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New Connections Between Frequency Response Functions for a Class of Nonlinear Systems ⋆

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Abstract: The notion of frequency response functions has been generalized to nonlinear systems in several ways. However, a relation between different approaches has not yet been established. In this paper, frequency domain representations for nonlinear systems are uniquely connected. Specifically, by means of novel analytical results, the generalized frequency response function (GFRF) and the higher order sinusoidal input describing function (HOSIDF) for polynomial Wiener-Hammerstein systems are explicitly related. Necessary and sufficient conditions for this relation to exist and results on uniqueness and equivalence of the HOSIDF and GFRF are provided. Finally, a numerically efficient computational procedure is presented that allows to compute the GFRF from the HOSIDF and *vice versa*.

1. INTRODUCTION

Important research has been done to extend frequency domain analysis and modeling techniques to nonlinear systems. For linear time invariant systems, frequency domain analysis has led to significant progress in analysis, modeling and controller design. Correspondingly, the frequency response function and representations such as the Bode, Nyquist and Nichols plot have become standard engineering tools. Currently, increased performance requirements force systems into regimes where nonlinear effects can no longer be neglected. Hence, frequency domain methods for nonlinear systems have attracted an increasing amount of attention as frequency domain based analysis can yield valuable insight into the mechanism that generates nonlinear behavior. However, the extension of frequency domain methods to nonlinear systems is nontrivial as nonlinear effects cannot be captured in the classical frequency response function.

Different methods to apply frequency domain analysis to nonlinear systems exist. Two methods that aim to model the complete dynamics of a class of nonlinear systems are the generalized frequency response function and the frequency response function for nonlinear systems [Billings and Tsang, 1989, Pavlov et al., 2007]. Moreover, the frequency domain representation of nonlinear dynamical systems allows to evaluate the validity of a linear approximation as often used in control design [Schoukens et al., 2005]. Finally, methods that model only a relevant subset of nonlinear effects in the frequency domain exist. An example of such method are the higher order sinusoidal input describing functions [Nuij et al., 2006, Rijlaarsdam et al., 2011a], which are recently shown to be especially suitable for optimal pre-compensator design for nonlinear systems [Rijlaarsdam et al., 2011b]. In the sequel, two methods are considered in particular: the Generalized Frequency Response Functions (GFRF) and the Higher Order Sinusoidal Input Describing Functions (HOSIDF).

The GFRF is based on the extension of the linear impulse response and frequency response function, to nonlinear systems that can be represented by a Volterra series [Schetzen, 1980]. These nonlinear impulse responses are referred to as Volterra kernels, which Fourier transforms yield the GFRFs. The GFRFs have been extensively investigated, including their analysis [Li and Billings, 2011], interpretation [Yue et al., 2005], identification [Billings and Tsang, 1989] and application [Jing et al., 2011].

The HOSIDFs are introduced in [Nuij et al., 2006] and further discussed in [Rijlaarsdam et al., 2011a]. Other than the sinusoidal input describing function [Gelb and Vander Velde, 1968], the HOSIDFs model the systems response to a sinusoidal input signal, at both the excitation frequency and harmonics of the excitation frequency. The HOSIDFs provide a natural extension of the widely used sinusoidal input describing function when nonlinear dynamics are present. Due to their straightforward identification and interpretation, the HOSIDFs enable on-site testing during system design, characterization of existing systems [Nuij et al., 2008] and nonlinear controller design [Rijlaarsdam et al., 2011b].

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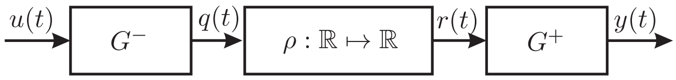

Fig. 1. $\overline{\mathbb{PWH}}$ system.

Although seemingly different approaches exist to analyze and model nonlinear systems in the frequency domain, the differences and equivalences between alternative methods have not yet been established. In this paper an explicit, analytical relation between the GFRF and HOSIDF is established for a specific class of nonlinear systems. Apart from providing valuable insight in the mechanisms that generate the HOSIDFs and GFRFs, these results allow to formalize statements on uniqueness and equivalence of both model types. Finally, this yields a numerically efficient analytical relation to compute the GFRF from the HOSIDFs and *vice versa*.

The paper is structured as follows: In Section 2 the required nomenclature is presented. Next, in Section 3 the Generalized FRF and the HOSIDF are introduced and compared. Then, in Section 4 the main contribution of the paper is presented by analytically relating the GFRF and HOSIDF for polynomial Wiener-Hammerstein systems. Finally, in Section 5 the main results are summarized.

