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An Approach to Stable Inversion of LPTV Systems with Application to a Position-Dependent Motion System

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***Abstract*— Model inversion is essential in many control approaches, including inverse model feedforward and iterative learning control (ILC). The aim of this paper is the development of inversion techniques for linear periodically time-varying (LPTV) systems, possibly multivariable. The proposed method involves stable inversion, where bounded solutions are computed through a two-point boundary value problem. A key aspect herein involves a new dichotomic split for the stable and unstable dynamics, which is nontrivial for general LTV systems. As a special case, well-known stable inversion techniques for LTI systems are recovered. The approach is successfully demonstrated on a position-dependent wafer stage system.**

I. INTRODUCTION

Model inversion is an important aspect in many control approaches, including feedforward and iterative learning control (ILC). Key advantages include the exploitation of preview to compensate for, e.g., reference induced, disturbances. Herein, model inversion is essential for perfect compensation. This can either be an exact inverse of the system as in inverse model feedforward, or the inverse of the process sensitivity as learning filter in ILC.

For minimum-phase systems the inverse is stable and yields bounded solutions. However, for nonminimum-phase systems, a direct inverse may be unstable and yields unbounded solutions. Several approaches have been developed to determine bounded solutions for the inverse of nonminimum-phase systems. This includes the approximate solutions ZPETC [25], ZMETC, and NPZ-Ignore, see [6] for a comparison, and developments into exact inversion techniques such as stable inversion [20], [12], [13], [29]. In stable inversion, the unstable part is seen as a noncausal operation, essentially by employing a bilateral Lapace/Z-transform [24]. Stable inversion is successfully implemented as exact inverse for LTI rational feedforward and ILC in [3], [4], [9].

Developments in accuracy and cost of control applications lead to LPTV behavior. For instance, periodic tasks for position-dependent systems [18], [27] lead to LPTV behavior. Examples of periodic tasks are meander patterns for the production of integrated circuits and printing tasks. Similarly, embedding multiple applications on a single platform results in periodic non-equidistant sampling of the applications which also leads to LPTV behavior.

Although a significant performance increase can be obtained for high performance control applications by recognizing their LPTV behavior, at present most control techniques are targeted at LTI systems. The aim of this paper is to facilitate LPTV control design by developing a stable inversion approach for LPTV systems. The key aspect is the nonminimum-phase zero-dynamics of such time-varying systems as it may lead to unbounded responses for the inverted system. Inversion of LTV systems, see, e.g., [8], requires a dichotomic split [11], [19] in a stable and unstable part. This is, however, nontrivial for general LTV systems, see also [13, sec. III], obstructing the use of LTV stable inversion. In the present paper, the periodicity of LPTV systems is exploited to enable such a split and by that the development of stable inversion techniques for LPTV systems.

The main contribution of this paper is the development of a general stable inversion approach for LPTV system applications. In [16], [23] feedback and in [28] feedforward optimization based approaches for multirate sampling are presented based on [8]. In contrast, in the present paper an exact inverse is obtained.

Two immediate applications are foreseen for stable inversion of LPTV systems. First, the development of rational noncausal feedforward as in [3] for LPTV systems, including non-equidistantly sampled systems, multirate systems, and position-dependent systems with periodic tasks. Second, the implementation of a learning filter for ILC [5] in LPTV applications. Other applications may be found in various fields that exhibit periodicity, e.g., due to rotating components as in aerospace, wind turbines, and automotive. Results related to the research reported here for stable systems include [21].

The outline of this paper is as follows. In section II, the problem of finding bounded solutions for LPTV systems is formulated. The main result, the stable inversion approach for LPTV systems, is presented in section III. In section IV, the stable inversion approach is demonstrated on a position-dependent wafer stage system. Conclusions are given in section V.

II. PROBLEM FORMULATION

Throughout the paper, square LPTV systems are considered. In this section, the class of LPTV systems is defined, the motivation for this type of systems is given, and the system inversion problem is posed.

A. LPTV systems

In this paper, linear periodic time-varying (LPTV) systems are considered, i.e., systems satisfying Definition 1 and

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Definition 2. See also [15, sec. 11.2.5] and [8], [22], for more on LPTV systems.

