

$$J_{k+1} = e^l_{k+1} W e^l_{k+1} + w^l_{k+1} W w^l_{k+1}$$

$$+ (w^l_{k+1} - w^l_k)^T W (w^l_{k+1} - w^l_k),$$ (25)

where $q = l$ for discrete time ILC and $q = h$ in multirate ILC. In addition, $w^l$, $w^h$, and $w^l_{\Delta w}$ are weighting matrices of appropriate sizes.

The resulting optimal multirate ILC controller is the main result of this section and is given by the following proposition.

Proposition 16. Given a multirate system $-J_{MR} : u^l \mapsto e^l$ and criterion (25). Then, the optimal multirate ILC controller $(Q_{MR}, L_{MR})$ is given by

$$w^l_{k+1} = Q_{MR} w^l_k + L_{MR} e^l_k,$$ (26)

$$Q_{MR} = (Q_{MR} W_{MR} + W_{w} + W_{w})^{-1} (Q_{MR} W_{f_{MR}} + W_{\Delta w})$$

$$L_{MR} = (Q_{MR} W_{f_{MR}} + W_{w} + W_{\Delta w})^{-1} Q_{MR} W_{w}.$$ (27)

The solution of the optimal multirate ILC problem seems to be novel. Specifically, the multirate solution involves non-square matrices, enabling the multirate ILC controller to map fast sampled error signals, i.e., $e^h$, into slow sampled command signals, i.e., $w^l$. In contrast, the discrete time ILC problem results in square matrices, requiring an identical sampling frequency of the error signal, i.e., $e^l$, and the command signal $w^l$. In fact, the discrete time result can directly be recovered using Proposition 15, since the discrete time problem is a special case of the ILC problem for $F = 1$. The proof of Proposition 16 follows along similar lines as the discrete time optimal ILC problem, see Frueh and Phan (2000), Gorinevsky (2002) and Gunnarsson and Norrlöf (2001).

4.3. Design aspects

To apply the multirate ILC approach in the previous sections, a model of the system $J_{MR}$ is required. Given $P_{d,h}$ and $C_{d,l}$, $J_{MR}$ can be computed by using the results of Proposition 11, Lemma 13, Lemma 14, and Proposition 15.

In (25), the weighting matrices $W_{f}, W_{w}$, and $W_{\Delta w}$ are used to emphasize the relative importance of the different criteria. Requiring that all matrices are positive definite is a sufficient condition for a well-posed optimization problem and guaranteed monotonic convergence in $u^l$ (Gunnarsson & Norrlöf, 2001), i.e.,

$$\|w^l_{(k+1)} - w^l_{(\infty)}\|_2 < \|w^l_k - w^l_{(\infty)}\|_2,$$ (29)

where $k \in \mathbb{Z}_+$ and $w^l_{(\infty)} = \lim_{k \to \infty} w^l_k$. Often, the weighting filters are selected as

$$W_{f} = r_f I, \quad W_{w} = r_w I, \quad W_{\Delta w} = r_{\Delta w} I,$$ (30)

where $r_f, r_w, r_{\Delta w} \in \mathbb{R}_+$. In the case of (30), $r_{\Delta w}$ affects the convergence speed, yet not the converged error for $k \to \infty$. Selecting a large $r_{\Delta w}$ is useful to attenuate the influence of trial-varying exogenous signals, including measurement noise. The parameters $r_f$ and $r_w$ can be used to weight the tracking error and the control effort. A large $r_f$ relative to $r_w$ results in a small tracking error, whereas a larger $r_w$ results in a smaller control effort and improved robustness against model uncertainty (Donkers, van de Wijdeven, & Bosgra, 2008).

Finally, it is noted that the discrete time ILC criterion $J_{f_{DT}}$ and multirate criterion $J_{MR}$ cannot be compared directly, since $e^l$ and $e^h$ are defined in different function spaces. Hence, suitable modifications should be made for comparing $J_{f_{DT}}$ and $J_{MR}$. Specifically, by considering $H e^l$, the optimal discrete time ILC problem in Definition 2 can be cast in the criterion in Definition 5. In this case, the weighting matrix $W_{f}$ in (30), corresponding to the criterion $J_{f_{DT}}$, should be scaled appropriately by $\frac{r_f}{r_f^N}$.