2. NOMENCLATURE AND PRELIMINARIES

In the following, signals are scalar, real valued and are denoted by non-capitalized roman letters, e.g. $x(t) \in \mathbb{R}$. The corresponding Fourier transform $\mathscr{F}\{\cdot\}$ is then given by:

$$\mathscr{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-2\pi i \xi t} dt$$

with $\xi \in \mathbb{R}$ the frequency in [Hz]. As $x(t)$ is real valued, $\mathscr{F}\{x(t)\}$ is point symmetric and all information is contained in the single sided spectrum:

$$\mathscr{X}(\xi) = \begin{cases} 2\mathscr{F}\{x(t)\} & \xi > 0 \\ \mathscr{F}\{x(t)\} & \xi = 0 \\ 0 & \xi < 0 \end{cases}$$

Frequent use is made of vectors containing only specific spectral components $X[\ell] = \mathscr{X}\big((\ell-1)\xi_0\big)$ at harmonics $k = \ell - 1$ of ξ_0. These are denoted in capitalized roman letters while matrices are denoted by capitalized Greek characters. Finally, all systems considered in this paper are single input, single output (SISO) and time invariant. The analysis presented in the sequel mainly focusses on the class of block structured dynamical systems defined below.

Definition 1. ($\overline{\mathbb{PWH}}$ Systems). Consider the polynomial Wiener-Hammerstein system depicted in Figure 1. This system consists of a series connection of a Linear Time Invariant (LTI) block $G^-(\xi)$ such that $q = G^- u$, a static nonlinear mapping $\rho : \mathbb{R} \mapsto \mathbb{R}$ and another LTI block $G^+(\xi)$ such that $y = G^+ r$. The system has one input $u(t) \in \mathbb{R}$, one output $y(t)$ and intermediate signals $q(t)$ and $r(t)$. The nonlinearity ρ is a static, polynomial mapping of degree P:

$$\rho : r(t) = \sum_{p=1}^{P} \alpha_p q^p(t) \tag{1}$$

with $\alpha_p \in \mathbb{R}$.

Next, the generalized frequency response function and the higher order sinusoidal input describing functions are introduced.

3. FREQUENCY DOMAIN REPRESENTATIONS OF NONLINEAR SYSTEMS

In this section the definitions of the GFRF and HOSIDFs are provided and a comparison between both methods is presented.

3.1 Generalized Frequency Response Function

In for example Schetzen [1980] a generalization of the impulse response of nonlinear systems that can be represented by the Volterra series is established. The input-output dynamics of such systems are captured in a series of Volterra kernels that are a nonlinear generalization of the impulse response of LTI systems. The response $y(t) \in \mathbb{R}$ of such system to an input $u(t) \in \mathbb{R}$ is given by:

$$y(t) = \sum_{p=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_p(\tau_1, \ldots, \tau_p) \prod_{m=1}^{p} u(t - \tau_m)\, d\tau_m \tag{2}$$

with $h_p(\tau_1, \tau_2, \ldots, \tau_p) : \mathbb{R}^p \mapsto \mathbb{R}$ the p^{th} order Volterra kernel. Applying the multiple Fourier transform to the p^{th} order Volterra kernel yields the corresponding p^{th} order GFRF.

Definition 2. ($\mathfrak{T}_p(\varpi_p)$: GFRF). Consider a system that can be represented by a Volterra series (2). Then its p^{th} order GFRF $\mathfrak{T}_p(\varpi_p) : \mathbb{R}^p \mapsto \mathbb{C}$, with $\varpi_p = (\xi_1,\ \xi_2,\ \ldots,\ \xi_p) \in \mathbb{R}^p$, is defined as:

$$\mathfrak{T}_p(\varpi_p) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_p(\tau_1, \ldots, \tau_p) \prod_{m=1}^{p} e^{-2\pi i \xi_m \tau_m}\, d\tau_m \tag{3}$$

(see: [Billings and Tsang, 1989]).

Hence, equivalent to (2), the output of such system subject to an input $u(t)$ with spectrum $\mathscr{U}(\xi) \in \mathbb{C}$ is given by the inverse p-dimensional Fourier transform.

$$y(t) = \sum_{p=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathfrak{T}_p(\varpi_p) \prod_{m=1}^{p} \mathscr{U}(\xi_m) e^{2\pi i \xi_m t}\, d\xi_m$$

Using results from Shanmugam and Jong [1975], the GFRFs of $\overline{\mathbb{PWH}}$ systems are an explicit function of the LTI dynamics $G^{\pm}(\xi)$ and the polynomial coefficients α_p. This result is later required for analysis and is summarized in the following lemma.

Lemma 1. (GFRF of $\overline{\mathbb{PWH}}$ systems). Consider a $\overline{\mathbb{PWH}}$ system as in Definition 1. Then the p^{th} order GFRF $\mathfrak{T}_p(\varpi_p)$ as in Definition 2 is given by:

$$\mathfrak{T}_p(\varpi_p) = \alpha_p\, \lambda_p(\varpi_p) \tag{4}$$

$$\lambda_p(\varpi_p) = G^+\left(\sum_{\ell=1}^{p} \varpi_p[\ell]\right) \prod_{\ell=1}^{p} G^-\left(\varpi_p[\ell]\right) \tag{5}$$

where $\varpi_p[\ell] = \xi_\ell$ denotes the ℓ^{th} element of $\varpi_p = (\xi_1, \xi_2, \ldots, \xi_p)$.