Definition 1 (Linearity). *System H is linear if for input αu, the output is αy, $\forall \alpha \in \mathbb{R}$.*

Definition 2 (Periodicity). *System H is periodic with period $\tau \in \mathbb{N}$ if it commutes with the delay operator defined by $(\mathcal{D}_\tau u)_k = u_{k-\tau}$, i.e., $\mathcal{D}_\tau H = H \mathcal{D}_\tau$.*

Linear time-invariant (LTI) systems are a special case of LPTV systems as given by Definition 3. Stability of LTI systems is defined in Definition 4.

Definition 3 (Linear time-invariant). *H is linear time-invariant (LTI) if it satisfies Definition 1 and Definition 2 is satisfied for all $\tau \in \mathbb{N}$.*

Definition 4 (LTI stability). *LTI system H (see Definition 3) is stable iff all eigenvalues λ_i of the state matrix A are within the unit circle, i.e., $|\lambda_i(A)| < 1$, $\forall i$.*

Unless states otherwise, Assumption 5 is imposed.

Assumption 5. *System H is square and LPTV with period $\tau \in \mathbb{N}$, i.e., H has the same number of outputs as inputs, and satisfies Definition 1 and Definition 2.*

The monodromy matrix [2, §3.1] plays a key role in obtaining the dichotomy and is defined in Definition 6.

Definition 6 (Monodromy matrix). *For an LPTV system (A_k, B_k, C_k, D_k) with period τ, the monodromy matrix is defined as*

$$\Psi := \Psi_0 = A_{\tau-1} A_{\tau-2} \dots A_0. \tag{1}$$

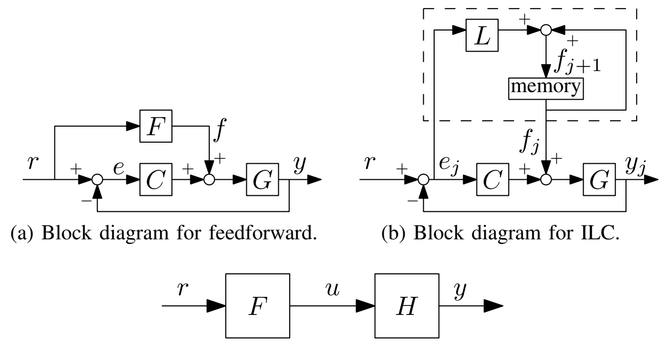
To facilitate the dichotomy, the monodromy matrix is assumed to be hyperbolic as stated in Assumption 7, see also [10, Theorem 3] and [12, Condition 1].

Assumption 7. *Monodromy matrix Ψ, see Definition 6, is assumed to be hyperbolic, i.e., contains no eigenvalues on the unit circle.*

Remark 8. *If Ψ is non-hyperbolic, similar techniques as in [14] can be exploited to approximate the output.*

B. Motivation: feedforward and ILC design

Consider the closed-loop system in Fig. 1(a), where G is the system to control and C a (stabilizing) feedback controller. The control goal is to minimize error $e = r - y$. High performance can be achieved by design of the feedforward filter F. A common approach is the use of inverse model feedforward in which $F \approx G^{-1}$. Alternatively, ILC can be used as shown in Fig. 1(b), where the feedforward signal is 'learned' in an iterative fashion through a learning filter $L \approx (SG)^{-1}$, with $SG = (I + GC)^{-1} G$ the process sensitivity. Although these techniques use different models, they can both be captured in the diagram shown in Fig. 1(c). The goal is the computation of a bounded output signal u for $F = H^{-1}$.



(a) Block diagram for feedforward. (b) Block diagram for ILC.

(c) Ideally, $F = H^{-1}$ such that $y = r$. However, if H is nonminimum phase, this leads to unbounded u.

Fig. 1. Both the design of feedforward in (a) and of ILC in (b) can be captured in the structure of (c), where $H = G$ for feedforward and $H = SG$ for ILC.

C. Problem formulation

The aim of this paper is to develop LPTV model inversion techniques for control applications as outlined in section II-B. A central aspect herein is that inverses may be unstable, e.g., in the LTI case due to nonminimum-phase zeros of the original system. Indeed, this situation occurs very often if sampling is considered, see, e.g., [1].

