Frequency Response Function Identification of Periodically Scheduled Linear Parameter-Varying Systems

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**ABSTRACT**

For Linear Time-Invariant (LTI) systems, Frequency Response Functions (FRFs) facilitate dynamics analysis, controller design, and parametric modeling, while many practically relevant systems are in fact more accurately described by Linear Parameter-Varying (LPV) models. The aim of this paper is to develop an FRF modeling framework for periodically scheduled Single-Input Single-Output (SISO) LPV systems, that enables the identification of LPV FRF models from global experiments. This is achieved by developing an appropriate definition of the harmonic FRF for input-output LPV systems and by developing a method to compute a suitable harmonic FRF estimator. The developed approach generalizes the Empirical Transfer Function Estimate (ETFE) to the class of periodically-scheduled LPV systems, and the classical ETFE is recovered for LTI systems as a special case. The developed method is successfully used to estimate a SISO LPV FRF of an experimental motion system, thereby confirming the potential of the developed framework.

1. Introduction

Increasing requirements in control applications lead to a situation where operating-condition dependent dynamics need to be explicitly addressed to achieve the control performance requirements, for instance through Linear Parameter Varying (LPV) control [22]. In wind turbine [37] and aerospace applications [41], the aerodynamic coefficients depend on exogenous parameters such as the wind speed, pitch angle, and altitude. In motion applications, such as industrial robotics [21] and lithographic systems [53], flexible dynamics and configuration-dependent inertia result in position-dependent effects [40, 12, 54]. These parameter variations can hamper the application of Linear Time-Invariant (LTI) control methods. The LPV framework aims to overcome the limitations of LTI control by explicitly accounting for parameter variations. Typical LPV approaches aim to formalize gain-scheduling control design [36] by formulating model-based synthesis problems that ensure stability and a guaranteed level of performance [3, 34]. The model-based nature of these methods spurred the development of LPV identification methods with a strong focus on parametric models [4, 28, 48, 42, 20, 44], which typically impose specific requirements on the prior knowledge regarding the dynamic order of the process.

In contrast to parametric models, nonparametric Frequency Response Function (FRF) models, such as the Empirical Transfer Function Estimate (ETFE) [26, eqn. 6.24], are commonly employed in practice, e.g., in industrial motion control design [39]. Benefits to using FRF models are that they are readily determined from input-output data, are often relatively fast and inexpensive to obtain [30], and provide an accurate description of the dynamics that is readily interpreted visually. More importantly, FRF estimates require no prior knowledge on the dynamic order, making them particularly well-suited for systems with flexible dynamics, whose order is typically extremely high [52]. In addition, the use of periodic excitation signals in the identification process enables estimation of nonparametric noise models, assessment of nonlinearities [31][30, §3], and analysis of time-variations [25]. However, the fundamental assumption that the plant is accurately described as a Linear Time-Invariant (LTI) system limits the use of FRF estimates in LPV applications.

LPV identification approaches can be classified as global or local, depending on the nature of the scheduling signals during the experiments, and local strategies are reported that estimate LPV models from a grid of local FRF estimates [46, 13, 32, 44, 51]. Each local estimate is obtained by freezing the scheduling signal at a constant value.

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Consequently, the behavior captured by the set of frozen estimates reflects only a part of the true operational conditions, which generally include non-constant scheduling behavior. Fundamentally, the dependence on the derivatives of the scheduling signal, known as dynamic dependence, cannot be captured by local experiments [2]. Moreover, a substantial amount of local FRF estimates is typically required to achieve the desired model accuracy [50, 15]. Each local FRF estimate needs to be estimated independently, and obtaining a sufficiently dense grid of local models can be relatively time-consuming, especially for a large number of scheduling variables. In addition, the plant may be unstable for a region of the frozen parameter space, which may complicate the estimation of the local models. Global strategies aim to overcome these limitations by using appropriate nonconstant scheduling signals that scan the parameter space and result in stable Linear Time-Varying (LTV) dynamics during the experiments, as will be demonstrated in this article.

LPV identification strategies can also be classified in the way in which the model is parametrized, and in contrast to LTI systems, the model can be (non)parametric in the scheduling dependence, and/or in the dynamics. For example, LPV subspace methods are typically parametric in the scheduling dependence, and are considered nonparametric in the dynamics since no specific structure is imposed on the state-space matrices, although estimation of the dynamic order is still required [12, 48]. In [43], a method is developed for input-output LPV models that employs a support vector machine approach to estimate the scheduling dependence nonparametrically, yet similarly requires prior knowledge of the dynamic order. The Bayesian approaches developed in [16] and [10] enable nonparametric estimation in both the dynamics and parameter dependence through the use of Gaussian processes. These approaches require careful tuning of the hyperparameters associated with the kernel functions and require approximate prior knowledge on the pole locations [6]. In contrast to these pre-existing approaches, the approach developed in this paper is nonparametric in the dynamics, parametric in the scheduling dependence, and does not require prior knowledge on the dynamic order.

Although LPV control has a large potential to improve the performance in many applications and substantial research has led to compatible system identification methods, presently, no systematic framework exists that enables the identification of nonparametric LPV FRF models from global experiments. The aim of this paper is to develop such a framework for a class of input-output LPV systems, which recovers the classical ETFE as a special case. This is achieved through the following contributions.

(C1) A new class of identifiable LPV FRF parametrizations is developed (Section 3).

(C2) A numerical parameter estimation framework is developed, which includes the formulation of a Weighted Non-linear Least Squares (WNLS) estimator and an explicit formulation of the corresponding Jacobian, which facilitates various iterative methods to solve the resulting nonlinear optimization problem (Section 4 and 5).

(C3) An LPV FRF of an experimental parameter-varying motion system is estimated using the developed approach and its generalizing properties are validated (Section 6).

Contribution C1 employs periodic scheduling to obtain Linear Periodic Time-Varying (LPTV) system behavior during the identification experiments, and a harmonic FRF model of the Discrete Time (DT) Harmonic Transfer Function (HTF) is estimated in an approach closely related to [27]. The key difference is that the LPTV model is specifically structured in terms of the scheduling parameters as in [12, 20], and the LPV parameters are subsequently inferred from the LPTV model, which is enabled by explicitly deriving conditions under which the LPV parameters are identifiable. Contribution C2 formulates a Maximum Likelihood Estimator (MLE) by appropriate weighting with the noise covariance [20, 19]. Bounds on the uncertainty of the MLE are determined based on its statistical properties. In addition, the Jacobian of the MLE cost function is determined in closed-form, which enables the formulation of various iterative optimization approaches. Contribution C3 is established by controlling a flexible beam system using an LPV controller, and estimating a global LPV FRF model of the closed-loop dynamics using the developed framework. It is shown that the LPV FRF contains a continuum of LTI FRF models, which includes representations of stable and unstable frozen dynamics. The accuracy of the stable dynamics is assessed using local ETFEs, and the instability boundary is confirmed experimentally. The predictive capabilities of the estimated global LPV FRF model are validated, indicating a level of performance that is comparable to the classical ETFE in its appropriate LTI setting.

This paper extends the results presented in [11] by providing proofs to the theorems and by presenting experimental results.

### 1.1. Notation

A DT signal \( s : \mathbb{Z} \rightarrow \mathbb{R} \) is called \( N \)-periodic if \( s(t+N) = s(t), \forall t \), for some \( N \in \mathbb{N} \). The \( N \)-point Discrete Fourier Transform (DFT) of \( x(t) \) is defined as, \( X(k) = \mathcal{F} \{ x(t) \} \triangleq \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} x(t) e^{-j\omega_k t}, \omega_k = \frac{2\pi k}{N}, k = 0, 1, ..., N - 1, \) and
the index set corresponding to real and complex DFT coefficients are denoted by

\[ k_\mathbb{R} \triangleq \left\{ k \in \mathbb{Z} \mid k = 0 \wedge \text{if } N \text{ is even } k = \frac{N}{2} \right\}, \quad \text{and} \quad k_\mathbb{C} \triangleq \left\{ k \in \mathbb{Z} \mid k = 0, \ldots, N - 1 \right\} \setminus k_\mathbb{R}. \]

The circular convolution of two \( N \)-length sequences \( X(k) \) and \( Z(k) \) is defined as, \( X(k) * Z(k) \triangleq \sum_{k=0}^{N-1} X(\langle n - k \rangle_N)Z(k) \), where \( \langle x \rangle_N \) denotes \( x \) modulo \( N \). A frequency lifted notation is used throughout where \( \hat{X} \) denotes the length \( N \) column with entries \( X(k) \), and \( \hat{X} \) denotes the diagonal matrix with \( \hat{X} \) on its diagonal, i.e.,

\[
\hat{X} = \begin{bmatrix} X(0) & \cdots & X(N-1) \end{bmatrix} \in \mathbb{C}^N, \quad X \triangleq \text{diag}\{ \hat{X} \} \in \mathbb{C}^{N \times N}.
\]

Moreover, \( [X]_r^t \) denotes the \( r \)-th column of \( X \), \( [X]^r \) is the matrix that remains by removing \( [X]_r^t \) from \( X \), and the same notation is used for \( \hat{X} \) where \( [\hat{X}]_r^t \) denotes the \( r \)-th element of \( \hat{X} \).