Table 1.
Variables in (8).

Description	**Variable**	**Definition**
HOSIDFs	$\breve{H}(\xi_0,\gamma) \in \mathbb{C}^P$	$\breve{H}(\xi_0,\gamma) = [\mathfrak{H}_1(\xi_0,\gamma)\ \mathfrak{H}_2(\xi_0,\gamma) \ldots\ \mathfrak{H}_P(\xi_0,\gamma)]^T$
gain compensation matrix	$\breve{\Upsilon}(\gamma) \in \mathbb{R}^{P\times P}$	$\breve{\Upsilon}_{k,k}(\gamma) = \gamma^k,\ k = 1, 2, \ldots, P$ and 0 otherwise.
harmonic selection matrix	$\breve{\Delta}(\xi_0) \in \mathbb{R}^{P\times P}$	diagonal matrix of δ-functions, s.t. $\breve{\Delta}(\xi_0)G(\xi) = \text{diag}([G(\xi_0) \ldots G(P\xi_0)])$.
input phase matrix	$\breve{\Phi}(\varphi_0) \in \mathbb{C}^{P\times P}$	$\breve{\Phi}_{k,k}(\varphi_0) = e^{ik\varphi_0},\ k = 1, 2, \ldots, P$ and 0 otherwise.
inter-harmonic gain matrix	$\breve{\Omega} \in \mathbb{R}^{P\times P}$	$\breve{\Omega}_{k,p} = 2 \begin{pmatrix} p \\ \frac{p-k}{2} \end{pmatrix} \sigma_{pk} \forall k \leq p, k \in \mathbb{N}_{\geq 1}$ and 0 otherwise. With $\sigma_p = p \bmod 2$, $\sigma_k = k \bmod 2, \sigma_{pk} = \sigma_p\sigma_k + (1-\sigma_p)(1-\sigma_k)$ and $\binom{a}{b} = \frac{a!}{b!(a-b)!}\ \forall\, a, b \in \mathbb{N},\ 0 \leq b \leq a$ and 0 otherwise.
input gain matrix	$\Gamma(\gamma) \in \mathbb{R}^{P\times P}$	$\Gamma_{p,p}(\gamma) = \left(\frac{\gamma}{2}\right)^p$ and 0 otherwise.
polynomial coefficients	$\boldsymbol{\alpha} \in \mathbb{R}^P$	$\boldsymbol{\alpha} = [\alpha_1\ \alpha_2\ \ldots\ \alpha_P]^T$.

(proof: see Shanmugam and Jong [1975]).

Summarizing, the GFRF extends the notion of the FRF for a class of nonlinear systems that can be represented by a Volterra series (2). For the class of $\overline{\mathbb{PWH}}$ systems, this GFRF is analytically related to the system parameter as investigated by Lemma 1. Next, a different approach to frequency domain analysis and modeling of nonlinear systems, the higher order sinusoidal input describing function, is investigated.

3.2 Higher Order Sinusoidal Input Describing Functions

In Nuij et al. [2006], Rijlaarsdam et al. [2011a] the dynamics of a class of SISO convergent nonlinear systems [Pavlov et al., 2004] are considered when such system is subject to a sinusoidal input:

$$u(t) = \gamma \cos(2\pi\xi_0 t + \varphi_0) \tag{6}$$

with $\gamma, \varphi_0 \in \mathbb{R}$ and $\xi_0 \in \mathbb{R}_{>0}$. The output of such system is composed of K harmonics of the input frequency, i.e.

$$\begin{aligned} y(t) &= \sum_{k=0}^{K} \mathfrak{H}_k(\xi_0,\gamma)\gamma^k \cos\big(k(2\pi\xi_0 t + \varphi_0)\big) \\ &= \sum_{k=0}^{K} |\mathfrak{H}_k(\xi_0,\gamma)|\gamma^k \cos\big(k(2\pi\xi_0 t + \varphi_0) + \angle\mathfrak{H}_k(\xi_0,\gamma)\big) \end{aligned}$$

Here, $\mathfrak{H}_k(\xi_0,\gamma) : \mathbb{R}_{>0} \times \mathbb{R} \mapsto \mathbb{C}$ is the k^{th} order Higher Order Sinusoidal Input Describing Function (HOSIDF), which describes the response (gain and phase) at harmonics of the excitation frequency ξ_0. The definition of the HOSIDF is formalized as in Rijlaarsdam et al. [2011a].

