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Abstract: Frequency Response Function (FRF) modeling of Linear Parameter Varying (LPV) systems facilitates analysis, controller design and parametric modeling of a large class systems, including position-dependent mechanical systems. The aim of this paper is to enable FRF identification of LPV systems using global experiments. This is achieved by developing an appropriate definition of the FRF for input-output LPV systems and by developing a method to compute a statistically optimal estimator of the FRF, which reduces to the Empirical Transfer Function Estimate (ETFE) for frozen scheduling. The developed method is successfully used to estimate a position-dependent FRF of a wide-format printer, confirming the potential of the approach.

Keywords: Linear Parameter-Varying Systems, Frequency-response methods

1. INTRODUCTION

Increasing requirements in control applications lead to a situation where operating condition dependent dynamics need to be explicitly addressed to achieve the performance requirements, and Linear Parameter Varying (LPV) control methods are a promising approach (Hoffmann and Werner, 2015). For example, lightweight system design and high accelerations lead to a situation where position-dependent effects arise due to flexible dynamics in combination with the inherent motion. Explicitly accounting for the configuration of the system, which can typically be accurately measured (Felici et al., 2007; Van Zundert et al., 2016), allows for an LPV system description and enables LPV control (Steinbuch et al., 2003; Wassink et al., 2005). This approach formalizes gain-scheduling control through various model-based methods that often rely on LMI conditions to synthesize controllers with stability and performance guarantees (Apkarian and Gahinet, 1995; Scherer, 2001). This spurred the development of LPV identification methods with a strong focus on parametric models (Bamieh and Giarre, 2002; Lovera and Mercere, 2007; Van Wingerden and Verhaegen, 2009; Tóth, 2010; Goos and Pintelon, 2016), in which the scheduling dependence is often modeled as a summation of basis functions. This naturally fits the modal description of mechanical systems, in which case the basis functions are often available as prior (De Rozario et al., 2017).

Frequency Response Function (FRF)-based design techniques are commonly employed in industrial motion control applications (Steinbuch et al., 2010) since FRF estimates are relatively fast and inexpensive to obtain (Pintelon and Schoukens, 2012), and provide an accurate description that is readily interpreted visually. A key property of the FRF estimate is that no prior knowledge on the dynamic order is required, which is typically extremely high for systems with flexible dynamics (Voorhoeve et al., 2015). Moreover, under mild assumptions on the disturbing noise, statistically optimal FRF estimates are readily obtained by using periodic repetitions (Pintelon and Schoukens, 2012). In these methods, it is assumed that the plant is accurately described as a Linear Time-Invariant (LTI) system and a number of extensions are reported that enable the suppression or estimation of the contribution of nonlinear effects (Rijlaarsdam et al., 2010; Pintelon and Schoukens, 2012). For example, operating the system in a jogging mode helps to reduce stick-slip friction (Van der Maas et al., 2016).

Non-parametric LPV identification methods enable estimation without completely specifying the model structure. Subspace methods (Felici et al., 2007) yield unstructured LPV state-space models for which statistical optimality is not guaranteed, and require the estimation of the state order. In Tóth et al. (2011), an approach is presented to estimate the scheduling dependence non-parametrically for a fixed dynamic order. This is dual to the problem considered here, in which the dynamic order is unknown while the dependence on the scheduling is assumed to be known. In Van der Maas et al. (2017), a method is developed in which both the dynamics and the scheduling dependence are estimated non-parametrically by continuously interpolating local FRF estimates that are obtained for constant scheduling, i.e., frozen configurations. In this way, the experiments only partly reflect the operational conditions since the dependence on the scheduling velocity cannot be captured and may complicate the use of jogging.

Although existing LPV system identification methods are substantially developed, at present, FRF identification of LPV systems based on global experiments while exploiting...
prior knowledge on the scheduling, is not yet investigated. This paper aims to fill this gap by extending the notion of the FRF to a class of affine input-output LPV systems, and by developing an identification method that (i) does not require prior knowledge of the dynamic order, (ii) requires only a limited number of global experiments, and (iii), is statistically optimal. This is achieved through the following sub-contributions.

(C1) A Maximum Likelihood (ML) estimator is formulated for the FRFs of the individual LTI transfers of affine input-output LPV systems, for the case in which the output is corrupted by noise (Section 3).

(C2) An iterative algorithm is developed to solve for the ML estimate and guidelines are presented for implementation and experiment design (Section 4).

