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Uniquely connecting frequency domain representations of given order polynomial Wiener–Hammerstein systems[☆]

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ABSTRACT

The notion of frequency response functions has been generalized to nonlinear systems in several ways. However, a relation between different approaches has not yet been established. In this paper, frequency domain representations for nonlinear systems are uniquely connected for a class of nonlinear systems. Specifically, by means of novel analytical results, the generalized frequency response function (GFRF) and the higher order sinusoidal input describing function (HOSIDF) for polynomial Wiener–Hammerstein systems are explicitly related, assuming the linear dynamics are known. Necessary and sufficient conditions for this relation to exist and results on the uniqueness and equivalence of the HOSIDF and GFRF are provided. Finally, this yields an efficient computational procedure for computing the GFRF from the HOSIDF and *vice versa*.

1. Introduction

Important research has been done to extend frequency domain analysis and modeling techniques to nonlinear systems. For linear and time invariant (LTI) systems, frequency domain techniques have resulted in a widespread acceptance in the engineering community for analysis, modeling and controller design. However, the linearity assumption can only be satisfied to a certain extent for physical systems.

The widespread acceptance of frequency domain techniques for LTI systems has been a strong motivation for the extension of these methodologies for nonlinear systems. In Billings and Tsang (1989) and Lang, Billings, Yue, and Li (2007) the generalized frequency response function (GFRF) for nonlinear systems has been defined. An alternative frequency response function for nonlinear systems is considered in Pavlov, van de Wouw, and Nijmeijer (2007). In Schoukens, Pintelon, Dobrowiecki, and Rolain (2005), frequency response based techniques for linear systems are extended to analyze the validity of a linear approximation of nonlinear systems. Finally, in Nuij, Bosgra, and Steinbuch (2006) and Rijlaarsdam, Nuij, Schoukens, and Steinbuch (2011a,b), generalizations of frequency response functions, called higher order sinusoidal input describing functions (HOSIDF), for nonlinear systems are investigated that only represent a relevant subset of nonlinear effects in terms of input signal classes.

Although seemingly different approaches have been independently developed to analyze and represent nonlinear systems in the frequency domain, the differences and equivalences between alternative methods have not yet been established. In this paper an explicit, analytical relation between the GFRF and HOSIDF is established for a specific class of nonlinear systems. Apart from providing valuable insight into the mechanisms that generate the HOSIDFs and GFRFs, these results allow one to formalize statements on uniqueness and equivalence of both model types. Finally, this leads to an efficient procedure for computing the GFRF from the HOSIDFs and *vice versa*.

The paper is organized as follows. In Section 2, the GFRF and the HOSIDF are defined. Then, in Section 3, the main contribution of the paper is presented, that explicitly relates the GFRF and HOSIDF for polynomial Wiener–Hammerstein systems. Finally, in Section 4 conclusions are presented.

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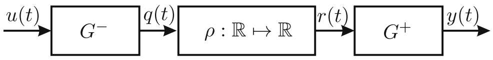

Fig. 1. $\overline{\mathbb{PWH}}$ system.

Notation. Throughout, signals are assumed scalar and real valued, and are denoted by lower case Roman letters, e.g. $x(t) \in \mathbb{R}$. The corresponding Fourier transform is defined as: $\mathscr{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-2\pi i\xi t}dt$, with $\xi \in \mathbb{R}$ the frequency in Hz. Next, define the corresponding single-sided spectrum $\mathscr{X}(\xi) = 2\mathscr{F}\{x(t)\}\ \forall \xi > 0$, $\mathscr{X}(\xi) = \mathscr{F}\{x(t)\}$ for $\xi = 0$ and $\mathscr{X}(\xi) = 0\ \forall \xi < 0$. Moreover, vectors containing specific spectral components $X[\ell] = \mathscr{X}((\ell - 1)\xi_0)$ at harmonics $k = \ell - 1$ of ξ_0 are denoted in capital Roman letters. Finally, all systems considered in this paper are single-input, single-output (SISO), time invariant and $\mathbb{R}_{>0} = \{x \in \mathbb{R} | x > 0\}$. The following class of block structured nonlinear systems is considered throughout this paper.