Definition 3. ($\mathfrak{H}_k(\xi,\gamma)$: HOSIDF). Consider a single input, single output, uniformly convergent, time invariant nonlinear system [Pavlov et al., 2004] subject to a sinusoidal input (6). Assume that the system is in steady state and define the output $y(t)$ and single sided spectra of the input and output $\mathscr{U}(\xi)$, $\mathscr{Y}(\xi) \in \mathbb{C}$. Then, the k^{th} higher order sinusoidal input describing function $\mathfrak{H}_k(\xi_0,\gamma) \in \mathbb{R}_{>0} \times \mathbb{R} \mapsto \mathbb{C},\ k = 0, 1, 2, \ldots$ is defined as:

$$\mathfrak{H}_k(\xi_0,\gamma) = \frac{\mathscr{Y}(k\xi_0)}{\mathscr{U}^k(\xi_0)}. \tag{7}$$

The following result reveals the HOSIDFs of $\overline{\mathbb{PWH}}$ systems as an explicit function of the LTI dynamics $G^{\pm}(\xi)$ and the polynomial coefficients α_p.

Lemma 2. (HOSIDFs of $\overline{\mathbb{PWH}}$ systems). Consider a $\overline{\mathbb{PWH}}$ system as in Definition 1. Then the corresponding HOSIDFs of order one and higher are given by $\breve{H}(\xi_0,\gamma) =$

$$\breve{\Upsilon}^{-1}(\gamma)\breve{\Delta}(\xi_0)\, G^{+}(\xi) \left[\breve{\Phi}(\angle G^{-}(\xi_0))\, \breve{\Omega}\, \Gamma(|G^{-}(\xi_0)|\gamma)\boldsymbol{\alpha}\right] \tag{8}$$

where the variables in (8) are defined in Table 1.

(proof: see [Rijlaarsdam et al., 2011a]).

Software to compute (8) and the variables in Table 1 is available online (www.davidrijlaarsdam.nl). Next, an example is provided that illustrates the application of Lemma 2.

Example 1. (HOSIDFs of a $\overline{\mathbb{PWH}}$ system). Consider a $\overline{\mathbb{PWH}}$ system as in Definition 1 with arbitrary LTI dynamics $G^{\pm}(\xi)$ and a polynomial nonlinearity of order three $\rho : r(t) = \alpha_1 q(t) + \alpha_2 q^2(t) + \alpha_3 q^3(t)$. Then, Lemma 2 yields the corresponding HOSIDFs of order one and higher as presented in (9).

Summarizing, the HOSIDFs model the response of convergent nonlinear dynamical systems subject to a sinusoidal input. For the class of $\overline{\mathbb{PWH}}$ systems, the HOSIDFs are related to the system parameters as in Lemma 2. In the next section, a detailed analysis and comparison of the GFRF and HOSIDF for $\overline{\mathbb{PWH}}$ systems is presented. This yields the first connection between both models.

4. CONNECTING THE GFRF AND HOSIDF FOR $\overline{\mathbb{PWH}}$ SYSTEMS

In this section an explicit relation between the GFRF and HOSIDFs for $\overline{\mathbb{PWH}}$ systems is derived, which yields the main result of this paper. To this end, consider a $\overline{\mathbb{PWH}}$ system with a polynomial nonlinearity (1) of degree P. Then, using the GFRFs as in Definition 2 and the results in Lemma 1, define,

$$\begin{aligned} T &= [\mathfrak{T}_1(\varpi_1)\ \mathfrak{T}_2(\varpi_2)\ \ldots\ \mathfrak{T}_P(\varpi_P)]^T \\ \Lambda &= \text{diag}([\lambda_1(\varpi_1)\ \lambda_2(\varpi_2)\ \ldots\ \lambda_P(\varpi_P)]) \end{aligned}$$

where $T(\varpi_1,\ \varpi_2,\ \ldots, \varpi_P) : \mathbb{R} \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^P \mapsto \mathbb{C}^P$ contains the GFRFs up to order P and $\Lambda(\varpi_1,\ \varpi_2,\ \ldots, \varpi_P) : \mathbb{R} \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^P \mapsto \mathbb{C}^{P\times P}$ is a diagonal expansion matrix containing the expansion terms $\lambda_p(\varpi_p)$, see (5), that map the LTI dynamics $G^{\pm}(\xi)$ and polynomial coefficients α_p to the GFRFs $\mathfrak{T}_p(\varpi_p)$. In the sequel an analytical rela-