2. Problem formulation

In this section, the considered class of LPV systems is introduced and the identification problem is formulated.

2.1. The considered class of LPV Systems

In this paper, the following class of DT Single-Input Single-Output (SISO) LPV systems is considered

\[
a(\rho(t), q^{-1})y(t) = b(\rho(t), q^{-1})u(t), \tag{2a}
\]

\[
a(\rho(t), q^{-1}) = a_0(q^{-1}) + \sum_{i=1}^{n_\rho} \phi_i(\rho(t))a_i(q^{-1}), \tag{2b}
\]

\[
b(\rho(t), q^{-1}) = b_0(q^{-1}) + \sum_{i=1}^{n_\rho} \psi_i(\rho(t))b_i(q^{-1}). \tag{2c}
\]

Here, \( \phi_i(\rho) \) and \( \psi_i(\rho) \) are bounded functions \( \phi_i, \psi_i : D \to \mathbb{R} \) of a known exogenous scheduling parameter \( \rho(t) \in D \subseteq \mathbb{R}^{n_\rho} \), that consists of any relevant time-varying parameters and their derivatives [35, 42]. The aim of this paper is to identify nonparametric FRF estimates of the functions \( a_i(q^{-1}) \) and \( b_i(q^{-1}) \) in (2b) and (2c), which are polynomials in \( q^{-1} \) of degree \( n_i \), i.e.,

\[
a_i(q^{-1}) = \sum_{j=0}^{n_i} a_{ij}q^{-j}, \quad b_i(q^{-1}) = \sum_{j=k_i}^{n_i} b_{ij}q^{-j}, \tag{3}
\]

where \( q \) is the shift operator, i.e., \( q^{-r}s(t) = s(t - r) \), and \( 0 \leq \kappa_i \leq n_i \). Note that for constant scheduling, i.e., \( \rho(t) = \tilde{\rho} \) \( \forall t \), (2) reduces to an LTI recursion relation with dynamic order \( \max_{i} n_i \) and relative degree \( \min_i \kappa_i \). The following example illustrates how (2) naturally results from first-principles modeling.

**Example 1.** Consider the following continuous-time model of a SISO mass-spring-damper system with parameter varying stiffness \( k(\rho(t)) = k_0 - k_1\rho(t) \),

\[
\left( m\frac{d^2}{dt^2} + d\frac{d}{dt} + (k_0 - k_1\rho(t)) \right)y(t) = u(t).
\]

A DT LPV representation is obtained by replacing \( \frac{d}{dt}y(t) \) and \( \frac{d^2}{dt^2}y(t) \) by the one-sided finite difference approximations 

\[
\frac{1}{T_s}(1 - q^{-1})y(t), \quad \text{and} \quad \frac{1}{T_s^2}(1 - 2q^{-1} + q^{-2})y(t), \tag{4}
\]

respectively. The result is of the form (2a), since it can be written as

\[
(a_0(q^{-1}) + a_1(q^{-1})\phi_1(\rho(t)))y(t) = b_0(q^{-1})u(t),
\]

where \( b_0(q^{-1}) = 1, a_1(q^{-1}) = 1, \phi_1(\rho(t)) = k_1\rho(t) \) and \( a_0(q^{-1}) = \frac{m}{T_s^2}(1 - 2q^{-1} + q^{-2}) + \frac{d}{T_s}(1 - q^{-1}) + k_0 \). The continuum of FRFs that is obtained by freezing \( \rho(t) = \tilde{\rho} \in [-1, 1] \) is shown in Figure 1 where \( m = 0.1, d = 1, k_0 = 500 \) and \( k_1 = 400 \).
Remark 1. The generalization of realization (2) to multivariable systems is beyond the scope of this paper, and would require an appropriate multivariable structure for \(a(\rho(t), q^{-1})\) and \(b(\rho(t), q^{-1})\), such as a parameter-dependent generalization of matrix-fraction descriptions [23].

Next, the classical ETFE approach to FRF estimation is recapitulated.

2.2. The Empirical Transfer Function Estimate

Estimating the FRF of an LTI system can be achieved in various different ways [30], and the ETFE is a well-known approach [26, eqn. 6.24]. Under the mild assumption that the input excites all frequencies, i.e., \(U(k) \neq 0\), the ETFE can be written using the lifted notation as

\[
\hat{\theta} \triangleq U^{-1} \hat{Y}, \quad \hat{\theta} \in \mathbb{C}^N. \tag{4}
\]

It is well known that if \(u(t)\) and \(y(t)\) are free of noise and leakage, and if \(y(t)\) is the steady-state output of an LTI system, then (4) equals FRF, i.e., \(\hat{\theta} = \hat{G}\), where \(G(e^{j\omega k}) = A^{-1}(e^{j\omega k}) B(e^{j\omega k})\), with \(A(e^{j\omega k}) = a(e^{j\omega k})\), \(B(e^{j\omega k}) = b(e^{j\omega k})\). Consequently, the FRF of an LTI system is readily estimated from the DFT coefficients \(U(k)\) and \(Y(k)\) without prior knowledge on the structural system parameters \(n\) and \(\kappa\). Moreover, \(Y(k)\) can be replaced by a sample mean of multiple realizations when \(y(t)\) is corrupted by noise, where different periods of a periodic \(y(t)\) can be considered as independent observations [30, §2.5.1]. Under mild assumptions on the distribution of the noise in the time domain, this approach renders (4) the Maximum Likelihood Estimator if the period length is sufficiently long. These properties are lost when the ETFE is applied to input-output data satisfying (2) unless \(\rho(t)\) is frozen. The aim of this paper is to develop an ETFE-like estimator for LPV systems described by (2) that possesses similar properties and recovers the classical ETFE for frozen scheduling. This identification problem is formalized next.

2.3. The identification problem

The following assumptions enable the formulation of a global FRF identification procedure for LPV systems.

Assumption 1. The scheduling signal \(\rho(t)\) and the functions \(\phi_i(\rho), i = 1, ..., n_\phi, \) and \(\psi_i(\rho), i = 1, ..., n_\psi, \) are known.

This assumption is common in LPV identification approaches that are parametric in the scheduling dependence, such as those developed in [4, 12, 44]. No particular set of functions \(\phi_i(\rho)\) and \(\psi_i(\rho)\) is assumed here and prior knowledge of these functions is typically obtained through first-principles modeling, or they are approximated using basis functions, such as polynomials [46, 25] or thin-plate splines [51].

Assumption 2. The scheduling signal \(\rho(t)\) is \(N_\rho\)-periodic for a given \(N_\rho \in \mathbb{N}\), i.e., \(\rho(t + N_\rho) = \rho(t), \forall t \in \mathbb{Z}\).

Assumption 3. The scheduling signal \(\rho(t)\) identical for each realization of the identification experiment.

By considering identical periodic scheduling, the LPV system (2) describes Linear Periodically Time-Varying (LPTV) dynamics that are identical for each experiment. The approach developed in this article is to identify a non-parametric model of the LPTV dynamics and to infer the LPV parameters from the resulting estimate.
Remark 2. Assumption 3 is required to establish the identifiability results in Section 3, but can be relaxed in practice.

Assumption 4. The input signal $u(t)$ is exactly known and is $N_u$-periodic for a given $N_u \in \mathbb{N}$, i.e., $u(t + N_u) = u(t)$, $\forall t \in \mathbb{Z}$.

Assumption 5. The system (2) is exponentially stable for the scheduling signal $\rho(t)$ satisfying Assumption 1 to 3.