Definition 1 ($\overline{\mathbb{PWH}}$ *Systems*)**.** Consider the system depicted in Fig. 1, which consists of a series connection of a linear time invariant (LTI) block $G^-(\xi)$ such that $q = G^-u$, a static nonlinear mapping $\rho : \mathbb{R} \mapsto \mathbb{R}$ and another LTI block $G^+(\xi)$ such that $y = G^+r$. The system has one input $u(t) \in \mathbb{R}$, one output $y(t)$ and intermediate signals $q(t)$ and $r(t)$. The nonlinearity ρ is a static, polynomial mapping of degree P and coefficients $\alpha_p \in \mathbb{R}$, i.e.

$$\rho : r(t) = \sum_{p=1}^{P} \alpha_p q^p(t). \tag{1}$$

2. Frequency response functions for nonlinear systems

In this section, two notions of frequency response functions for nonlinear systems are defined. First, the GFRF is defined. Hereto, let the nonlinear system be represented by its Volterra series (Schetzen, 1980). In this case the input–output dynamics are captured in a series of Volterra kernels, where the corresponding pth-order Volterra kernel is given by $h_p(\tau_1, \tau_2, \ldots, \tau_p) : \mathbb{R}^p \mapsto \mathbb{R}$ which is a nonlinear generalization of the impulse response of LTI systems. The response $y(t) \in \mathbb{R}$ of such a system with input $u(t) \in \mathbb{R}$ equals

$$y(t) = \sum_{p=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_p(\tau_1, \ldots, \tau_p) \prod_{m=1}^{p} u(t - \tau_m)\, d\tau_m. \tag{2}$$

The multiple Fourier transform of the pth-order Volterra kernel then yields the corresponding pth-order GFRF.

Definition 2 ($\mathfrak{T}_p(\varpi_p)$*: GFRF*)**.** Consider a system that can be represented by a Volterra series (2). Then its pth-order GFRF $\mathfrak{T}_p(\varpi_p) : \mathbb{R}^p \mapsto \mathbb{C}$, with $\varpi_p = (\xi_1, \xi_2, \ldots, \xi_p) \in \mathbb{R}^p$, is defined as

$$\mathfrak{T}_p(\varpi_p) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_p(\tau_1, \ldots, \tau_p) \prod_{m=1}^{p} e^{-2\pi i\xi_m \tau_m}\, d\tau_m \tag{3}$$

(see Billings & Tsang, 1989; Eykhoff, 1974; George, 1959; Schetzen, 1980).

Next, the following intermediate result enables the representation of GFRFs of $\overline{\mathbb{PWH}}$ as an explicit function of the LTI dynamics $G^{\pm}(\xi)$ and the polynomial coefficients α_p.

Lemma 1 (*GFRF of* $\overline{\mathbb{PWH}}$ *Systems*)**.** *Consider a* $\overline{\mathbb{PWH}}$ *system as in Definition 1. Then the pth-order GFRF* $\mathfrak{T}_p(\varpi_p)$ *as in Definition 2 is given by*

$$\mathfrak{T}_p(\varpi_p) = \alpha_p\, \lambda_p(\varpi_p) \tag{4}$$

$$\lambda_p(\varpi_p) = G^+\left(\sum_{\ell=1}^{p} \varpi_p[\ell]\right) \prod_{\ell=1}^{p} G^-(\varpi_p[\ell]) \tag{5}$$

where $\varpi_p[\ell] = \xi_\ell$ *denotes the* ℓ*th element of* $\varpi_p = (\xi_1, \xi_2, \ldots, \xi_p)$.
(*Proof:* Shanmugam & Jong, 1975)