$$H(\xi_0,\gamma) = \tag{9}$$

$$\overbrace{\begin{bmatrix} \frac{1}{\gamma} & 0 & 0 \\ 0 & \frac{1}{\gamma^2} & 0 \\ 0 & 0 & \frac{1}{\gamma^3} \end{bmatrix}}^{\breve{\Upsilon}^{-1}(\gamma)} \overbrace{\begin{bmatrix} G^+(\xi_0) & 0 & 0 \\ 0 & G^+(2\xi_0) & 0 \\ 0 & 0 & G^+(3\xi_0) \end{bmatrix}}^{\breve{\Delta}(\xi_0)\, G^+(\xi)} \overbrace{\begin{bmatrix} e^{i\angle G^-(\xi_0)} & 0 & 0 \\ 0 & e^{2i\angle G^-(\xi_0)} & 0 \\ 0 & 0 & e^{3i\angle G^-(\xi_0)} \end{bmatrix}}^{\breve{\Phi}(\angle G^-(\xi_0))} \overbrace{\begin{bmatrix} 2 & 0 & 6 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}^{\breve{\Omega}} \overbrace{\begin{bmatrix} \frac{1}{2}|G^-(\xi_0)|\gamma & 0 & 0 \\ 0 & \frac{1}{4}|G^-(\xi_0)|^2\gamma^2 & 0 \\ 0 & 0 & \frac{1}{8}|G^-(\xi_0)|^3\gamma^3 \end{bmatrix}}^{\Gamma(|G^-(\xi_0)|\gamma)} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \mathfrak{H}_1(\xi_0,\gamma) \\ \mathfrak{H}_2(\xi_0,\gamma) \\ \mathfrak{H}_3(\xi_0,\gamma) \end{bmatrix} = \begin{bmatrix} \alpha_1 G^+(\xi_0)G^-(\xi_0) + \frac{3\gamma^2\alpha_3}{4} G^+(\xi_0)|G^-(\xi_0)|^2 G^-(\xi_0) \\ \frac{1}{2}\alpha_2 G^+(2\xi_0)\big(G^-(\xi_0)\big)^2 \\ \frac{1}{4}\alpha_3 G^+(3\xi_0)\big(G^-(\xi_0)\big)^3 \end{bmatrix}$$

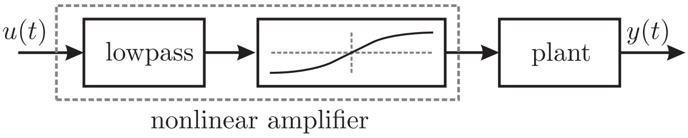

Fig. 2. Example of $\overline{\mathbb{PWH}}$ system representing a nonlinear amplifier driving a linear time invariant 4[th] order plant.

tion between the GFRF and HOSIDFs is derived. Next, consider the sets $\mathbb{W}_p \subseteq \mathbb{R}^p$, such that $\mathbb{W}_p =$

$$\left\{ (\xi_1,\ldots,\xi_p) \in \mathbb{R}^p \,\middle|\, G^+\left(\sum_{\ell=1}^{p} \xi_\ell\right) \neq 0 \text{ and } G^-(\xi_\ell) \neq 0 \right\}$$

and define $\mathbb{W} = \mathbb{W}_1 \times \mathbb{W}_2 \times \cdots \times \mathbb{W}_P$ and $\boldsymbol{\varpi}_P = \{\varpi_1,\ \varpi_2,\ \ldots, \varpi_P\}$.

The first step in connecting the GFRF and HOSIDF is to relate the polynomial coefficients to the GFRF, see also Lemma 1.

Lemma 3. (Polynomial coefficients & GFRF). Consider a $\overline{\mathbb{PWH}}$ system. Then, if and only if $\boldsymbol{\varpi}_P \in \mathbb{W}$, the following bijective mapping $\mathbb{R}^P \mapsto \mathbb{C}^P$ from the polynomial coefficients to the GFRF exists:

$$T(\boldsymbol{\varpi}_P) = \Lambda(\boldsymbol{\varpi}_P)\boldsymbol{\alpha} \tag{10}$$

Hence, then the mapping is unique, and a unique inverse mapping $\mathbb{C}^P \mapsto \mathbb{R}^P$ exists:

$$\boldsymbol{\alpha} = \Lambda^{-1}(\boldsymbol{\varpi}_P)T(\boldsymbol{\varpi}_P) \tag{11}$$

(proof: see [Rijlaarsdam et al., 2012]).

Remark 1. Consider for example, the case where $\varpi_1 = \xi_1$ is a zero of either $G^+(\xi)$ of $G^-(\xi)$, then $\boldsymbol{\varpi}_P \notin \mathbb{W}$ and (10) is not bijective.

Next, considering Lemma 2 and 3 and substituting (11) in (8) yields a mapping from the GFRFs to the corresponding HOSIDFs and *vice versa*. This yields an explicit relation between the HOSIDF and GFRF model which is formalized in the following theorem.

Theorem 1. (Connecting GFRF and HOSIDF). Consider a $\overline{\mathbb{PWH}}$ system and assume that the following holds:

(1) $\boldsymbol{\varpi}_P \in \mathbb{W}$, and
(2) $\xi_0 \in \mathbb{R}_{>0}$ and finite, and
(3) $\gamma \neq 0$.