Example

In this section, the ILC approach of the previous sections is applied to a simulation model of a motion system. The setup is described in Section 5.1. In Section 5.2, both the discrete time and multirate ILC algorithms, as proposed in Section 4 are applied to a situation where sampling zeros are present. Then, in Section 5.3, the discrete time and multirate ILC algorithms are applied to the situation where aliased disturbances are present.

5.1. Setup

The considered system is a positioning system, which is represented by the differential equation

$$\ddot{x} = ku, \quad y = [x \ \dot{x}]^T,$$ (31)

where the mass $m$ and motor constant $k$ are normalized, i.e., $k = m = 1$. In addition, for the sake of the example, it is assumed that both the position and velocity are measured. The system (31) is unstable, hence a stabilizing discrete time feedback controller $C_{d,l}$, which is a lead-lag controller that again for the sake of the example only uses the position measurement, is implemented with a sampling time $h^l = 0.01$ s, i.e., $f^l = \frac{1}{h^l} = 100$ Hz. Specifically,

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{h} & \frac{1}{h} \end{bmatrix}, \quad C_{d,l} = \begin{bmatrix} 126z - 123.6 \\ z - 0.8282 \end{bmatrix}.$$ (32)

The sampling time of the feedback controller is restricted, since at each sample instant a new control input has to be computed. However, it is possible to record measurement data with a sampling time of \( h = 0.0025 \text{s} \), hence \( F = 4 \) and the sampling frequency \( f^h = \frac{1}{h} = 400 \text{Hz} \). The measurement data recorded with a sampling time \( h^i \) is available for the ILC algorithm. The ILC setup and feedback interconnection are depicted in Fig. 2, where

\[
e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} r - x \\ \dot{r} - x \end{bmatrix}.
\]

(33)

In the sequel, only scalar ILC is considered, i.e., sampled measurements of either \( e_1 \) or \( e_2 \) are used as input for the ILC algorithm. In all cases, the considered exogenous variables are trial invariant, except for the command signal \( w^i \). Throughout this section, \( r_{3w} = 10^{-12} \). In addition, the parameter \( r_c = 1 \) in \( \gamma_{\text{MR}} \) and \( r_c = 4 \) in \( \gamma_{\text{DT}} \), see Section 4.3. To compare the presented multirate ILC approach with standard discrete time ILC, both multirate and discrete time ILC are applied, which is indicated by the subscripts DT and MR, respectively. For instance, the resulting error at sampling frequency \( f^l \) after 10 iterations of the discrete time ILC algorithm is denoted by \( e_{\text{DT}(10)}^l \).

5.2. Example 1: Sampling zeros

5.2.1. Example 1.1

In this example, the position error \( e_1 \) is used in both the discrete time and multirate ILC algorithms, where \( r_{3w} = 0 \) in (30). In the first trial, i.e., \( k = 0 \), the command input \( w^i(0) = 0 \).

The corresponding errors \( e_{\text{DT}(0)}^l \) and \( e_{\text{MR}(0)}^l \), measured at sampling frequencies \( f^l \) and \( f^h \), respectively, are depicted in Fig. 3(a).

Firstly, the discrete time ILC algorithm is applied. The error after 10 iterations is depicted in Fig. 3(b), both at sampling frequencies \( f^l \) and \( f^h \). The error \( e_{\text{DT}(10)}^l \) is not used by the discrete time ILC algorithm, only for analysis of the results. The ILC algorithm achieves zero tracking error \( e_{\text{DT}(10)}^l \) after 10 iterations. However, this zero tracking error is at the expense of a poor intersample behavior. These results are confirmed by evaluating the criteria \( \gamma_{\text{DT}} \) and \( \gamma_{\text{MR}} \), see Table 1. The poor intersample behavior cannot be observed from \( e^l \) and \( \gamma_{\text{DT}} \), hence such criteria are not suitable for analyzing the performance of sampled-data systems.