Assumption 5 ensures exponential decay of the transient response of the LPTV system [5, §1.2.3]. In combination with Assumption 2 and 4, $y(t)$ exponentially converges to a steady-state periodic signal with period length $N$, where $N$ is the least common multiple of $N_u$ and $N_\rho$ [20].

Assumption 6. The output $y(t)$ of (2) is measured during steady state.

Under the presented assumptions, the signals $\rho(t), u(t)$ and $y(t)$ are $N$-periodic, such that no leakage is introduced by the DFT [30]. The absence of leakage is essential in the classical ETFE and is of similar importance in the LPV FRF identification approach as developed in this article. The identification problem is formalized as follows.

Problem 1. Under Assumption 2 to 6, develop an estimator of the coefficients $A_i(e^{j\omega_k}), B_i(e^{j\omega_k})$, corresponding to the polynomials $a_i(q), b_i(q)$, of (3), that enables accurate output simulation.

The simulation accuracy in Problem 1 is measured using an appropriate metric, such as the Best Fit Ratio (BFR), as is discussed in Section 6.

Remark 3. Estimating parametric models of the polynomials $a_i(q), b_i(q)$ in (3) from the nonparametric estimates $A_i(e^{j\omega_k}), B_i(e^{j\omega_k})$ can be achieved using LTI identification techniques [30, §8] and results in a parametric LPV model as given by (2). Such a model can be used to synthesize LPV controllers using the approach developed in [1]. Alternatively, the nonparametric estimates $A_i(e^{j\omega_k}), B_i(e^{j\omega_k})$ can be used directly for LPV controller design using a model matching approach based on the frozen model set as is developed in [7].

In this section, the considered class of LPV systems is introduced and the identification problem is formalized under a set of assumptions that are common in LPV system identification. In the next section, the considered class of LPV systems is parametrized.

3. LPV FRF representation and parametrization

In this section, the periodic steady-state behavior of (2) is represented in the frequency domain and an identifiable parametrization of the resulting representation is derived, which constitutes contribution C1.

3.1. Frequency domain representation

A periodic scheduling signal renders the LPV dynamics given by (2) to be LPTV, and applying a periodic input to (2) asymptotically results in a periodic output. This steady-state behavior allows the following frequency domain representation, which is obtained by applying the DFT to (2) [27, §2.2]

$$A_0(e^{j\omega_k})Y(k) + \sum_{i=1}^{n_\phi} \Phi_i(k) \ast \left(A_i(e^{j\omega_k})Y(k)\right) = B_0(e^{j\omega_k})U(k) + \sum_{i=1}^{n_\Psi} \Psi_i(k) \ast \left(B_i(e^{j\omega_k})U(k)\right), \quad k = 0, \ldots, N - 1,$$

(5)

where $A_i(e^{j\omega_k}), B_i(e^{j\omega_k}) \in \mathbb{C}$ are the parameters to be estimated, and $\Phi_i(k)$ and $\Psi_i(k)$ are the DFT coefficients corresponding to the signals $\phi_i(\rho(t))$ and $\psi_i(\rho(t))$. In contrast to LTI systems, a sinusoidal input generates multiple output harmonics due to the time variance, as is reflected by the circular convolution operator ‘$\ast$’, which induces mixing between frequencies. Stacking the $N$ equations represented by (5) results in

$$A \tilde{Y} = B \tilde{U}, \quad A, B \in \mathbb{C}^{N \times N},$$

$$A \triangleq A_0 + \sum_{i=1}^{n_\phi} T_{\phi_i} A_i, \quad B \triangleq B_0 + \sum_{i=1}^{n_\Psi} T_{\psi_i} B_i, \quad A_i = \text{diag}\{A_i(e^{j\omega_k})\}, \quad B_i = \text{diag}\{B_i(e^{j\omega_k})\},$$

(6a)

(6b)
where $\check{Y}, \check{U} \in \mathbb{C}^N$ are the columns obtained by stacking $Y(k)$ and $U(k)$, as in (1). The matrices $T_{\phi_i}$ are circulant

$$T_{\phi_i} \triangleq \begin{bmatrix} \Phi_i(0) & \Phi_i(N-1) & \ldots & \Phi_i(1) \\ \Phi_i(1) & \Phi_i(0) & \ldots & \Phi_i(2) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_i(N-1) & \Phi_i(N-2) & \ldots & \Phi_i(0) \end{bmatrix}, \quad \text{and} \quad T_{\psi_i} \triangleq \begin{bmatrix} \Psi_i(0) & \Psi_i(N-1) & \ldots & \Psi_i(1) \\ \Psi_i(1) & \Psi_i(0) & \ldots & \Psi_i(2) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_i(N-1) & \Psi_i(N-2) & \ldots & \Psi_i(0) \end{bmatrix}. \quad (7)$$

Note that $T_{\phi_i} \check{X}$ is a matrix representation of the circular convolution $\Phi_i(k) * X(k)$. Under the assumption that $\mathcal{A}$ is invertible, (6a) identically satisfies

$$\check{Y} = G \check{U}, \quad G = \mathcal{A}^{-1}B \in \mathbb{C}^{N \times N}. \quad (8)$$

The matrix $G$ represents the DT HTF of the LPTV dynamics evaluated on the DFT grid $\Omega$ [27, §2.5] [24, 33], which identically follows by frequency lifting of the LPTV dynamics [5, §6.4.1, 6.4.2] resulting in an $N \times N$ LTI system. In the sequel, $G_\phi$ denotes the HTF estimate corresponding to the true LPTV system. Next, an identifiable parametrization of (8) is developed.

### 3.2. Identifiable parametrization

Consider (8) in terms of the unknown parameters

$$G(\theta) \triangleq A^{-1}(\theta)B(\theta), \quad (9a)$$

$$\theta^T = \begin{bmatrix} \check{A}_0^\top & \check{A}_{n_p}^\top & B_0^\top & B_{n_p}^\top \end{bmatrix} \in \mathbb{C}^{n_\theta}, \quad (9b)$$

where $n_\theta = N(n_{\phi} + n_w + 2)$. The aim is not to identify the full HTF matrix $G$, but to estimate the LPV parameters $\check{A}_i, \check{B}_i$. The specific structure of $G(\theta)$ in terms of the scheduling function, expressed by $T_{\phi_i}$ and $T_{\psi_i}$, enables inference of these LPV parameters, provided that the signals $\phi_i(\rho(t)), \psi_i(\rho(t))$ are sufficiently informative [42, §9.3.5.2]. This requirement can be expressed in conditions under which parametrization (9) is identifiable. Identifiability ensures that each parameter vector $\theta$ is associated to a unique model, which is necessary for the existence of a consistent estimate of $\theta$ [14]. This is formalized by the following definition [26, Def. 4.8].

**Definition 1.** A model set $G(\theta)$ is said to be globally identifiable if it is globally identifiable at almost all $\theta^*$, where $G(\theta)$ is said to be globally identifiable at $\theta^*$ if

$$G(\theta) = G(\theta^*) \Rightarrow \theta = \theta^*.$$

The following Theorem establishes how the identifiability of $G(\theta)$ depends on the signals $\phi_i(\rho(t)), \psi_i(\rho(t))$.

**Theorem 1.** Consider the model set $G(\theta)$, given by (9a), (9b), (6a), and (6b). For $n_{\phi} \geq 1$ and or $n_w \geq 1$, $G(\theta)$ is globally identifiable if

$$\text{rank} \left( \begin{bmatrix} \check{P}_0 & \Phi_1 & \ldots & \Phi_{n_{\phi}} \end{bmatrix} \right) = n_{\phi} + 1, \quad (10)$$

$$\text{rank} \left( \begin{bmatrix} \check{P}_0 & \check{\Psi}_1 & \ldots & \check{\Psi}_{n_w} \end{bmatrix} \right) = n_w + 1, \quad (11)$$

with $\check{P}_0 \triangleq \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^\top \in \mathbb{R}^N$, and if $[\theta]_\tau = 1$ for some $\tau \in [1, N] \subseteq \mathbb{N}$. \quad (12)

For $n_{\phi} = n_w = 0$, $G(\theta)$ is globally identifiable if either $\check{A}_0 = 1_N$ or $\check{B}_0 = 1_N$, where $1_N \triangleq [1 \ldots 1]^\top \in \mathbb{N}^N$.