If and only if condition 1-3 hold, the GFRFs and HOSIDFs are uniquely related by the following bijective mapping $\mathbb{C}^P \mapsto \mathbb{C}^P$:

$$\breve{H}(\xi_0,\gamma) = \breve{\mathfrak{R}}(\boldsymbol{\varpi}_P,\xi_0,\gamma)\, T(\boldsymbol{\varpi}_P,\xi_0,\gamma) \tag{12}$$

$$T(\boldsymbol{\varpi}_P,\xi_0,\gamma) = \breve{\mathfrak{R}}^{-1}(\boldsymbol{\varpi}_P,\xi_0,\gamma)\breve{H}(\xi_0,\gamma) \tag{13}$$

with $\breve{\mathfrak{R}}(\boldsymbol{\varpi}_P,\xi_0,\gamma) =$

$$\breve{\Upsilon}^{-1}(\gamma)\breve{\Delta}(\xi_0)\, G^+(\xi)\breve{\Phi}(\angle G^-(\xi_0))\breve{\Omega}\Gamma(|G^-(\xi_0)|\gamma)\Lambda^{-1}(\boldsymbol{\varpi}_P)$$

(proof: see [Rijlaarsdam et al., 2012]).

Remark 2. Although violation of conditions 1-3 implies that (12) is not unique, this does not imply that the GFRFs are not identifiable.

Remark 3. The HOSIDF of order zero is not required to compute the GFRFs since $\mathfrak{H}_0(\xi_0,\gamma)$ can be computed from the even HOSIDFs.

The results from Theorem 1 directly yield results on uniqueness of the HOSIDFs and GFRFs and their properties for linear systems.

Lemma 4. (GFRF & HOSIDF for Linear Systems).
Consider a $\overline{\mathbb{PWH}}$ system and assume conditions 1-3 in Theorem 1 are satisfied. Then the following statements are equivalent:

(1) The system is linear.
(2) Only the HOSIDF of order one is nonzero, i.e. $\mathfrak{H}_k = 0\ \forall\ k \neq 1$.
(3) Only the GFRF of order one is nonzero, i.e. $\mathfrak{T}_p = 0\ \forall\ p \neq 1$.

(proof: see [Rijlaarsdam et al., 2012]).

Theorem 1 yields the first connection between the GFRF and HOSIDF. The results yield clear insight in the mechanism that generates the GRFRs from the HOSIDFs and *vice versa*. Moreover, Lemma 4 formalizes the equivalence of the HOSIDF and GFRF for linear systems. Finally, since the relations are analytical, the numerical values of the GFRFs from the HOSIDFs can in general be computed efficiently. The following example illustrates this by applying Theorem 1 to a case study of a nonlinear amplifier driving a LTI mechanical system.

Example 2. (Analysis of a Nonlinear Amplifier).
When supplying an excitation signal to an electromechanical system, the signal is often amplified to reach a sufficient level of excitation. Ideally, the amplifier used to drive the system is linear. However, in practice many amplifiers suffer from nonlinear effects such as saturation.

Consider a simplified model of a saturating amplifier driving a fourth order linear, time invariant mechanical plant. Such system fits the structure of a $\overline{\mathbb{PWH}}$ system (Figure 2), where the dynamics of the amplifier are modeled by a low-pass characteristic and a static polynomial nonlinearity is used to model the nonlinear saturation effect. Next, suppose the static polynomial nonlinearity is unknown, but the linear dynamics $G^{\pm}(s) \triangleq G^{\pm}(2\pi i\xi)$ are known and given by: $G^-(s) = \frac{10000}{s^2+2513s+1.579\cdot 10^6}$, $G^+(s) = \frac{750000s^2+1.875\cdot 10^6 s+3.75\cdot 10^8}{s^4+7.8s^3+1601s^2+400s+50000}$.

Next, the mapping (13) in Theorem 1 is applied to compute the GFRFs from the HOSIDFs. The required HOSIDFs $H(\xi_0, \gamma)$ are identified at a single frequency amplitude combination by exciting the system in Figure 2 with a sinusoidal input (6) with amplitude $\gamma = 1$ and frequency $\frac{\xi_0}{2\pi} = 10$ $[Hz]$. Both the excitation signal and the response are depicted in Figure 3 (a). These results reveal that the response contains second and third harmonics of the input signal and a DC term. Hence, as an approximation, the first three GFRFs are selected to model the systems dynamics.

To compute the GFRFs the results from Theorem 1 are applied which requires the identification of the first three HOSIDFs at a single frequency, amplitude combination. From the simulation depicted in Figure 3 (a), these are readily computed using Definition 3 and equal $\check{H}(\xi_0 = 10, \gamma = 1) = [\mathfrak{H}_1(10,1)\ \mathfrak{H}_2(10,1)\ \mathfrak{H}_3(10,1)]^T = [-1.7 - 0.1i\ \ -1.0 \cdot 10^{-4} + 1.5 \cdot 10^{-5}i\ \ -6.6 \cdot 10^{-7} + 1.8 \cdot 10^{-7}i]^T$.