Next, a different approach to frequency domain analysis and modeling of nonlinear systems, using the higher order sinusoidal input describing function (HOSIDF), is introduced. In Nuij et al. (2006); Rijlaarsdam et al. (2011b) the dynamics of a class of SISO uniformly convergent nonlinear systems (Pavlov, Pogromsky, van de Wouw, & Nijmeijer, 2004) are considered when such system is subject to a sinusoidal input:

$$u(t) = \gamma \cos(2\pi\xi_0 t + \varphi_0) \tag{6}$$

with $\gamma, \varphi_0 \in \mathbb{R}$ and $\xi_0 \in \mathbb{R}_{>0}$. The output of such system is composed of K harmonics of the input frequency, i.e. $y(t) = \sum_{k=0}^{K} \mathfrak{H}_k(\xi_0, \gamma)\gamma^k \cos(k(2\pi\xi_0 t + \varphi_0))$. Here, $\mathfrak{H}_k(\xi_0, \gamma) : \mathbb{R}_{>0} \times \mathbb{R} \mapsto \mathbb{C}$ is the kth-order HOSIDF. This describes the response in terms of gain and phase at harmonics of the excitation frequency, ξ_0, and is defined as in Rijlaarsdam et al. (2011b).

Definition 3 ($\mathfrak{H}_k(\xi, \gamma)$*: HOSIDF*)**.** Consider a SISO, uniformly convergent, time invariant nonlinear system subject to a sinusoidal input (6). Next, define the output $y(t)$ and single-sided spectra of the input and output $\mathscr{U}(\xi), \mathscr{Y}(\xi) \in \mathbb{C}$. Then, the kth-order higher order sinusoidal input describing function $\mathfrak{H}_k(\xi_0, \gamma) \in \mathbb{R}_{>0} \times \mathbb{R} \mapsto \mathbb{C}$, $k = 0, 1, 2, \ldots$, is defined as

$$\mathfrak{H}_k(\xi_0, \gamma) = \frac{\mathscr{Y}(k\xi_0)}{\mathscr{U}^k(\xi_0)}. \tag{7}$$

The following result reveals that the HOSIDFs of $\overline{\mathbb{PWH}}$ systems can be written as an explicit function of the LTI dynamics $G^{\pm}(\xi)$ and the polynomial coefficients α_p.

Lemma 2 (*HOSIDFs of* $\overline{\mathbb{PWH}}$ *Systems*)**.** *For any* $\overline{\mathbb{PWH}}$ *system, the corresponding HOSIDFs of order 1 and higher are given by*

$$\breve{H}(\xi_0, \gamma) = \breve{\Upsilon}^{-1}(\gamma)\breve{\Delta}(\xi_0)\, G^+(\xi)[\breve{\Phi}(\angle G^-(\xi_0))\, \breve{\Omega}\, \Gamma(|G^-(\xi_0)|\gamma)\boldsymbol{\alpha}] \tag{8}$$

where the variables in (8) *are defined in Table 1.*
(*Proof:* Rijlaarsdam et al., 2011b).

3. Connecting the GFRF and HOSIDF

In this section, the GFRFs and HOSIDFs for $\overline{\mathbb{PWH}}$ systems are explicitly related, which constitutes the main result of this paper. Hereto, consider a $\overline{\mathbb{PWH}}$ system with a polynomial nonlinearity (1) of degree P and known linear blocks $G^{\pm}(\xi)$. Then, using Definition 2 and Lemma 1, define

$$T = [\mathfrak{T}_1(\varpi_1)\ \mathfrak{T}_2(\varpi_2) \ldots \mathfrak{T}_P(\varpi_P)]^T$$
$$\Lambda = \mathrm{diag}([\lambda_1(\varpi_1)\ \lambda_2(\varpi_2) \ldots \lambda_P(\varpi_P)])$$

where $T(\varpi_1, \varpi_2, \ \ldots, \varpi_P) : \mathbb{R} \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^P \mapsto \mathbb{C}^P$ contains the GFRFs up to order P and $\Lambda(\varpi_1, \varpi_2, \ \ldots, \varpi_P) : \mathbb{R} \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^P \mapsto \mathbb{C}^{P\times P}$ is a diagonal expansion matrix containing the expansion terms $\lambda_p(\varpi_p)$ (see (5)) that map the LTI dynamics $G^{\pm}(\xi)$

Table 1
Variables in Eq. (8).