A proof of Theorem 1 is provided in Appendix A. Theorem 1 reveals that identifiability requires the DFT vectors of the scheduling signals $\phi_i(\rho(t))$ and $\psi_i(\rho(t))$ to be independent, which can be achieved by choosing an appropriate scheduling trajectory $\rho(t)$. Theorem 1 formalizes the persistence of excitation requirement that appears as an assumption in [12, 48, 20]. Additionally, when the system is LPV, i.e., if $n_{\phi} \geq 1$ and or $n_w \geq 1$, then only a single element of $\theta$ needs to be fixed to 1, since all parameters are coupled by the time variations. This bears close resemblance to SISO rational transfer function parametrizations, which are similarly identifiable when a single coefficient is fixed. When $n_{\phi} = n_w = 0$, the LTI case is recovered, in which case $N$ constraints are required to obtain an identifiable parametrization, i.e., only the ratio $A_0^{-1}(e^{j\omega_k})B_0(e^{j\omega_k})$ can be identified.

In this section, the considered class of LPV FRF representations is parameterized and conditions are provided to ensure global identifiability. In the next section, an estimator of $\theta$ is formulated.
4. Parameter estimation

In this section, an approach is developed to solve Problem 1 by formulating a Weighted Nonlinear Least-Squares (WNLS) estimator. It is shown how this estimator reduces to the classical ETFE, and its statistical properties are derived under specific assumptions on the disturbing noise, which partly constitutes contribution C2.

4.1. Weighted nonlinear least squares estimation

Consider the output error

$$\hat{E}(\theta) \triangleq \hat{Y} - G(\theta)\hat{U},$$

with $G(\theta)$ and $\theta$ as given by (9). Given a set of $n_e$ output observations $\mathcal{Y} \triangleq \{\hat{Y}^{(i)}\}_{i=1}^{n_e}$ corresponding to a set of known inputs $\{\hat{U}^{(i)}\}_{i=1}^{n_e}$. The WNLS estimator is then defined as

$$\hat{\theta} \triangleq \arg \min_{\theta \in \mathbb{C}^{n_0}} \mathcal{V}(\theta),$$

$$\mathcal{V}(\theta) \triangleq \sum_{i=1}^{n_e} \mathcal{E}^{(i)}(\theta)^H \mathcal{E}^{(i)}(\theta),$$

$$\mathcal{E}^{(i)}(\theta) \triangleq \mathcal{W} \hat{E}^{(i)}(\theta),$$

where $\hat{E}(\theta)$ is given by (13) and $\mathcal{W} \in \mathbb{C}^{N \times N}$ is a weighting matrix.

4.2. Connection to the ETFE

Before considering the general nonlinear optimization problem posed by (14), it is shown how (14) reduces to the classical ETFE as given by (4). Recall from Theorem 1 that the LTI case requires that either $\hat{A}_0 = \mathbb{1}_N$ or $\hat{B}_0 = \mathbb{1}_N$ to obtain an identifiable parametrization. Imposing either condition renders $G(\theta)$ diagonal, i.e., $G(\theta) = \text{diag} \{\theta\}$, such that (13) is linear in the parameters and (14) admits a closed-form solution. In addition, if $\mathcal{W}$ is diagonal, then (14) is a set of uncoupled linear equations whose solution $\hat{\theta}$ is identical to the classical ETFE as given by (4). Hence, the WNLS estimator reduces the ETFE as a special case.

4.3. Maximum likelihood estimation

In this section, the output is considered to be corrupted by measurement noise, and it is shown how $\mathcal{W}$ can be chosen such that (14) equals the Maximum Likelihood Estimate (MLE), which possesses favorable statistical properties. The probability distribution of $\hat{E}(\theta)$ follows from the forthcoming assumptions.

Assumption 7. The output satisfies $Y^{(i)}(k) = Y_o^{(i)}(k) + V^{(i)}(k)$, where $Y_o^{(i)}(k)$ is the true deterministic output corresponding to $U^{(i)}$, and $V^{(i)}(k)$ is circular complex Gaussian distributed for $k \in k_{\Re}$ and Gaussian distributed for $k \in k_{\Im}$, with nonzero covariance $\sigma_{\hat{V}}^2(k)$, and different realizations $i$ are mutually independent [30, §2.4.1].

This assumption is common in frequency domain identification and is satisfied asymptotically ($N \to \infty$) for $k \in k_{\Re}$ if $v_i(t) = H(z)e_i(t)$, with $H(z)$ a stable filter, and with $e_i(t)$ independent identically-distributed noise with existing moments of any order [30, §7, §16.16].

Assumption 8. The matrix $G_o$ corresponding to the true LPTV dynamics, as given by (8), is an element of the model set, i.e., $\exists \theta_o$ such that $G(\theta_o) = G_o$.

Under this assumption it holds that $\mathbb{E}\{\hat{Y}\} = G(\theta_o)\hat{U} = \hat{Y}_o$, such that $\hat{E}(\theta_o) = \hat{V}$. Consequently, $\hat{E}(\theta_o)$ is a circular complex Gaussian distributed vector with positive definite covariance matrix $C_{\hat{V}} = \text{diag} \{\sigma_{\hat{V}}^2(k)\}$. Noting that the latter allows the decomposition $\mathcal{W} \mathcal{W}^T = C_{\hat{V}}^{-1}$ enables the following result [20, 19].

Lemma 1. Consider $\hat{\theta}$ as given by (14a), under Assumption 1 to 8. If $\mathcal{W}$ is such that $\mathcal{W} \mathcal{W}^T = C_{\hat{V}}^{-1}$, where $C_{\hat{V}} = \text{diag} \{\sigma_{\hat{V}}^2(k)\}$, then $\hat{\theta}$ is the MLE, which is consistent, asymptotically ($n_e \to \infty$) normally distributed, and asymptotically efficient.
A proof follows by noting that the likelihood of $\hat{E}(\theta)$ satisfies
\[
\ell(Y|\theta) = \prod_{i=1}^{n_x} \frac{1}{\pi N \det(C_\phi)} e^{-\hat{E}(\theta)^H C_\phi^{-1} \hat{E}(\theta)}.
\]
Maximizing $\ell(Y|\theta)$ is equivalent to minimizing $-\log \ell(Y|\theta)$. If $\mathcal{W}$ is such that $\mathcal{W}^T = C_\phi^{-1}$, then the latter is proportional to (14b), i.e., $\mathcal{V}(\theta) \propto -\log \ell(Y|\theta)$. Hence, $\hat{\theta}$ is the MLE [18] and a proof of its statistical properties follows along the same lines as in [30, Thm. 9.21].

Consistency in Lemma 1 implies that the true parameter is obtained asymptotically by increasing the number of experiments $n_x$. Asymptotic efficiency implies that the parameter covariance matrix asymptotically attains the Cramér-Rao lower-bound [45], i.e.,
\[
C_\theta \succeq \left( \mathbb{E} \left( \Gamma^H \Gamma \right) \right)^{-1}, \quad \Gamma = \frac{\partial \log \ell(Y|\theta, \theta^*)}{\partial \theta^k} \bigg|_{(\theta^*, \theta^*)}, \quad (15)
\]
which enables determination of uncertainty bounds on $\hat{\theta}$ [30, §9.11.4].

**Remark 4.** The functions $\ell(Y|\theta)$ and $\mathcal{V}(\theta)$ are real functions that are not analytic in the complex parameter $\theta$. To ensure that their derivatives exist, they are treated as a real valued functions of $\theta$ and its complex conjugate $\theta^*$, i.e., $\ell: \mathbb{C}^{n_\theta} \times \mathbb{C}^{n_\theta} \to \mathbb{R}$, such that (15) exists [9].

### 4.4. Sample maximum likelihood estimation

The MLE can typically not be determined since $C_\phi$ is generally unknown. A practical alternative is to replace $C_\phi$ by a data-based estimate. A well-developed approach is to record consecutive periods of the output that corresponds to a periodically repeated input sequence [30, §10]. Under mild conditions, these periods may be considered to be independent observations and enable nonparametric estimation of the noise spectrum $\sigma_\nu^2(k)$, as is discussed in Section 6.2. The estimator $\hat{\theta}$ that results by using this estimate of $C_\phi$ is referred to as the Sample Maximum Likelihood estimator (SML), which can similarly be shown to be consistent [20, Thm. 14].

In this section, a WNLS estimator is formulated, which effectively solves Problem 1. In the next section, the computational aspects of determining $\hat{\theta}$ are treated.