Next, by application of (13) the corresponding GFRFs are computed for a range of frequencies as depicted in Figures 3 (b) - 3 (d). Using the results in Shanmugam and Jong [1975] these results are compared to the exact GFRFs, which yields that the results are very close to the true value. This follows from further computations which reveal that the maximum relative error equals 10^{-7} when the GFRFs drop below -80 $[dB]$ which is close to the computational precision. The corresponding first and third HOSIDFs are computed using Rijlaarsdam et al. [2011a] and are depicted in Figure 3 (e) - 3 (f) for comparison.

The GFRFs generally require direct identification, identification of the corresponding Volterra kernels [Boyd et al., 1983], or identification of an underlying nonlinear model [Peyton Jones and Billings, 1989, Billings and Peyton Jones, 1990, Zhang et al., 1995]. Hence, Theorem 1 significantly simplifies the identification process and provides useful insight in the mechanism that generates the GFRF.

5. CONCLUSION

In this paper a novel connection between two frequency domain methods for the analysis and modeling of nonlinear systems is presented. Specifically, a unique relation between the Generalized Frequency Response Function (GFRF) and the Higher Order Sinusoidal Input Describing Function (HOSIDF) is presented. These methods are compared and an explicit analytical relation between both is derived for polynomial Wiener-Hammerstein systems. Necessary and sufficient conditions are derived for this mapping to exist and the results are illustrated by a numerical example. Moreover, for this system class uniqueness of the GFRF and HOSIDF models is shown and equivalent properties for linear time invariant systems are presented. This analysis yields a clear insight into the mechanisms that generate the GFRFs and HOSIDFs. Finally, the results provide a numerically efficient method to compute the GFRF from the HOSIDFs and *vice versa*.

REFERENCES

S.A. Billings and J. C. Peyton Jones. Mapping non-linear integro-differential equations into the frequency domain. *Int J Control*, 52:863–879, 1990.

S.A. Billings and K.M. Tsang. Spectral analysis for non-linear systems, part i: Parametric non-linear spectral analysis. *Mech Syst Signal Pr*, 3(4):319–339, 1989.

S. Boyd, Y.S. Tang, and L. O. Chua. Measuring volterra kernels. *IEEE T Circuits Syst*, CAS-30(8):571–577, 1983.

A. Gelb and W. Vander Velde. *Multiple input describing functions and nonlinear system design*. McGraw Hill, 1968.

X. Jing, Z. Lang, and S. Billings. Nonlinear influence in the frequency domain: Alternating series. *Syst Control Lett*, 2011. doi: 10.1016/j.sysconle.2011.01.003.

L.M. Li and S.A. Billings. Analysis of nonlinear oscillators using volterra series in the frequency domain. *J Sound Vib*, 330:337–355, 2011.

P.W.J.M. Nuij, O.H. Bosgra, and M. Steinbuch. Higher-order sinusoidal input describing functions for the analysis of non-linear systems with harmonic responses. *Mech Syst Signal Pr*, 20(8):1883–1904, 2006.

P.W.J.M. Nuij, M. Steinbuch, and O.H. Bosgra. Experimental characterization of the stick/sliding transition in a precision mechanical system using the third order sinusoidal input describing function. *Mechatronics*, 18 (2):100–110, 2008.

A. Pavlov, A. Pogromsky, N. van de Wouw, and H. Nijmeijer. Convergent dynamics, a tribute to Boris Pavlovich Demidovich. *Syst Control Lett*, 52(3-4):257–261, 2004.

A. Pavlov, N. van de Wouw, and H. Nijmeijer. Frequency response functions for nonlinear convergent systems. *IEEE T Autom Control*, 52(6):1159–1165, 2007.

J. C. Peyton Jones and S. A. Billings. Recursive algorithm for computing the frequency response of a class of non-linear difference equation models. *Int J Contr*, 50(5): 1925–40, 1989.

D. Rijlaarsdam, T. Oomen, P. Nuij, J. Schoukens, and M. Steinbuch. Uniquely connecting frequency domain representations of given order polynomial Wiener-Hammerstein systems. *Automatica*, page (provisionally accepted), 2012.

D.J. Rijlaarsdam, P.W.J.M. Nuij, J. Schoukens, and M. Steinbuch. Spectral analysis of block structured non-linear systems and higher order sinusoidal input describing functions. *Automatica*, 47(12):2684–2688, 2011a.

D.J. Rijlaarsdam, P.W.J.M. Nuij, J. Schoukens, and M. Steinbuch. Frequency domain based nonlinear feed forward control design for friction compensation. *Mech Syst Signal Pr*, 27(2):551–562, 2011b.