Description	Variable	Definition
$\breve{H}(\xi_0,\gamma)\in\mathbb{C}^P$	HOSIDFs	$\breve{H}(\xi_0,\gamma)=[\mathfrak{H}_1(\xi_0,\gamma)\ \mathfrak{H}_2(\xi_0,\gamma)\dots\ \mathfrak{H}_P(\xi_0,\gamma)]^T$.
$\breve{\Upsilon}(\gamma)\in\mathbb{R}^{P\times P}$	Gain compensation matrix	$\breve{\Upsilon}_{k,k}(\gamma)=\gamma^k,\ k=1,2,\dots,P$ and 0 otherwise.
$\breve{\Delta}(\xi_0)\in\mathbb{R}^{P\times P}$	Harmonic selection matrix	Diagonal matrix of δ-functions, s.t. $\breve{\Delta}(\xi_0)G(\xi)=\text{diag}\,([G(\xi_0)\dots G(P\xi_0)])$.
$\breve{\Phi}(\varphi_0)\in\mathbb{C}^{P\times P}$	Input phase matrix	$\breve{\Phi}_{k,k}(\varphi_0)=e^{ik\varphi_0},\ k=1,2,\dots,P$ and 0 otherwise.
$\breve{\Omega}\in\mathbb{R}^{P\times P}$	Inter-harmonic gain matrix	$\breve{\xi}_{k,p}=2\begin{pmatrix}p\\ \frac{p-k}{2}\end{pmatrix}\sigma_{pk}\ \forall k\le p,\ k\in\mathbb{N}_{\ge1}$ and 0 otherwise. With $\sigma_p=p \bmod 2,\ \sigma_k=k \bmod 2,\ \sigma_{pk}=\sigma_p\sigma_k+(1-\sigma_p)(1-\sigma_k)$ and $\begin{pmatrix}a\\ b\end{pmatrix}=\frac{a!}{b!(a-b)!}\forall a,b\in\mathbb{N},\ 0\le b\le a$ and 0 otherwise.
$\Gamma(\gamma)\in\mathbb{R}^{P\times P}$	Input gain matrix	$\Gamma_{p,p}(\gamma)=\left(\frac{\gamma}{2}\right)^p$ and 0 otherwise.
$\boldsymbol{\alpha}\in\mathbb{R}^P$	Polynomial coefficients	$\boldsymbol{\alpha}=[\alpha_1\ \alpha_2\ \dots\ \alpha_P]^T$.

and polynomial coefficients α_p to the GFRFs $\mathfrak{T}_p(\varpi_p)$. Next, consider the sets $\mathbb{W}_p\subseteq\mathbb{R}^p$ such that

$$\mathbb{W}_p=\left\{(\xi_1,\dots,\xi_p)\in\mathbb{R}^p\ \middle|\ G^+\left(\sum_{\ell=1}^{p}\xi_\ell\right)\neq0 \text{ and } G^-(\xi_\ell)\neq0\right\}$$

and define $\mathbb{W}=\mathbb{W}_1\times\mathbb{W}_2\times\cdots\times\mathbb{W}_P$, which includes all frequencies that do not correspond to an imaginary axis zero of the LTI dynamics $G^\pm(\xi)$ and define $\boldsymbol{\varpi}_P=\{\varpi_1,\varpi_2,\ \dots,\varpi_P\}$. The first step in connecting the GFRF and HOSIDF is to relate the polynomial coefficients α_p to the GFRF; see also Lemma 1.