### 5. Nonlinear optimization

The optimization problem (14) is generally nonlinear and can practically not be solved analytically. In this section, a closed-form expression of the Jacobian of $\mathcal{V}(\theta)$ is derived, which facilitates a range of iterative optimizers that are common in system identification, and which are briefly reviewed. Combined with the results in the previous section, a complete numerical estimation framework is established, which constitutes contribution C2.

#### 5.1. Closed-form Jacobian

The following condition is necessary and sufficient for $\theta$ to be stationary point of $\mathcal{V}(\theta)$ [9, Theorem. 3]
\[
\frac{\partial \mathcal{V}(\theta, \theta^*)}{\partial \theta^T} \bigg|_{(\theta, \theta^*)} = \sum_{i=1}^{n_x} \mathcal{E}^{(i)}(\theta)^H J^{(i)}(\theta) = 0, \quad (16)
\]
\[
J^{(i)}(\theta) \triangleq \frac{\partial \mathcal{E}^{(i)}(\theta)}{\partial \theta^T}, \quad (17)
\]
where $J(\theta)$ is referred to as the model Jacobian. Numerical methods aimed at finding the roots corresponding to (16) significantly benefit from a closed-form expression of $J(\theta)$, which is given by the following theorem.

**Theorem 2.** Define the following matrix functions
\[
T_\phi(X^{(i)}) \triangleq \begin{bmatrix} X^{(i)} & T_{\psi_1}X^{(i)} & \cdots & T_{\psi_{n\phi}}X^{(i)} \end{bmatrix}, \quad (18)
\]
\[
T_\psi(X^{(i)}) \triangleq \begin{bmatrix} X^{(i)} & T_{\phi_1}X^{(i)} & \cdots & T_{\phi_{n\psi}}X^{(i)} \end{bmatrix}, \quad (19)
\]
A proof of Theorem 2 is presented in Appendix B. A key aspect of (20) is that none of the matrix factors involve the Kronecker product \( \otimes \). This product naturally arises in matrix differentiation and causes substantial inflation of the matrix dimensions, thereby hampering efficient numerical storage and manipulation \([17, \S 12]\). Lemma 2, as presented in Appendix B, is paramount to the elimination of the Kronecker product from (20), as is shown in the proof of Theorem 2. Moreover, note that the parameter constraint (12), i.e., \( \theta \text{r} = 1 \), is readily imposed by removing the \( r \)th column from \( J(\theta) \), resulting in \( |J(\theta)|^r \).

In this section, an analytic expression of the Jacobian is determined that is key to various iterative optimizers as are treated next.

### 5.2. Iterative optimization

In this section, iterative optimization methods that are common in system identification are presented to provide a complete framework.

The following algorithm is sometimes referred to as an Instrumental Variable (IV) approach \([8, 47]\) due the asymmetric nature of the linear system of equations that similarly arise in IV the methods in \([30, \S 9.13]\). This IV method is obtained by iteratively solving a linearized version of (16). More specifically, (16) is rewritten as

\[
\sum_{i=1}^{n_r} \left( \mathcal{W}A^{(i)}(\theta_j)^{-1}Q^{(i)}(\theta_{j+1}) \right)^H J^{(i)}(\theta_j) = 0, \tag{21a}
\]

\[
Q^{(i)}(\theta_{j+1}) = A^{(i)}(\theta_{j+1})Y^{(i)} - B^{(i)}(\theta_{j+1})U^{(i)}, \tag{21b}
\]

which is iteratively solved for \( \theta_{j+1} \), where \( \theta_j \) the results from the previous iteration. A key property of (21a) is that its stationary points are identical to those of the original cost function \([8]\), i.e., (16) holds for \( \theta_{j+1} = \theta_j = \theta_\infty \).

Presently, global convergence guarantees have not been established theoretically, but empirical results indicate that the IV method can pass over local minima \([55, 49]\), making the IV estimate a suitable starting parameter for subsequent gradient-based optimization \([51]\). A concise reformulation of (21) is obtained by imposing identifiability through the constraint \( \theta \text{r} = 1 \), in which case \( Q \), as given by (21b), can be written as

\[
Q^{(i)}(\theta) = [T_Q^{(i)}]^T + |T_Q^{(i)}|^{1/r} \theta_\text{r},
\]

\[
T_Q^{(i)} \triangleq \begin{bmatrix} T_\Phi(\bar{Y}^{(i)}) & T_\psi(\bar{U}^{(i)}) \end{bmatrix}.
\]  

\[
\sum_{i=1}^{n_r} \left( \mathcal{W}A^{(i)}(\theta_j)^{-1}Q^{(i)}(\theta_{j+1}) \right)^H J^{(i)}(\theta_j) = 0, \tag{21a}
\]

\[
\sum_{i=1}^{n_r} \left( \mathcal{W}A^{(i)}(\theta_j)^{-1}Q^{(i)}(\theta_{j+1}) \right)^H J^{(i)}(\theta_j) = 0, \tag{21a}
\]

\[
\sum_{i=1}^{n_r} \left( \mathcal{W}A^{(i)}(\theta_j)^{-1}Q^{(i)}(\theta_{j+1}) \right)^H J^{(i)}(\theta_j) = 0, \tag{21a}
\]

Employing \( J^{(i)}(\theta) \), as presented in Theorem 2, the Hermitian transpose of (21) can be written as

\[
\Gamma(\theta_j) \theta_{j+1} = \gamma(\theta_j), \tag{23a}
\]

\[
\Gamma(\theta) = J(\theta)^H W(\theta)^H W(\phi) Q.
\]  

\[
\gamma(\theta_j) = J(\theta_j)^H W(\theta_j)^H \begin{bmatrix} |T_Q^{(1)}|^{1/r} & \cdots & |T_Q^{(n_r)}|^{1/r} \end{bmatrix}^T,
\]

\[
J(\theta) = \begin{bmatrix} |T_J^{(1)}(\theta)|^{1/r} & \cdots & |T_J^{(n_r)}(\theta)|^{1/r} \end{bmatrix}, \quad Q = \begin{bmatrix} \mathcal{W}A^{(1)}(\theta)^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{W}A^{(n_r)}(\theta)^{-1} \end{bmatrix}.
\]

In summary, the IV algorithm is applied by iteratively solving (23), starting from some initial \( \theta_0 \). A related algorithm results by replacing \( J(\theta_j) \) by \( Q \) in (23a). This approach bears similarities with the Sanathanan-Koerner (SK) method \([47]\) in the sense that \( \Gamma(\theta_j) \) in (23b) is symmetric, enabling a QR decomposition \([17, \S 5.2]\) that benefits the numerical
conditioning \cite{iv}. When the SK iterations have converged, the original WNLS cost is obtained, but the resulting estimate is generally not a stationary point of $\mathcal{V}(\theta)$.

The Gauss Newton (GN) method is obtained through a local linearization of $\mathcal{V}(\theta)$ and similarly results in (23a) where $Q$ is replaced by $J(\theta_j)$, and where $\gamma(\theta_j)$ is instead given by

$$\gamma(\theta_j) = J(\theta_j)^H \left[ \varepsilon^{(1)}(\theta_j)^T \cdots \varepsilon^{(n_p)}(\theta_j)^T \right]^T.$$ \(\Delta \theta_j\)

Furthermore, $\theta_{j+1}$ in (23a) is replaced by the parameter update $\Delta \theta$, and the parameter estimate is instead updated as $\theta_{j+1} = \theta_j + \Delta \theta$, starting from a suitably chosen $\theta_0$, such as the IV or SK estimate. In the Levenberg Marquardt (LM) method, a damping term is introduced to $\Gamma(\theta_j)$ \cite{lm, §9.L.4}. As the damping increases, the search direction converges to the steepest descent direction, thereby increasing the robustness far from the optimum. By appropriately updating the damping parameter, convergence can be enforced while benefiting from the relatively high convergence rate of the GN method close to the optimum \cite{gn}.

5.3. Practical aspects

The matrix $\Gamma(\theta_j)$ must be regular for (23) to have a unique solution, which implies that $Q$ and $J(\theta)$ must have full column rank. Consequently, $n_p \geq n_{\phi} + n_{\psi} + 2$ experiments are required with different realizations of $\hat{U}$. Due to the structure of $Q$ and $J(\theta)$, as determined by (18), (19), (22) and (20b), $\hat{U}$ and $\hat{Y}$ may not contain zero entries, implying that full bandwidth excitation is required.