(C3) An LPV FRF of a position-dependent motion system is estimated using the developed method (Section 5).

C1 is related to Goos and Pintelon (2016) and Felici et al. (2007) in the sense that periodic scheduling is exploited to obtain Linear Periodic Time-Varying (LPTV) system behavior during the identification, from which the LPV dynamics are inferred. Related to Lataire et al. (2012), prior knowledge on the scheduling functions is exploited to limit user intervention. Interestingly, the Empirical Transfer Function Estimate (ETF) (Ljung, 1999, eqn. 6.24) is recovered as a special case for LTI systems. C2 is related to the iterative optimization approach presented in Blom and Van den Hof (2010), which is generalized to complex parameters in this paper.

1.1 Notation

A Discrete Time (DT) signal $s : Z \mapsto \mathbb{R}$ is called periodic with period $N$ if $s[n+N] = s[n]$ $\forall n$. The $N$-point Discrete Fourier Transform (DFT) of a signal $x$ is defined as, $X[k] = \mathcal{F}(x[n]) \triangleq \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\omega_k n}$, $\omega_k = \frac{2\pi k}{N}$, $k = 0, 1, ..., N-1$. The circular convolution of two length $N$ sequences $f$ and $g$ is defined as, $f[n] * g[n] \triangleq \sum_{k=0}^{N-1} f[(n-k) \mod N]g[k]$. Throughout, $\hat{X}$ denotes the column vector with entries of the length $N$ sequence $X[k]$, i.e., $\hat{X} = [X[0], ..., X[N-1]]^T \in \mathbb{C}^N$, and $\hat{X} \triangleq \text{diag}(\{X\}) \in \mathbb{C}^{N \times N}$ is the diagonal matrix with $X$ on its diagonal. Moreover, $[X]^T$ is the transpose of $X$, $[X]^T$ is the matrix that remains by removing $[X]^T$ from $X$ and the same notation is used for $\hat{X}$ where $[\hat{X}]^T$ the $\tau$th row of $\hat{X}$.

2. PROBLEM FORMULATION

In this section, the considered class of LPV systems is introduced and the identification problem is presented.

2.1 The Considered Class of LPV Systems

The following DT single-input single-output LPV systems are considered,

\begin{align}
\dot{x}(t) & = A(r) x(t) + B(r) u(t), \\
\dot{x}(t) & = A(r) x(t) + B(r) u(t),
\end{align}

where $r$ is the scheduling parameter with $r[n] \in D \subseteq \mathbb{R}^{n_r}$ and $p_i$ are bounded functions of $r$, i.e., $p_i : D \mapsto \mathbb{R}$. Furthermore, $a_i(q)$ and $b_i(q)$ are polynomials with orders $n_a^{(i)}$ and $n_b^{(i)}$, i.e., for $i = 0, ..., n_r$,

\begin{align}
a_i(q) = \sum_{j=0}^{n_a^{(i)}} a_{ij} q^{-j}, & \quad b_i(q) = \sum_{j=0}^{n_b^{(i)}} b_{ij} q^{-j},
\end{align}

with $q$ the shift operator, i.e., $q^{-\tau} s[n] = s[n-\tau]$. Note that these filters can be equivalently represented in the frequency domain by the DFT coefficients of their impulse responses, i.e., for $i = 1, ..., n_r$,

\begin{align}
A_i[k] = \mathcal{F}(a_i(q) \delta[n]), & \quad B_i[k] = \mathcal{F}(b_i(q) \delta[n]),
\end{align}

where $\delta[n]$ denotes the discrete impulse. Key advantages of (1) are that this description encompasses many practical systems and that controller synthesis algorithms are well-developed for this model structure (Ali et al., 2010).

2.2 The Empirical Transfer Function Estimate

Consider the ETFE (Ljung, 1999, eqn. 6.24),

\begin{align}
\theta_{\text{eff}} & \triangleq U^{-1} \hat{Y}, & \theta_{\text{eff}} & \in \mathbb{C}^N,
\end{align}

where the notation as presented in Section 1.1 is used. It is well known that for LTI systems in steady state with period $N$ and noise free $u$ and $y$, the following holds; $\theta_{\text{eff}} = G$, where $G[k] = A_0^{-1}[k]B_0[k]$. Consequently, the FRF of an LTI system is readily estimated from the DFT coefficients $U[k]$ and $Y[k]$ without requiring knowledge on the polynomial orders $n_a^{(0)}$, $n_b^{(0)}$ of the system. Moreover, when $u$ and $y$ are periodic and corrupted by Gaussian disturbances, then (4) is the Maximum Likelihood solution if $U[k]$ and $Y[k]$ are replaced by their sample averages, which are obtained by averaging multiple independent periods (Pintelon and Schoukens, 2012).