Lemma 3 (*Polynomial Coefficients & GFRF*)**.** *Consider a $\overline{\mathbb{PWH}}$ system. Then, if and only if $\boldsymbol{\varpi}_P\in\mathbb{W}$, the following bijective mapping $\mathbb{R}^P\mapsto\mathbb{C}^P$ from the polynomial coefficients α_p to the GFRF exists:*

$$T(\boldsymbol{\varpi}_P)=\Lambda(\boldsymbol{\varpi}_P)\boldsymbol{\alpha}. \tag{9}$$

Proof. The mapping (9) follows directly from (4)–(5) and is bijective if and only if Λ is of full rank. Since Λ is a diagonal matrix, it is of full rank if and only if $|\lambda_p(\varpi_p)|\neq0\ \forall p$. Next, the results in (5) yield that $|\lambda_p(\varpi_p)|=0$ if and only if $G^+(\sum_{\ell=1}^{p}\varpi_p[\ell])=0$ or $\prod_{\ell=1}^{p}G^-(\varpi_p[\ell])=0$. Hence, $|\lambda_p(\varpi_p)|=0$ if and only if $\varpi_p\notin\mathbb{W}_p$. Hence, the mapping (9) is bijective if and only if $\boldsymbol{\varpi}_P\in\mathbb{W}$. □

Next, considering Lemmas 2 and 3 and substitution of the inverse of (9) in (8) yields a mapping from the GFRFs to the corresponding HOSIDFs and *vice versa*.

Theorem 1 (*Connecting GFRF and HOSIDF*)**.** *Consider a $\overline{\mathbb{PWH}}$ system with known LTI dynamics $G^\pm(\xi)$. If and only if the following properties hold:*

(i) $\boldsymbol{\varpi}_P\in\mathbb{W}$, *and*
(ii) $\xi_0\in\mathbb{R}_{>0}$, *and*
(iii) $\gamma\neq0$,

then the GFRFs and HOSIDFs are uniquely related by the bijective mapping $\mathbb{C}^P\mapsto\mathbb{C}^P$:

$$\breve{H}(\xi_0,\gamma)=\breve{\mathfrak{R}}(\boldsymbol{\varpi}_P,\xi_0,\gamma)T(\boldsymbol{\varpi}_P,\xi_0,\gamma) \tag{10}$$

with

$$\breve{\mathfrak{R}}(\boldsymbol{\varpi}_P,\xi_0,\gamma)=\breve{\Upsilon}^{-1}(\gamma)\breve{\Delta}(\xi_0)G^+(\xi)\breve{\Phi}(\angle G^-(\xi_0))\breve{\Omega}\Gamma(|G^-(\xi_0)|\gamma)\Lambda^{-1}(\boldsymbol{\varpi}_P).$$

Proof. The mapping (10) is bijective if and only if $\breve{\mathfrak{R}}$ is of full rank. The matrix $\breve{\mathfrak{R}}$ is of full rank if and only if all matrices in (10) have full rank. Matrices $\breve{\Upsilon}$, $\breve{\Delta}(\xi_0)\,G^+(\xi)$, $\breve{\Phi}(\angle G^-(\xi_0))$ and $\Gamma(|G^-(\xi_0)|\gamma)$ are diagonal and are defined and of full rank for finite $\xi_0\in\mathbb{R}_{>0}$ and $\gamma\neq0$. Moreover, matrix Λ is of full rank if and only if $\boldsymbol{\varpi}_P\in\mathbb{W}$ (Lemma 3). Finally, analysis reveals that $\breve{\Omega}$ is upper triangular; see Lemma 2. Next, consider an arbitrary row $\breve{\Omega}_{\ell_1}$ of $\breve{\Omega}$ with its first nonzero element at the kth column in that row. Now, because of the rule according to which $\breve{\Omega}$ is generated, any row $\breve{\Omega}_{\ell_2},\ \ell_2>\ell_1$ has a zero element at the kth position. Hence, there is at least one element $\breve{\Omega}_{\ell_1,k}\neq\zeta\breve{\Omega}_{\ell_2,k},\ \zeta\in\mathbb{R}\setminus\{0\}$ and thus $\breve{\Omega}_{\ell_1}\neq\zeta\breve{\Omega}_{\ell_2}$. Since ℓ_1 and ℓ_2 are arbitrary, this proves that $\breve{\Omega}$ has full rank. If $\gamma=0$ or $\xi_0\le0$ or $\boldsymbol{\varpi}_P\notin\mathbb{W}$, then $\breve{\mathfrak{R}}$ is singular or undefined since at least one of the matrices in (10) is singular or undefined. Hence, if and only if $\gamma\neq0$ and $\xi_0>0$ and finite, and $\boldsymbol{\varpi}_P\in\mathbb{W}$, the mapping (10) is defined and is bijective. □

Remark 1. Violation of conditions (i)–(iii) implies that (10) cannot be used to identify the GFRFs from the HOSIDFs. However, this does not imply that the GFRFs cannot be otherwise identified.