Numerically solving the system of complex equations (23) is facilitated by reformulating it as a real system

$$\begin{bmatrix} \Re\{\Gamma(\theta_j)\} & \Im\{\Gamma(\theta_j)\} \\ -\Im\{\Gamma(\theta_j)\} & \Re\{\Gamma(\theta_j)\} \end{bmatrix} \begin{bmatrix} \Re\{\theta_{j+1}\} \\ \Im\{\theta_{j+1}\} \end{bmatrix} = \begin{bmatrix} \Re\{\gamma(\theta)\} \\ \Im\{\gamma(\theta)\} \end{bmatrix}.$$ \(\Delta \theta_j\)

The number of variables can be reduced by a factor 2 by imposing the DFT symmetry on the elements of $\theta$, i.e., by imposing the following on $\alpha_k, \beta_k$

$$\begin{align*}
\Im\{X(k)\} &= 0, & \text{for } k \in k_\Re, \\
\Re\{X(k)\} &= \Re\{X(N-k)\}, \\
\Im\{X(k)\} &= -\Im\{X(N-k)\}, & \text{for } k \in k_\Im.
\end{align*}$$

In this section, the LPV FRF identification framework is completed by developing the numerical aspects to optimize for $\hat{\theta}$. In the next section, the application of this framework is demonstrated in practice.

6. Application

In this section, an LPV FRF of an experimental parameter-varying motion system is estimated using the developed approach, and its generalizing capability is validated. In addition, it is shown that the continuum of frozen FRFs is accurately estimated, where this continuum includes unstable behavior that can otherwise not be estimated from frozen experiments. These results collectively constitute contribution C3.

6.1. The control system & experiment design

The system under consideration is a flexible beam system $H$ that is controlled by an LPV controller $K(\rho)$ in feedback that operates at a sampling rate of $f_s = 200$ Hz, as shown in Figure 2. The beam is elastically suspended and is actuated using a voice-coil motor. The displacement of the beam is measured using an optical sensor, which is placed close to the actuator location, resulting in a co-located motion system. The controller $K(\rho)$ mimics a damper and a parameter-varying stiffness, as is visualized by the frozen transfers shown in Figure 3. The excitation signal is added to the controller output, creating the closed-loop LPV dynamics $G(\rho)$.

6.1.1. Stability analysis and scheduling signal design

The stability of the closed-loop is assessed for frozen values of the scheduling parameter $\hat{\rho} \in [0, 1]$ by using the Nyquist test \cite[Thm. 4.7]{nyquist}. The result is shown in Figure 4, which indicates that for $\hat{\rho} < 0.35$, the open-loop FRF passes the point $-1$ on the right as $\omega$ increases. Since the open loop is stable, it is predicted that the closed-loop is
The optimization (14) is solved as follows. First, the IV algorithm is applied, as is developed in Section 5.2, where poorly identifiable. The best dynamics are predominantly LTI at higher frequencies, which makes the parameters associated to the time-variations significantly improves the initial estimate, yet it does not converge, as is shown in Figure 5. This is likely because the \( \hat{\eta} Y \) shows that (9) is globally identifiable for (25). The SML estimator, as developed in Section 4.4, is formulated using the feedback interconnection between the beam \( H \) and the LPV controller \( K(\rho) \).

stable for \( \rho < 0.35 \), and unstable for \( \rho > 0.35 \). This prediction is confirmed by the impulse responses as is shown in Figure 4. Note that for \( \rho > 0.35 \) it is not possible to estimate the FRF of the closed-loop from frozen experiments since the system is unstable. Using the developed method, estimation of the unstable frozen transfer functions as part of the continuum is enabled by using an appropriate scheduling signal that results in stable LPTV dynamics during the experiments. To this end, the controller is identically scheduled during the experiments as

\[
\rho(t) = \sin(8\pi \frac{t}{N}) + \sin(10\pi \frac{t}{N}),
\]

such that the frozen unstable region is briefly "visited" periodically. The impulse response of \( G(\rho(t)) \) with \( \rho(t) \) given by (25) reveals exponential stability of the LPTV closed-loop dynamics, as is shown in Figure 4.

6.1.2. The identification data
The identification data is obtained by conducting \( n_e = 20 \) experiments using full-band random-phase multisines. The fundamental period consist of \( N = 512 \) samples [30, Def. 3.1], and the input spectrum is suppressed for frequencies up to 20 Hz to avoid excessive displacements. For each experiment, the first 2 periods are disregarded to decrease the influence of the transient response and the remaining 20 periods are measured and used to estimate the sample mean \( \hat{Y}^{(i)} \) and the sample covariance \( \hat{\sigma}_Y^{2(i)}(k) \) [30, §10] as

\[
\hat{Y}^{(i)}(k) \triangleq \frac{1}{m} \sum_{j=1}^{m} Y_j^{(i)}(k), \quad \hat{\sigma}_Y^{2(i)}(k) \triangleq \frac{1}{1-m} \sum_{j=1}^{m} [Y_j^{(i)}(k) - \hat{Y}^{(i)}(k)]^2.
\]

where \( m = 20 \) denotes the total number of periods. To avoid spectral leakage, which may induce a nonnegligible bias in the parameter estimate, it is important to ensure that the measured period is identical to the true period of the system, and that the true period remains constant. For motion systems, the system periodicity can typically be controlled relatively accurately by ensuring periodic motion of the constituent components.

6.2. Parametrization & optimization
The HTF estimate \( G \) is parametrized as (9) and (6) with basis functions \( \phi_1(\rho) = \varphi_1(\rho) = \rho \). Applying Theorem (25) shows that (9) is globally identifiable for (25). The SML estimator, as developed in Section 4.4, is formulated using the lifted means \( \hat{Y}^{(i)} \) in (14) and by taking \( \mathcal{W} \mathcal{W}^T = C_{\varphi}^{-1} \), where \( C_{\varphi} \) is taken as the sample mean of \( \hat{\varphi}_Y^{(i)} = \text{diag}\{\hat{\sigma}_Y^{2(i)}(k)\} \). The optimization (14) is solved as follows. First, the IV algorithm is applied, as is developed in Section 5.2, where the parameters are initialized as \( \hat{A}_0 = I, \hat{A}_1 = 0, \hat{B}_1 = 0 \), and with \( \hat{B}_0 \) a complex random vector. The IV algorithm significantly improves the initial estimate, yet it does not converge, as is shown in Figure 5. This is likely because the dynamics are predominantly LTI at higher frequencies, which makes the parameters associated to the time-variations poorly identifiable. The best \( \theta \) is obtained after 3 iterations, which is subsequently refined by using the LM algorithm with smoothed damping [29], which results in a significantly improved estimate.
Frequency Response Function Identification of Periodically Scheduled Linear Parameter-Varying Systems

Figure 3: The magnitude of the frozen transfers of the LPV controller $K(\rho)$ for $\tilde{\rho} = -1$ (---), $\tilde{\rho} = 0$ (-----), and $\tilde{\rho} = 0.35$ (----), and the range $\tilde{\rho} \in [-1, 1]$. This shows that $K(\rho)$ behaves as a damper for $\rho = -1$ and as a stiffness for $\rho = 1$.

Figure 4: Left: Nyquist plot of the open-loop transfer $HK(\tilde{\rho})$ for $\tilde{\rho} = -1$ (---), $\tilde{\rho} = 0$ (-----), and $\tilde{\rho} = 0.35$ (----), and the range $\tilde{\rho} \in [-1, 1]$. This indicates that the closed-loop stability boundary is close to $\tilde{\rho} \approx 0.35$. Right: The impulse responses of the frozen closed loop $G(\tilde{\rho})$ for $\tilde{\rho} = 0.35$ (----) and $\tilde{\rho} = 0.4$ (-----) confirm that the stability boundary is in the range $0.35 < \tilde{\rho} < 0.4$. The impulse response of $G(\rho(t))$, with $\rho(t)$ given by (25) (-----), indicates exponential stability of the LPTV closed-loop dynamics.

Figure 5: The cost $\mathcal{V}(\theta_j)$ during the IV iterations (---) and the subsequent LM iterations (----) that start from $\theta_2$.