The aim of this paper is to develop an ETFE type estimator for the class of LPV systems given by (1), which includes the classical LTI ETFE as a special case.

2.3 The Identification Problem

To formulate a global FRF identification procedure for LPV systems, the following assumptions are made.

Assumption 1. The system can be excited by a freely chosen periodic signal, $u(n+N_u) = u(n)$, that is synchronized with $r$, i.e., $\frac{N_u}{n_r} \in \mathbb{Q}$.

In the case of mechanical systems, as is considered in Section 5, $r$ is typically a collection of position variables which can often be accurately measured. The functions $p_i(r)$ can be obtained from first principles modeling or can be approximated using a general basis, e.g., based on splines of polynomials, which often leads to accurate results (De Rozario et al., 2017; Van der Maas et al., 2017).

Assumption 2. The scheduling signal $r$ can be set to be periodic, i.e., $r(n+N_p) = r(n) \forall n$.

This is achieved for motion systems by superimposing the position of certain parts to perform a periodic motion (Felici et al., 2007; Van Zundert et al., 2016).

Assumption 3. The system can be excited by a freely chosen periodic signal, $u(n+N_u) = u(n)$, that is synchronized with $r$, i.e., $\frac{N_u}{n_r} \in \mathbb{Q}$.
Assumption 4. The output $y$ of the system (1) under assumptions 2 and 3 is bounded for bounded inputs $u$ and the periodic steady-state response is observed (Bittanti and Colaneri, 2009).

Under assumptions 2 to 4 the steady state output $y$ is periodic with period length $N$, where $N$ is the least common multiple of $N_a$ and $N_p$ (Goos and Pintelon, 2016). Consequently, no leakage is introduced by transforming the periodic signals $\rho, u$ and $y$ to the frequency domain by using the DFT (Pintelon and Schoukens, 2012).

Assumption 5. The input is exactly known and the output is perturbed by stationary filtered white noise, i.e.,

$$ u[k] = u_o[k], \quad y[k] = y_o[k] + v[k], \quad (5) $$

where $y_o$ is deterministic and $v$ is assumed to be such that asymptotically $(N \to \infty)$ $V[k]$ is circular complex, normally distributed and uncorrelated over the frequency, i.e., the regular covariance matrix of $Y[k]$ is given by,

$$ C_Y \triangleq \mathbb{E}\{VV^H\} = \begin{bmatrix} \sigma_y^2 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \sigma_y^2 & \cdots \\ \end{bmatrix} . \quad (6) $$

Assumption 5 is commonly made for control systems where $y$ is measured and the digital input $u_o$ is known exactly and is not measured during the experiments.

The identification problem is formalized as follows.

**Problem 1.** Develop an estimator of the DFT coefficients $A_i[k], B_i[k]$, in (3), which does not require knowledge on the orders $n_a^{(i)}$ and $n_b^{(i)}$, and which maximizes the likelihood of the output error under assumptions 2 to 5.

In this section, the considered class of LPV systems is introduced and the identification problem is formalized under a set of assumptions that are commonly made in LPV system identification. In the next section, an approach is developed to solve Problem 1.

3. LPV FRF ESTIMATION: A GLOBAL APPROACH

In this section, a solution to Problem 1 is developed. First, the periodic steady-state behavior of (1) is represented in the frequency domain and an identifiable parametrization of the resulting representation is derived. This is followed by the formulation an ML estimator which constitutes sub-contribution C1.