The results from Theorem 1 directly provide results on uniqueness of the HOSIDFs and GFRFs and their properties for linear systems.

Lemma 4 (*GFRF & HOSIDF for Linear Systems*)**.** *Consider a $\overline{\mathbb{PWH}}$ system and assume conditions* (i)–(iii) *in Theorem 1 are satisfied. Then the following statements are equivalent:*

(a) *The system is linear.*
(b) *All HOSIDFs except the first are zero: $\mathfrak{H}_k=0\ \forall k\neq1$.*
(c) *All GFRFs except the first are zero: $\mathfrak{T}_p=0\ \forall p\neq1$.*

Proof. Consider a linear $\overline{\mathbb{PWH}}$ system, i.e. $\rho:r(t)=\alpha_1q(t)$. Then, as an LTI system has a sinusoidal response to a sinusoidal input (6), $\mathfrak{H}_k=0\ \forall k\neq1$. Next, using (10), the structure of $\breve{\Omega}$ and the fact that all other matrices are diagonal yields that $\mathfrak{T}_p=0\ \forall p\neq1$. Conversely, if $\mathfrak{T}_p=0\ \forall p\neq1$ this implies a linear $\overline{\mathbb{PWH}}$ system and by the same arguments $\mathfrak{H}_k=0\ \forall k\neq1$. □

Theorem 1 provides the first connection between the HOSIDF and the GFRF. This yields a clear insight into the mechanism that generates the GFRFs from the HOSIDFs and *vice versa*. The results presented in this paper yield a bijective mapping between the HOSIDF, which is a representation valid only for sinusoidal inputs, and the GFRF, which is valid for a more general class of input signals. Therefore, the existence of a mapping from the GFRF to the HOSIDF is not surprising. However, the existence of a mapping from the HOSIDF to the GFRF is nontrivial, especially as no knowledge about the nonlinearity is required to define this mapping. It is shown that only knowledge on the linear dynamics is required to connect the GFRF and HOSIDF. That is, the HOSIDF at a single amplitude–frequency combination provides sufficient information for identifying the nonlinearity and uniquely connecting the HOSIDF and GFRF.

The following example illustrates the main results of the paper.

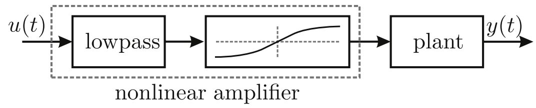

Fig. 2. Example of a $\overline{\mathbb{PWH}}$ system representing a nonlinear amplifier driving a linear time invariant fourth-order plant.

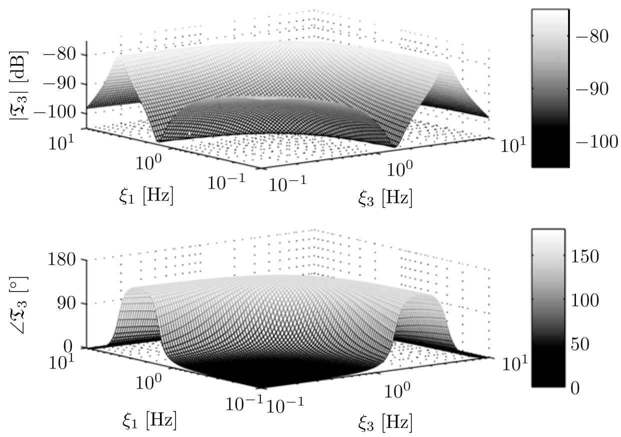

Fig. 3. Third generalized FRF ($\xi_2 = 1.96$ Hz. (arbitrary)).