6.3. The estimated frozen behavior

The resulting LPV FRF model includes a continuous description of the frozen behaviors, i.e., the FRFs that are obtained for a constant $\rho(t) = \tilde{\rho}$. The Bode diagrams of the frozen behavior is displayed in Figure 6 as a function of $\tilde{\rho}$. The magnitude plot reveals large variations of the first resonance peak, which is relatively strongly damped at $\tilde{\rho} = -1$ and appears to be undamped at $\tilde{\rho} \approx 0.36$. The phase plot reveals that this pole pair crosses the imaginary axis at $\tilde{\rho} \approx 0.36$, resulting in an unstable system. This is in agreement with the stability analysis of the frozen closed-loop system presented in Section 6.1.1. The local irregularities or spikes, which can be seen mostly at higher frequencies, are related to the sub-optimal nature of $\theta_j$. Specifically, it is observed that a random initial estimate $\theta_0$ features numerous spikes, most of which vanish as $\theta_j$ converges. However, if $\theta_j$ converges to a local optimum, which is typically the case since global optimality cannot be guaranteed, some small spikes may remain that are inherently associated to the local optimum. A practical approach to mitigating these local irregularities is to perform the nonlinear optimization with different initial estimates and taking the best result.

The benefit of the LPV FRF is assessed by comparing it to the classical ETFE when both are applied to the same data set. The purpose of this comparison is to show that irregularities in the ETFE, that are due to time variations, can
Figure 6: The magnitude (top) and phase (bottom) of the frozen LPV FRF $G(\hat{\theta}, \bar{\rho}, e^{j\omega})$ display large variations along the $\bar{\rho}$-axis for low frequencies, indicating a resonance that is significantly damped at $\bar{\rho} = -1$ and undamped around $\bar{\rho} = 0.36$. The magnitude is almost independent of $\bar{\rho}$ for higher frequencies, indicating LTI behavior at those frequencies. The phase displays a $360^\circ$ shift around $\bar{\rho} = 0.36$, indicating a pole pair that crosses the unit circle.

be removed using the LPV FRF. The accuracy of both models is assessed by comparing them to estimates of the closedloop FRFs that are obtained from separate experiments with identical inputs, during which the controller $K(\bar{\rho})$ is frozen for $\bar{\rho} \in \{-1, 0, 0.35\}$. The result is visualized in Figure 7. This shows that the ETFE provides a relatively accurate description for higher frequencies, since the dynamics are predominantly LTI there, as is evidenced by the identical curves for all $\bar{\rho}$. For lower frequencies the modeling error is significant, as is evidenced by the large gap between the mismatch and the noise estimate. The frozen LPV FRF performs significantly better and closely resembles the estimated frozen FRFs leaving a much smaller gap between the mismatch and the noise.
Figure 7: This figure compares the classical ETFE (left column) to the LPV FRF (right column) by displaying the magnitude of the following FRF data. Three benchmark FRFs are estimated by freezing the LPV controller $\bar{\rho} \in \{-1, 0, 0.35\}$, which are displayed in both columns (---). Both the ETFE (---) and LPV FRF (----) are estimated using the data that is obtained when the LPV controller is scheduled as given by (25), and their difference with the benchmark data is shown (---) as well as an estimate of the noise spectrum $\sigma_n(k)$ (-----). This shows that the LPV FRF provides an accurate estimate of the frozen FRFs, whereas the the ETFE provides a distorted estimate at low frequencies.

6.4. The dynamic behavior & validation

The main potential of the LPV FRF model is its predictive capability for dynamic scheduling. The 20 experiments are repeated to assess this potential, where the controller is instead scheduled with a smoothed triangle wave. Figure 8 shows the predicted output and the averaged true output of a single experiment for both the identification data and the validation data, revealing relatively high predictive capabilities for both sets, where the performance for the identification set is naturally slightly higher. This observation is quantified by determining the Best Fit Ratio (BFR)

$$BFR = \max \left\{ 1 - \frac{1}{N} \sum_{t=0}^{N-1} \frac{|y(t) - \hat{y}(t)|}{\overline{y(t)}}, 0 \right\} \cdot 100\%,$$

(26)

where $y(t)$ is the true output, $\hat{y}(t)$ is the predicted output, and $\overline{y}(t)$ is the average output. The average BFR over 20 experiments for the training and validation data are shown in Table 1, where the first 2 periods are neglected to exclude transient behavior. In addition, the table displays the average BFR that is achieved by the ETFE in case the controller is frozen at $\bar{\rho} = 0$. Comparing these results indicates that the simulation accuracy of the LPV FRF for dynamic scheduling is close to the performance of the ETFE for frozen scheduling.

To illustrate the importance of choosing the right scheduling functions $\phi_1(\rho)$ and $\psi_1(\rho)$, two additional LPV FRF models are estimated with $\phi_1(\rho) = \psi_1(\rho) = \rho^2$ and $\phi_1(\rho) = \psi_1(\rho) = \rho^3$. The resulting BFRs for both the identification and the validation data are displayed in Table 2. The results indicate that the simulation performance of these models is comparable to that of the ETFE ($\phi_1(\rho) = \psi_1(\rho) = 0$), and is much lower than the first LPV FRF model ($\phi_1(\rho) =$...
The output simulated by the LPV FRF model (---) and the periodically averaged true output (-----) for a particular experiment from the training set and validation set, where the validation set is generated by using triangular scheduling, as is shown in the bottom plot. The predictive performance, as measured by the BFR given by (26), equals 96.81% for the identification set and 93.40% for the validation set. This indicates that the LPV FRF model can accurately predict the time-varying behavior corresponding to different scheduling situations. For comparative analysis using 20 different experiments, see Table 1.

Table 1
The sample mean and standard deviation of the BFR, as given by (26), over 20 experiments for the LPV FRF for the identification set and validation set, where a different scheduling signal is used during the validation experiments. Similar data is presented for the ETFE in case the scheduling is frozen at $\bar{\rho} = 0$. The results indicate the simulation performance of the LPV FRF model for time-varying dynamics is close to the performance of the ETFE for LTI dynamics.

<table>
<thead>
<tr>
<th></th>
<th>BFR Identification set [%]</th>
<th>BFR Validation set [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mean standard deviation</td>
<td>mean standard deviation</td>
</tr>
<tr>
<td>LPV FRF</td>
<td>95.86 0.85</td>
<td>92.12 1.41</td>
</tr>
<tr>
<td>ETFE</td>
<td>95.24 0.87</td>
<td>95.02 0.80</td>
</tr>
</tbody>
</table>

$\psi_1(\rho) = \rho$. Hence, it is essential to choose the correct basis functions, and the resulting simulation performance provides a good indicator to assess their suitability.

In this section, the developed identification method is applied to estimate an LPV FRF model of an experimental motion system. This is achieved by solving for the WNLS estimator by means of the developed iterative optimization routine, which results in an estimated model whose predictive capabilities for dynamic scheduling have been found to be comparable to the ETFE in the classical sense, thereby successfully solving Problem 1 by providing a generalization of the classical ETFE to the class of LPV systems.

7. Conclusion

A framework is developed that enables the estimation of LPV dynamics without requiring prior knowledge of the dynamic order of the system. This is achieved by developing the concept of frequency response functions models for periodically scheduled SISO LPV systems and constructing an approach to identify these nonparametric models from global experiments. In this way, the developed method generalizes the classical LTI ETFE to the class of LPV systems, and it is shown that the classical ETFE is indeed recovered for LTI systems. The experimental results show that the predictive capabilities of the LPV FRF estimator are comparable to those of classical ETFE in its appropriate LTI setting. Furthermore, in contrast to pre-existing methods based on local FRF identification, it is shown that
Table 2
The sample mean and standard deviation of the BFR, as given by (26), for various LPV FRF models with different basis functions \( \phi_i(\rho) \) and \( \psi_i(\rho) \). The displayed result are based on 20 experiments for both identification set and validation set, where a different scheduling signal is used during the validation experiments.

<table>
<thead>
<tr>
<th>Basis Function</th>
<th>Identification set [%]</th>
<th>Validation set [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mean</td>
<td>standard deviation</td>
</tr>
<tr>
<td>( \phi_i(\rho) = \psi_i(\rho) = 0 )</td>
<td>56.74</td>
<td>6.19</td>
</tr>
<tr>
<td>( \phi_i(\rho) = \psi_i(\rho) = \rho )</td>
<td>95.86</td>
<td>0.85</td>
</tr>
<tr>
<td>( \phi_i(\rho) = \psi_i(\rho) = \rho^2 )</td>
<td>55.60</td>
<td>7.12</td>
</tr>
<tr>
<td>( \phi_i(\rho) = \psi_i(\rho) = \rho^3 )</td>
<td>55.89</td>
<td>7.05</td>
</tr>
</tbody>
</table>

the developed method enables the estimation of the entire continuum of local LTI models, which can include locally unstable representations. The development of nonparametric LPV FRF models is considered to be an important step toward enabling FRF-based LPV controller design, thereby enabling more widespread adoption of the LPV paradigm in control engineering practice.