3.1 A Frequency Domain Representation

The periodic steady-state behavior of (1) allows an intuitive frequency domain representation. Under assumptions 2 to 4, transformation of (1) using the DFT results in,

$$ A[k]Y[k] = B[k]U[k], \quad (7a) $$

$$ A[k] = A_0[k] + \sum_{i=1}^{n_p} P_i[k] \ast A_i[k], \quad (7b) $$

$$ B[k] = B_0[k] + \sum_{i=1}^{n_p} P_i[k] \ast B_i[k], \quad (7c) $$

where $A_i[k], B_i[k] \in \mathbb{C}$ are the parameters to be estimated. Relation (7) can be equivalently represented by,

$$ A \dot{Y} = BU, \quad A, B \in \mathbb{C}^{N \times N}, \quad \begin{align} A & \triangleq A_0 + \sum_{i=1}^{n_p} T_p A_i, \quad B \triangleq B_0 + \sum_{i=1}^{n_p} T_p B_i, \end{align} \quad (8a) $$

$$ \begin{bmatrix} P_1[0] & \cdots & P_1[N-1] \\ P_2[1] & \cdots & P_2[N-2] \\ \vdots & \ddots & \vdots \\ P_N[N-1] & \cdots & P_N[0] \end{bmatrix} \quad \begin{align} T_p[i] & = \begin{bmatrix} P_1[0] & \cdots & P_1[1] \\ P_1[1] & \cdots & P_1[2] \\ \vdots & \ddots & \vdots \\ P_1[N-1] & \cdots & P_1[0] \end{bmatrix} \\ \end{align} \quad (9) $$

The next section addresses the challenge of obtaining a parametrization of (8) that associates each parameter with a unique model.

3.2 An Identifiable Parametrization

The aim of this section is to develop an identifiable parametrization of (8) to enable consistent estimation, i.e., each parameter is associated with a unique model. To this end, let

$$ G(\theta) = \hat{A}^{-1}(\theta)B(\theta), \quad [A_0] = 1, \quad (10a) $$

$$ \theta = [\theta_a^T \theta_b^T]^T \in \mathbb{C}^{n_e}, \quad n_e = 2(N(n_p + 1) - 1), \quad (10b) $$

$$ \begin{align} \theta_a^T & = [\hat{A}_0]^{-1} \hat{A}_1 \ldots \hat{A}_{n_p}, \quad (10c) \\ \theta_b^T & = [\hat{B}_0^T \hat{B}_1^T \ldots \hat{B}_{n_p}^T], \quad (10d) \end{align} $$

This parametrization, the $\tau$th DFT coefficient of $A_0$ is fixed to unity to suppress the following degree of freedom

$$ A^{-1}B = \hat{A}^{-1}\alpha^{-1}\hat{B} = \hat{A}^{-1}\hat{B}, \quad \alpha \in \mathbb{C}, \quad (10e) $$

The question remains whether (10) is identifiable and how this depends on the matrices $T_p$. This corresponds to the notion that the scheduling functions $p_i$ must be sufficiently exciting (Tóth, 2010, §9.3.5.2). Since under Assumption 1 the signals $p_i(\rho[n])$ are assumed to be known, the matrices $T_p$ are treated in this paper to be a part of the model. The following theorem presents the condition on the signals $p_i(\rho[n])$ such that global identifiability of (10) is ensured.

**Theorem 1.** Parametrization (10) is globally identifiable (Ljung, 1999, Def. 4.8) if and only if the columns of the following matrix are independent,

$$ [P_1 \ P_1 \ldots P_{n_p}] , \quad P_0 = [1 \ 0 \ \ldots \ 0]^T \in \mathbb{R}^N. \quad (11) $$

The proof exploits the circulant and diagonal structure of $T_p$, $A_i$, and $B_i$, respectively, and will be presented elsewhere. This theorem states that for almost all $\theta$ there are no different values that correspond to equal models if the DFT vectors of the scheduling signals $p_i(\rho[n])$ are independent. Recall that for position-dependent motion systems this can be achieved by choosing an appropriate scheduling trajectory. Note that only a single constraint is required, since all frequencies are coupled by the scheduling. When scheduling is constant, i.e., in the LTI case $p_i = 0 \ \forall i$, $N$ constraints are required instead to obtain an identifiable parametrization and $A_0[k]$ and $B_0[k]$ cannot be determined separately.

In this section, the considered class of systems is parameterized and conditions are provided to ensure global identifiability. In the next section an ML estimator of $\theta$ is formulated.
3.3 Maximum Likelihood Estimator

The aim of this section is to solve Problem 1 by formulating a likelihood maximizing estimator of $\theta$. In addition, special cases of the resulting optimization problem are considered which shows that the ETFE is obtained when the approach is applied to LTI systems.