Secondly, the multirate ILC algorithm is applied, with \( e_{\text{MR}(10)}^l \) the input to the algorithm. Though the multirate ILC algorithm results after 10 iterations in a larger error at the sampling frequency \( f^h \), the multirate ILC algorithm results in improved intersample behavior compared with the discrete time ILC, see Fig. 3(c) and Table 1. The poor intersample behavior in the discrete time ILC case can be attributed to the cancelation of a sampling zero (Åström et al., 1984). Discretization of the system in (32) yields

\[
p_{11}^{\text{DT}} = \frac{h}{2}(z + 1)(z - 1)^2,
\]

(34)

where a sampling zero at \( z = 1 \) appears due to the relative degree 2 of the system \( P_{11} \). The discrete time ILC algorithm in fact cancels this sampling zero, resulting in the poor intersample behavior. The location of the sampling zero in (34) is invariant under a changing sampling frequency, hence modifying the sampling frequency in the discrete time approach does not change the results.

5.2.2. Example 1.2

In Section 5.2.1, it is concluded that neglecting sampling zeros in discrete time ILC can result in poor intersample behavior. By analyzing Fig. 3, it can be concluded that the intersample oscillation requires a nonzero command signal \( w^i \). Hence, an ad hoc solution to resolve the intersample behavior issue in discrete time ILC in case of sampled zeros is to include input weighting in criterion (25). In Fig. 4 the results are depicted for \( r_{3w} = 10^{-11} \); compared with Example 1.1 in Section 5.2.1, inclusion of input weighting in ILC results in a significantly lower error at the higher sampling frequency \( f^h \) at the expense of a slightly larger error at the lower sampling frequency \( f^l \), and hence better intersample behavior. Still, the multirate ILC algorithm outperforms the discrete time algorithm if the error is considered at sampling frequency \( f^h \), see Table 1.

5.2.3. Example 1.3

In Sections 5.2.1 and 5.2.2, it was shown that standard ILC may result in the cancelation of sampling zeros, which in turn results in poor intersample behavior. In this section, it is shown that the poor intersample behavior can also be resolved by reducing the relative degree of the system such that sampling zeros do not appear. To achieve this, the velocity error \( e_2 \), see (33), is considered in the ILC approach. The initial error \( e_2 \) at trial \( k = 0 \) is depicted in Fig. 5(a). The resulting errors after 10 trials of discrete time and multirate ILC with \( r_{3w} = 0 \) are depicted in Fig. 5(b) and Fig. 5(c), respectively. The corresponding criteria \( \gamma_{\text{DT}} \) and \( \gamma_{\text{MR}} \) are presented in Table 2. Clearly, the discrete time ILC algorithm results in zero tracking error at sampling frequency \( f^l \), whereas the intersample behavior remains acceptable. When considering the results in Table 2, it is concluded that the multirate ILC approach results in a better balanced tradeoff between the error at sampling frequency \( f^l \) and intersample behavior, evaluated at a sampling frequency \( f^h \).
5.2.3

et al., Fig. 6 and Fig. 7. Oomen et al., 1996 errors after 10 trials multirate ILC. (circles), error after 10 trials multirate ILC.

5.2.3

This implies that disturbances are commonly trial invariant. In the example, such errors are modeled by a velocity error \( e_v \) at sampling frequency \( f_v \), see Fig. 6(a). In addition, \( w_{0i} \).

As in Section 5.2.3, the discrete time ILC algorithm is applied using \( e_{vD} \). The error after 10 iterations is depicted in Fig. 6(b), both at sampling frequencies \( f_1 \) and \( f_h \). In addition, the criteria \( J_D \) and \( J_MR \) are given in Table 3. Discrete time ILC perfectly attenuates the disturbance at the sampling frequency \( f_1 \). However, this is at the expense of a poor intersample response. In fact, the error signal has deteriorated compared with the initial situation, since its criterion value \( J_MR \) has increased by approximately 20% at sampling frequency \( f_h \). Though the degradation is merely 20%, the error at sampling frequency \( f_1 \) leads to the wrong conclusion that discrete time ILC performs a perfect task!

Application of the multirate ILC algorithm results in a comparable error at sampling frequency \( f_1 \) and \( f_h \), see Fig. 6 and Table 3. In particular, it is concluded that the multirate ILC algorithm results in better performance at both sampling frequencies \( f_1 \) and \( f_h \) compared with the initial situation. In addition, the error at sampling frequency \( f_1 \), at which the command signal \( u_l \) is defined, is well balanced with the error at sampling frequency \( f_h \).