8. Acknowledgment

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A. Proof of Theorem 1

**Proof.** A proof to Theorem 1 results by showing that

\[ A^{-1}(\theta)B(\theta) = A^{-1}(\theta^*)B(\theta^*) \Rightarrow \theta = \theta^*, \]

for almost all \( \theta \), given (10), (11), and (12). To this end, note that the matrices \( A(\theta) \) and \( B(\theta) \) are invertible generically, i.e., for almost all \( \theta \). Consequently, \( G(\theta) = G(\theta^*) \) is identical to

\[ A(\theta^*)A^{-1}(\theta) - B(\theta^*)B(\theta)^{-1} = 0. \]

The latter is only satisfied if both terms are equal to a certain matrix, denoted here by \( S \), i.e., \( S - S = 0 \). This results in two identities

\[ A(\theta^*) = SA(\theta) \quad \text{and} \quad B(\theta^*) = SB(\theta), \]

that imply that \( S \) needs to be full rank since \( A(\theta) \) and \( B(\theta) \) are generically full rank. Clearly, the transformation by \( S \) leaves \( G \) unchanged, i.e., \( A^{-1}S^{-1}SB = A^{-1}B = G \). Only if \( S \) is diagonal will it generically preserve the structure of \( A \) and \( B \) as specified by (6b). To see this, consider the case where \( n_\phi = n_\psi = 0 \). Clearly,

\[ A_0(\theta^*) = SA_0(\theta), \tag{27} \]

is diagonal if and only if \( S \) is diagonal. The latter results as follows. Given that \( A_0(\theta) \) is diagonal, then \( A_0(\theta^*) \) is diagonal if \( S \) is diagonal, and \( S \) is diagonal if \( A_0(\theta^*) \) is diagonal. If \( A_0 = I_N \), then \( A_0(\theta) = A_0(\theta^*) = I \forall \theta \), implies \( S = I \). Consequently, \( B(\theta^*) = B(\theta) \) which implies \( \theta = \theta^* \). By Definition 1, this implies that \( G(\theta) \) is globally identifiable. Note that same result is obtained mutatis mutandis when \( B_0 = I_N \).

Next, consider the case where \( n_\phi \geq 1 \). Condition (10) implies that \( \Phi_1(1) \neq 0 \), such that \( T_{\phi_1} \) is a non-diagonal circulant matrix, as given by (7). In order to preserve the structure of \( A \) as specified by (6b), \( S \) must be diagonal and it must commute with \( T_{\phi_1} \), i.e., it must hold that

\[ T_{\phi_1}A_1(\theta^*) = ST_{\phi_1}A_1(\theta) = T_{\phi_1}SA_1(\theta). \]
A diagonal matrix only commutes with a non-diagonal circulant matrix if its proportional to the identity matrix, i.e., $S = \beta I$, $\beta \in \mathbb{C}$. Given (12), i.e., $[\theta]_r = 1$ for some $r \in [1, N] \subseteq \mathbb{N}$, (27) implies that $\beta = 1$. Hence, it must hold that $A(\theta^*) - A(\theta) = 0$, which is written as

$$
\begin{bmatrix}
I & T_{\phi_1} & \cdots & T_{\phi_{n_\phi}}
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
A_0(\theta)
\end{bmatrix}
\begin{bmatrix}
A_1(\theta)
\end{bmatrix}
\vdots
\begin{bmatrix}
A_{n_\phi}(\theta)
\end{bmatrix}
\end{bmatrix}
- 
\begin{bmatrix}
\begin{bmatrix}
A_0(\theta^*)
\end{bmatrix}
\begin{bmatrix}
A_1(\theta^*)
\end{bmatrix}
\vdots
\begin{bmatrix}
A_{n_\phi}(\theta^*)
\end{bmatrix}
\end{bmatrix} = 0. \tag{28}
$$

Due to the diagonal and circulant structure of the matrices $I, A_i$ and $T_{\phi_i}$, the $k^{th}$ column of (28) is given by

$$
a_0(k)\Phi_0 + \sum_{i=1}^{n_\phi} a_i(k)\Phi_i = 0, \quad a_i(k) \triangleq \bar{A}_i(k, \theta) - \tilde{A}_i(k, \theta^*) \quad i = 0, 1, \ldots, n_\phi.
$$

Given (10), the column vectors $\bar{P}_i, \tilde{\Phi}_i$ are linearly independent, which implies that $a_j(k) = 0 \forall k$, which in turn implies that $\bar{A}_i(k, \theta) = \tilde{A}_i(k, \theta^*)$. The same can be shown mutatis mutandis for $\bar{B}_i(k, \theta)$, implying $\theta = \theta^*$, which by Definition 1, implies that $G(\theta)$ is globally identifiable.

## B. Proof of Theorem 2

A proof of Theorem 2 requires the following auxiliary result.

**Lemma 2.** Given $W \in \mathbb{C}^{s \times m}$, $Z \in \mathbb{C}^{s \times l}$, then,

$$
W(I_n \otimes Z) = \begin{bmatrix}
\text{vec}(Z^T w_1) & \cdots & \text{vec}(Z^T w_m)
\end{bmatrix}^T
$$

where $w_i = \text{reshape}([W^T]^i, s, n) \in \mathbb{C}^{s \times n}$

with $[W^T]^i$ the $i^{th}$ column of $W^T$ and where $x = \text{vec}(X)$ stacks the columns of $X \in \mathbb{C}^{s \times n}$ in the column vector $x$ and where $X = \text{reshape}(x, s, n)$ does the inverse by restructuring $x$ into $X$ [17, §12].

A proof to this lemma is immediate from the identity $\text{vec}(ABC) = (C^T \otimes A)\text{vec}(B)$. A proof to Theorem 2 is the following.

**PROOF.** The dependence on $\theta$ and $i$ is omitted for the sake of clarity in this proof. To determine (17), consider $\mathcal{E}$ as given by (14c) and (13), and take the derivative

$$
\frac{\partial \mathcal{E}}{\partial \theta^T} = -\mathcal{W} \frac{\partial (A^{-1}B\check{U})}{\partial \theta^T}. \tag{29}
$$

Consider the product rule for matrix differentiation with respect to some $X \in \mathbb{C}^{s \times l}$

$$
\frac{\partial (AB)}{\partial X} = \frac{\partial A}{\partial X} (I_l \otimes B) + (I_s \otimes A) \frac{\partial B}{\partial X}. \tag{30}
$$

Applying (30) to (29) yields

$$
\frac{\partial (A^{-1}B\check{U})}{\partial \theta^T} = N_y + N_u, \quad N_y = \frac{\partial A^{-1}}{\partial \theta^T} (I_{n_q} \otimes B\check{U}), \quad N_u = A^{-1} \frac{\partial B\check{U}}{\partial \theta^T}.
$$

Consider the following identity for $F \in \mathbb{C}^{p \times p}$

$$
\frac{\partial F^{-1}}{\partial X} = -(I_s \otimes F^{-1}) \frac{\partial F}{\partial X} (I_l \otimes F^{-1}). \tag{31}
$$

Applying (31) to $N_y$ yields

$$
N_y = -A^{-1} \frac{\partial A}{\partial \theta^T} (I_{n_q} \otimes A^{-1}B\check{U}).
$$
where it is used that \((A \otimes B)(C \otimes D) = (AC) \otimes (BD)\). Applying (30) to \(N_u\) yields

\[
N_u = A^{-1} \frac{\partial B}{\partial \theta^\top} (I_{n_y} \otimes \dot{U}).
\]

Furthermore, it follows readily that

\[
\frac{\partial A}{\partial \theta^\top} = \left( \frac{\partial A_0}{\partial \theta^\top} + \sum_{i=1}^{n_d} T_{\phi_i} \frac{\partial A_1}{\partial \theta^\top} \right), \quad \frac{\partial A_1}{\partial \theta^\top} = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{bmatrix},
\]

which follows similarly for \(\frac{\partial B}{\partial \theta^\top}\). Using this relation and applying Lemma 2 to \(N_y\) and \(N_u\) results in (20a), (20b) and (20c).

References

Frequency Response Function Identification of Periodically Scheduled Linear Parameter-Varying Systems