The estimator that solves Problem 1 maximizes the likelihood of the following the equation error,

$$E(\theta) \triangleq \bar{Y} - A^{-1}(\theta)B(\theta)\tilde{U}.$$  \hfill (12)

Under the assumption that $G(\theta)$ represents the true system, it holds that $E\{\bar{Y}\} = A^{-1}(\theta)B(\theta)\tilde{U}$ such that $E(\theta) = \bar{V}$ and $E(\theta) = C_Y$. Consequently, $E(\theta)$ is complex circular normally distributed and for a given data set of $n_e$ observations, $\bar{Y} = \{\bar{Y}^{(i)}\}_{i=1}^{n_e}$, corresponding to the known input and scheduling vectors, $\{\tilde{U}^{(i)}, \bar{P}_1^{(i)}, \ldots, \bar{P}_n^{(i)}\}_{i=1}^{n_e}$, $E(\theta)$ satisfies the following joint likelihood function,

$$\ell(\theta|\bar{Y}) = \prod_{i=1}^{n_e} \frac{1}{\sqrt{2\pi} \det(C_Y)} e^{(-E(\theta)^{\top} C_Y^{-1} E(\theta))}.$$  \hfill (13)

Maximizing the likelihood is equivalent to minimizing $-\log\ell(\theta|\bar{Y})$, such that the ML estimator is given by,

$$\hat{\theta}_{ml} = \arg\min_{\theta \in \mathbb{C}^n} \ell(\theta|\bar{Y}),$$ \hfill (14a)\hspace{1cm}$$V_{ml} \triangleq \frac{1}{n} \sum_{i=1}^{n} \ell_i(\theta)^{\top} \ell_i(\theta),$$ \hfill (14b)

where $C_Y = \bar{V} \bar{V}^{\top}$. \hfill (14c)

Since $n_0$ is independent of $n_e$, it can be shown that $\hat{\theta}_{ml}$ is consistent, asymptotically ($n_e \to \infty$) normally distributed and asymptotically efficient (Goodwin and Payne, 1977). The latter implies that the parameter covariance matrix asymptotically attains the Cramér-Rao lower-bound, i.e.,

$$C_{\hat{\theta}} \succeq \mathbb{E}\{\Gamma^{\top}\Gamma\}^{-1}, \quad \Gamma = \frac{\partial \log \ell(\theta|\bar{Y})}{\partial \theta}.$$ \hfill (15)

Note that $\ell$ is treated as a real valued function of $\theta$ and its complex conjugate $\theta^*$, i.e. $\ell : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$, to ensure the derivative in (15) exists as is elaborated in (Brandwood, 1983) and (Van den Bos, 1994).

In practice the true output covariance matrix $C_Y$ is typically unknown and the ML estimator can be approximated by the Sampled Maximum Likelihood estimator (SML). In this approach, the periodic nature of the output $y$ is exploited by replacing $\bar{Y}$ with the sample mean $\bar{Y}$ and $C_Y$ by the sample covariance matrix $C_Y$ estimate, as is elaborated in Pintelon and Schoukens (2012, chap. 10).

Note that (14) is a nonlinear optimization problem, since $A(\theta)$ is a full matrix in the general LPV case. In case $A(\theta)$ is independent of $\rho$, (12) is linear-in-the-parameters such that (14) results in a linear system of equations that can readily be solved analytically. This special case of LPV systems is encountered in mechanical systems that perform a motion w.r.t. sensors in a fixed reference frame (De Rozario et al., 2017). Furthermore, in the LTI case, (12) is diagonal since both $A(\theta)$ and $B(\theta)$ are independent of $\rho$, hence (14) decouples per frequency and $\bar{\theta}_{ml}$ reduces to the classical ETFE which is independent of $C_Y$.

In this section, an ML estimator of $\theta$ is formulated, with favorable asymptotic properties. This effectively provides a solution to Problem 1 and in the next section, an iterative approach is developed to solve (14).

4. ITERATIVE OPTIMIZATION

The optimization problem (14) is generally nonlinear and can practically not be solved analytically. In this section, the approach as presented in (Blom and Van den Hof, 2010) is generalized to complex parameters and is developed to solve (14), which constitutes sub-contribution C2.