Example 1 (*Analysis of a Nonlinear Amplifier*)**.** Consider the nonlinear amplifier $\rho G^{-}(s)$ in Fig. 2, where ρ is an unknown static polynomial function that represents the nonlinearity and

$$G^{-}(s) = \frac{10000}{s^2 + 2513s + 1.579 \cdot 10^6} \tag{11}$$

represents the low-pass characteristic of the amplifier. The amplifier generates an input to the fourth-order LTI electromechanical system (plant)

$$G^{+}(s) = \frac{750000s^2 + 1.875 \cdot 10^6 s + 3.75 \cdot 10^8}{s^4 + 7.8s^3 + 1601s^2 + 400s + 50000}. \tag{12}$$

As illustrated in Fig. 2, this system fits the structure of a $\overline{\mathbb{PWH}}$ system. Hence, the results of Theorem 1 apply given the knowledge of $G^{-}(s)$ and $G^{+}(s)$.

Next, the mapping (10) in Theorem 1 is applied to compute the GFRFs from the HOSIDFs. The required HOSIDFs $H(\xi_0, \gamma)$ are identified at a single frequency–amplitude combination by exciting the system in Fig. 2 with a sinusoidal input (6) with an arbitrarily chosen amplitude $\gamma = 1$ and frequency $\xi_0 = 10$ [Hz]. From the simulation data, the HOSIDFs are then readily computed using Definition 3 and are given by $\check{H}(\xi_0 = 10, \gamma = 1) = [\mathfrak{H}_1(10, 1)\ \mathfrak{H}_2(10, 1)\ \mathfrak{H}_3(10, 1)]^T = [-1.7 - 0.1i\ \ -1.0 \cdot 10^{-4} + 1.5 \cdot 10^{-5}i\ \ - 6.6 \cdot 10^{-7} + 1.8 \cdot 10^{-7}i]^T$.

Next, by application of (10), the corresponding GFRFs are computed for a range of frequencies and the third GFRF is depicted in Fig. 3. A comparison of the GFRFs obtained with the results obtained using the exact approach in Shanmugam and Jong (1975) reveals a close correspondence, e.g., the maximum error is close to the computational precision. The corresponding HOSIDFs are computed as in Rijlaarsdam et al. (2011b) and the third-order HOSIDF is depicted in Fig. 4.

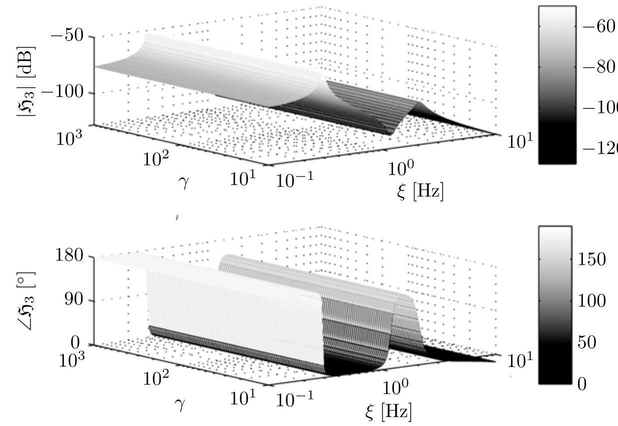

Fig. 4. Third HOSIDF.

4. Conclusion

In this paper a novel connection between two frequency domain methods for the analysis and modeling of nonlinear systems is presented. Specifically, a unique relation between the generalized frequency response function (GFRF) and the higher order sinusoidal input describing function (HOSIDF) is established. An explicit analytical relation between the two is derived for polynomial Wiener–Hammerstein systems and necessary and sufficient conditions are derived for this bijective mapping to exist. Moreover, properties of the GFRFs and HOSIDFs, for linear and time invariant systems are presented. This analysis yields clear insight into the mechanisms that generate the GFRFs and HOSIDFs and provides an efficient method for computing the GFRFs from the HOSIDFs and *vice versa*.

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