M. Schetzen. *The Volterra and Wiener theories of nonlinear systems*. John Wiley & Sons, Inc., 1980.

J. Schoukens, R. Pintelon, T. Dobrowiecki, and Y. Rolain. Identification of linear systems with nonlinear distortions. *Automatica*, 41(3):491–504, 2005.

K.S. Shanmugam and M. T. Jong. Identification of nonlinear systems in frequency domain. *IEEE T Aerosp*, 11(6):1218–1225, 1975.

R. Yue, S.A. Billings, and Z.-Q. Lang. An investigation into the characteristics of non-linear frequency response functions. part 1: Understanding the higher dimensional frequency spaces. *Int J Control*, 78(13):1031–1044, 2005.

H. Zhang, S.A. Billings, and Q.M. Zhu. Frequency response functions for nonlinear rational models. *Int J Control*, 61(5):1073–1097, 1995.

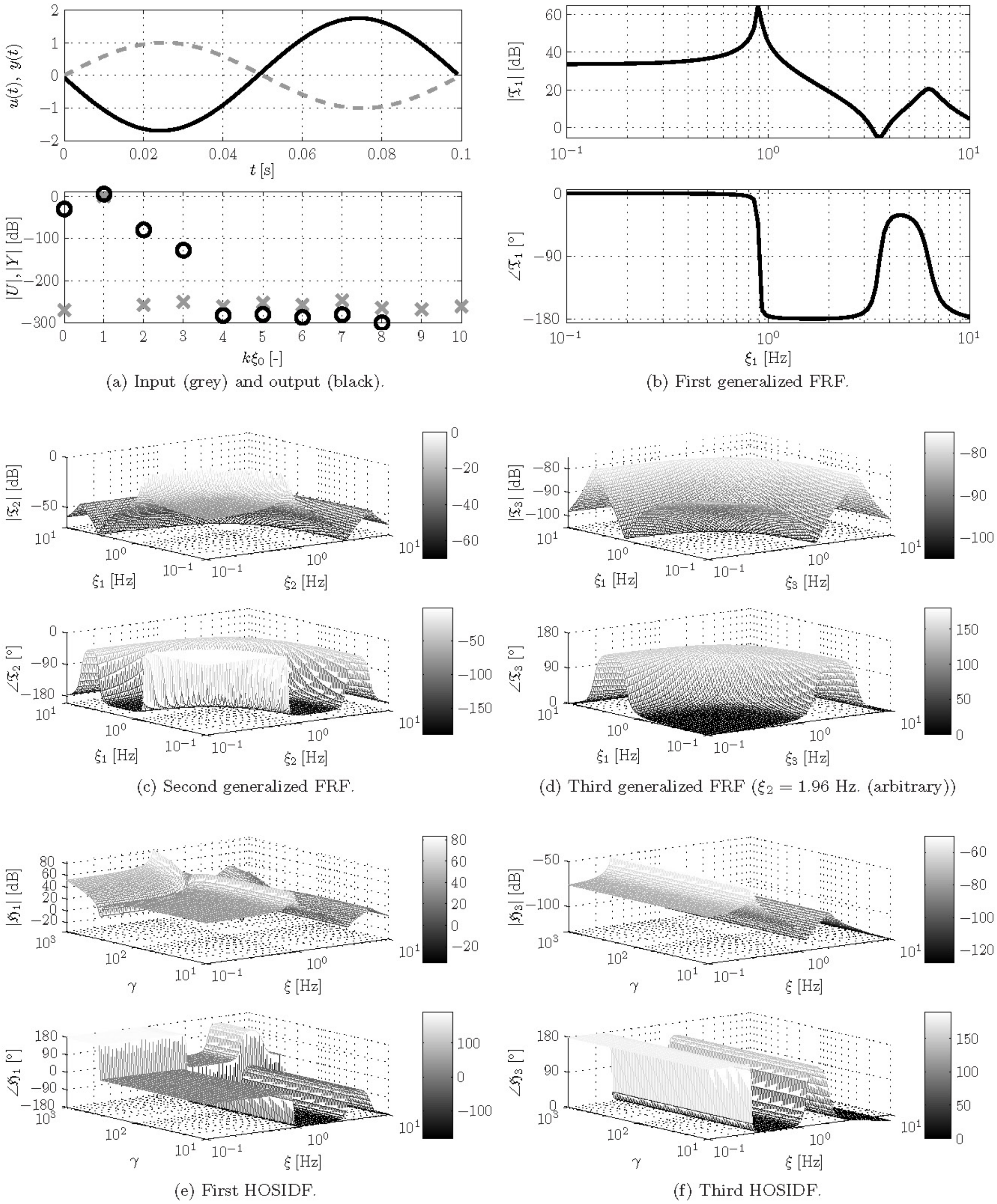

Fig. 3. The systems response, the generalized FRFs and the HOSIDFs.