4.1 The Iterative Optimization Method

The aim is to converge to a stationary point $\hat{\theta}$ of $V_{ml}$. To overcome the fact that $V_{ml}$ is not analytic in $\theta$, $V_{ml}$ is regarded as a real valued function of $\theta$ and its complex conjugate $\theta^*$, as in the previous section. A necessary and sufficient condition for $\hat{\theta}$ to be a stationary point is (Brandwood, 1983, Theorem. 3),

$$\frac{\partial V_{ml}(\theta, \theta^*)}{\partial \theta^\top \mid_{(\hat{\theta}, \hat{\theta}^*)}} = \sum_{i=1}^{n} \ell_i(\theta)^{\top} J_i(\theta) = 0,$$ \hfill (16)

which is referred to as the Jacobian matrix. In the iterative approach, (16) is rewritten as,

$$\sum_{i=1}^{n} \left( W^{-1} A_i(\theta_j) Q_i(\theta_{j+1}) \right)^{\top} J_i(\theta_j) = 0, \quad (18a)$$

which is iteratively solved for $\theta_j$, with $\theta_j$ the result from the previous iteration. A key property of this method is that the fixed points of (18) correspond to the stationary points of the original cost function, i.e. (16) holds. The following theorem presents the key results to express (18) in terms of the data in a compact manner.

Theorem 2. With $\theta$ defined as in (10), $Q(\theta)$, as given by (18b), can be written in linear regression form as,

$$Q(\theta) = |Y|^{\top} + \Phi_\theta, \quad \Phi \triangleq [\Phi_\theta, \Phi_u],$$ \hfill (19)

$$\Phi_y = \begin{bmatrix} |Y|^{\top} T_p Y \cdots T_{p_y} Y \end{bmatrix} \in \mathbb{C}^{N \times N(p_y + 1)}^{-1},$$ \hfill (20)

$$\Phi_u = \begin{bmatrix} U T_{p_1} U \cdots T_{p_y} U \end{bmatrix} \in \mathbb{C}^{N \times N(p_y + 1)},$$ \hfill (21)

and $J(\theta)$ as given by (17) can be written as,

$$J(\theta) = W^{-1} A^{-1}(\theta) [\Psi(\theta) \Phi_u],$$ \hfill (22)

$$\Psi(\theta) = \begin{bmatrix} |Z|^{\top} T_p Z \cdots T_{p_y} Z \end{bmatrix} \in \mathbb{C}^{N \times N(p_y + 1)}^{-1},$$ \hfill (23)

with $Z(\theta)$ the diagonal matrix whose diagonal equals,

$$Z(\theta) = A^{-1}(\theta) B(\theta) \tilde{U}.$$ \hfill (24)

The proof follows by applying matrix differentiation and Kronecker algebra identities, and will be published elsewhere. Using this theorem, the Hermitian transpose of (18) can be written as the following system of equations,

$$\Gamma(\theta_j) \theta_{j+1} = \gamma(\theta_j)$$ \hfill (25a)

$$\Gamma(\theta_j) = \Psi(\theta_j)^{\top} W(\theta_j) H W(\theta_j) \Phi,$$ \hfill (25b)

$$\gamma(\theta_j) = \Psi(\theta_j) H W(\theta_j) H b,$$ \hfill (25c)
Fig. 1. The aim is to estimate the transfer between forces \( u \) and the \( y \)-acceleration of the carriage of the Océ Arizona flatbed printer, which depends on the position \( \rho \) due to flexible dynamics.

\[
b = \begin{bmatrix} Y(1) \cdots Y(n) \end{bmatrix}^T, \\
\Psi = \begin{bmatrix} \Phi_1 \Phi_2 \cdots \Phi_n \end{bmatrix}, \\
W = \begin{bmatrix} w_1 \cdots w_n \end{bmatrix},
\]

and the algorithm is summarized as iteratively solving (25), starting from an initial estimate \( \theta_0 \).