5.3. Example 2: Aliased disturbances

In Section 5.2.3, it was shown that ILC based upon velocity measurements, i.e., based upon \( e_v \) does not suffer from sampling zeros that can lead to poor intersample behavior, both in the discrete time and multirate case. In this section, Example 1.3 of Section 5.2.3 is investigated again with a more realistic disturbance signal.

In typical motion tasks, an identical trajectory is performed. This implies that disturbances are commonly trial invariant. In the example, such errors are modeled by a velocity error \( e_v \) at sampling frequency \( f_v \), see Fig. 6(a). In addition, \( w_{0i} \).

As in Section 5.2.3, the discrete time ILC algorithm is applied using \( e_{vD} \). The error after 10 iterations is depicted in Fig. 6(b), both at sampling frequencies \( f_1 \) and \( f_h \). In addition, the criteria \( J_D \) and \( J_MR \) are given in Table 3. Discrete time ILC perfectly attenuates the disturbance at the sampling frequency \( f_1 \). However, this is at the expense of a poor intersample response. In fact, the error signal has deteriorated compared with the initial situation, since its criterion value \( J_MR \) has increased by approximately 20% at sampling frequency \( f_h \). Though the degradation is merely 20%, the error at sampling frequency \( f_1 \) leads to the wrong conclusion that discrete time ILC performs a perfect task!

Application of the multirate ILC algorithm results in a comparable error at sampling frequency \( f_1 \) and \( f_h \), see Fig. 6 and Table 3. In particular, it is concluded that the multirate ILC algorithm results in better performance at both sampling frequencies \( f_1 \) and \( f_h \) compared with the initial situation. In addition, the error at sampling frequency \( f_1 \), at which the command signal \( u_l \) is defined, is well balanced with the error at sampling frequency \( f_h \).

5.3. Example 2: Aliased disturbances

The results in Fig. 6 and Table 3 are confirmed by the cumulative power spectral density (Zhou et al., 1996), see Fig. 7. When comparing the initial error \( e_{0r} \) and \( e_{0h} \), see Fig. 7(a), with the error after discrete time ILC, i.e., \( e_{DT(10)} \) and \( e_{DT(10)} \). In Fig. 7(b), it turns out that the ILC algorithm produces a command input \( u_l \) such that the power of \( e_{DT(10)} \) increases compared with \( e_{0h} \), yet the downsampled error \( e_{DT(10)} \) is perfectly zero. The multirate ILC algorithm reduces the error compared with the initial situation at both the low and high sampling frequency, as can be observed in Fig. 7(a) and Fig. 7(c). The main performance improvement is achieved in the low frequency band, since these disturbances are not aliased and hence can effectively be attenuated (Oomen et al., 2007b).

6. Conclusions

A novel ILC framework for sampled-data systems aiming at high continuous time performance is presented. The presented approach extends common, discrete time ILC approaches by explicitly addressing the intersample behavior in the learning algorithm. A multirate approach is pursued to enable actual implementation in a digital computer environment. In the limiting case, the multirate problem converges to the sampled-data problem. In practice, a small sampling frequency ratio \( F \) is sufficient to approximate the sampled-data ILC problem due to the low-pass characteristic of the zero-order-hold interpolator. In addition, key issues in sampled-data and multirate control,
including the time-varying nature of the multirate ILC setup, have been dealt with appropriately in the optimal ILC framework.

Two realistic simulation examples reveal that the proposed multirate ILC approach outperforms discrete time ILC. Specifically, discrete time ILC results in poor intersample behavior, both in case of sampling zeros and aliased disturbances. Multirate ILC deals with both these phenomena appropriately by balancing the error at a low sampling frequency and the intersample behavior. The presented approach also handles the case of aliased poles (Oomen et al., 2007a), yet these are not expected to lead to poor intersample behavior and therefore are not explicitly addressed in the examples.

Extensions of the framework include a noninteger sampling frequency ratio $F$, however, in this case Proposition 11 and the block Toeplitz results in Section 4 do not apply directly.

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