4.2 Practical Aspects

For (25) to have a unique solution, \( \Gamma(\theta_j) \) must be regular, which implies that \( \Phi \) and \( \Psi \) must have full column rank. Consequently, the number of experiments must satisfy \( n_k \geq 2(n_k + 1) \) and different realizations of \( U \) are required, which can be readily generated by using multi-sines with varying phases. Due to structure of \( \Phi \) and \( \Psi \) as given by (21), (20) and (23), \( \bar{U} \) and \( \bar{Y} \) may not contain any zero entries. In practice, the excitation bandwidth is often smaller than the Nyquist frequency, i.e. \( U[k] = 0 \) for some \( k < \frac{N}{2} \). Similarly in this case, if the scheduling functions \( p_i \) are slowly varying, \( Y \) may be close to the noise level for some frequencies (Goos and Pintelon, 2016). This results in columns of zeros in \( \Phi \) and \( \Psi \) such that some unknowns \( \theta \) cannot be determined. By removing these from (25a), a regular system of equations may be obtained. Furthermore, recall that the DFT coefficients satisfy the following conditions, \( \Im{X[0]} = 0 \), and for \( k = 1, \ldots, N-1 \), \( \Re{X[k]} = \Re{X[N-k]} \), \( \Im{X[k]} = -\Im{X[N-k]} \). By imposing these conditions on the complex system of equations (25a) the dimension of \( \theta \) is reduced by a factor 2 and the ML problem is obtained when the DC and Nyquist components are used, i.e., \( k = 0 \) and \( k = \frac{N}{2} \in \mathbb{N} \) (Pintelon and Schoukens, 2012, §9.1.2).

In this section, an iterative method is developed to solve for \( \hat{\theta}_{nl} \) and practical guidelines are provided to efficiently construct the required regular system of equations. This completes the formulation of the LPV FRF identification procedure that is applied to a motion system next.

5. APPLICATION TO A MOTION SYSTEM

In this section, the developed identification method is used to estimate a position-dependent FRF of a simulated wide-format printer, which constitutes sub-contribution C3.

5.1 The Plant

Consider the wide-format printer as shown in Figure 1. The identification objective is to estimate a position-dependent FRF of the \( y \)-direction acceleration of the print head as induced by the actuator forces \( u \). A simplified model of the LPV dynamics is given by (1) where,

\[
p_1 = \rho^2, \quad p_2 = 2\rho^3 - \rho, \quad (26)
\]

and for which the polynomial coefficients are as follows.

\[
\begin{array}{ccccccc}
\rho_{1(q)} & \rho_{2(q)} & \rho_{3(q)} & \rho_{4(q)} & \rho_{5(q)} & \rho_{6(q)} & \rho_{7(q)} \\
-1.9081 & -0.9081 & 0.9081 & -1.9081 & 0.9081 & -1.9081 & 0.9081 \\
-2.9444 & 0.0276 & -0.3462 & -2.2171 & 1.2646 & -1.6353 & 0.2377 \\
0.9277 & -0.0054 & -0.0222 & 0.7077 & 0.3855 & 0.4155 & \\
\end{array}
\]

5.2 The Identification Procedure

First, the carriage is set to perform the periodic motion, \( \rho[k] = \sin(2\pi \frac{k}{10}) + \frac{1}{2} \cos(4\pi \frac{k}{10} + \frac{\pi}{2}) \). \( N = 400 \). (27)

By means of Theorem 1, it is verified that (10) is globally identifiable for (26) and (27). Then, 14 experiments are conducted using full-band random phase multi-sines with constant spectrum, during which \( y \) is perturbed as in Assumption 5. After the transients have sufficiently decayed, 20 periods are used to estimate the averages \( \hat{Y}^{(i)} \) and covariances \( \hat{C}^{(i)} \). The noise coloring filter is modeled after a typical accelerometer spectrum, resulting in an average Signal-to-Noise Ratio (SNR) as shown in Figure 2.

First, the ETFE is computed using (4). The averaged result is shown in Figure 3, which shows distortion at higher frequencies, even though the SNR is relatively high. This is caused by the inability of the ETFE to account for the position-dependent dynamics. Next, the LPV FRF is computed by using the iterative algorithm as presented in Section 4 to solve for the SML estimate as presented in Section 3. In Figure 4, the cost is shown as a function of the iterations for the cases with \( n_k \in \{6, 8, 10, 12, 14\} \) observations. This shows that the iterative algorithm achieves a significant improvement over an arbitrary initial estimate. Moreover, when the number of observations is increased,
over-fitting is reduced as is evidenced by the observation that $V_{sml}(\hat{\theta})$ tends to $V_{sml}(\theta_0)$. Figures 5 and 6 show the magnitude of the estimated FRF for constant values of $\rho \in [-1,1]$ and the absolute relative error between the estimated FRF $G(\hat{\rho})$ and the true frozen FRF $G(\theta_0)$, respectively. This shows that an accurate estimate of the position-dependent dynamics is obtained.

6. CONCLUSION

In this paper, a method is developed to estimate the FRF of LPV systems by using a limited number of global experiments. Application of the presented approach to a wide-format printer shows that accurate position-dependent models can indeed be obtained.

REFERENCES