III. Stable inversion

In section III-A, stable inversion for general LTV systems is presented. These results hinge on the knowledge of a dichotomy between stable and unstable dynamics. The dichotomy for LPTV systems is presented in section III-B. The regular inversion solution and stable inversion solution for LTI systems are recovered as special cases in section III-C.

A. LTV stable inversion

First, the stable inversion procedure is presented for general LTV systems, i.e., not necessarily satisfying Definition 2. A state-space realization of the inverse LTV system $F = H^{-1}$ is provided by Lemma 9.

Lemma 9. *Let H have state-space realization $(A_k^H, B_k^H, C_k^H, D_k^H)$, then $F = H^{-1}$ has a state-space realization (A_k, B_k, C_k, D_k), with*

$$\left[\begin{array}{c|c} A_k^H - B_k^H (D_k^H)^{-1} C_k^H & B_k^H (D_k^H)^{-1} \\ \hline -(D_k^H)^{-1} C_k^H & (D_k^H)^{-1} \end{array}\right].$$

Proof. Follows directly from interchanging the input and output. □

If one can find a dichotomic split into a stable and unstable part, then Theorem 10 provides the bounded stable inversion solution.

Theorem 10 (Stable inversion of LTV systems). *Consider a hyperbolic LTV system F over the time interval $k_s < k < k_e$,*

$k, k_s, k_e \in \mathbb{N}$. *Let the state-space of F be split as*

$$\begin{aligned} x^s_{k+1} &= A^{ss}_k x^s_k + A^{su}_k x^u_k + B^s_k r_k, \\ x^u_{k+1} &= A^{us}_k x^s_k + A^{uu}_k x^u_k + B^u_k r_k, \\ u_k &= C^s_k x^s_k + C^u_k x^u_k + D_k r_k, \end{aligned} \tag{2}$$

where x^s_k is picking up the stable part, i.e., $\lim_{k\to\infty} \|x^s_k\| = 0$, with $x^s_{k_s} = 0$ and x^u_k the unstable part, i.e., $\lim_{k\to-\infty} \|x^u_k\| = 0$, with $x^u_{k_e} = 0$. Then, to find the bounded solution, solve for P_k backward in time using

$$\begin{aligned} &P_k = \left(A^{uu}_k - P_{k+1}A^{su}_k\right)^{-1}\left(P_{k+1}A^{ss}_k - A^{us}_k\right), \\ &P_{k_e} = 0, \end{aligned} \tag{3}$$

and for g_k backward in time using

$$\begin{aligned} &g_k = \left(P_{k+1}A^{su}_k - A^{uu}_k\right)^{-1}\left(B^u_k r_k - P_{k+1}B^s_k r_k - g_{k+1}\right), \\ &g_{k_e} = 0. \end{aligned} \tag{4}$$

Then, x^s_k can be solved forward in time from

$$\begin{aligned} &x^s_{k+1} = \left(A^{ss}_k + A^{su}_k P_k\right) x^s_k + B^s_k r_k + A^{su}_k g_k, \\ &x^s_{k_s} = 0, \end{aligned} \tag{5}$$

and x^u_k follows from

$$x^u_k = P_k x^s_k + g_k. \tag{6}$$

The output follows directly from

$$u_k = C^s_k x^s_k + C^u_k x^u_k + D_k r_k. \tag{7}$$

Proof. Follows along similar lines as the continuous-time results in [7]. The sweep method is applied, i.e., it is assumed there exist P_k, g_k such that (6) holds. Eliminating x^u from the state equations in (2) yields a linear relation of the form $\mathcal{A}x^s_k = \mathcal{B}$, $\forall x^s_k$. Since the relation holds for all x^s_k, the solution is found from equating $\mathcal{A} = 0$ and $\mathcal{B} = 0$ yielding (3) and (4), respectively. □

Hence, if a dichotomy of the system in a stable and unstable part is available as in (2), then Theorem 10 provides a bounded solution of the LTV system. The solution is exact for $k_s \to -\infty$ and $k_e \to \infty$. Note that the solution is noncausal since (3) and (4) are solved backwards in time. In particular, it is the solution of a two-point boundary value problem, see (3), (4), and (5). The question remains how to determine such a dichotomy. In the following section, the dichotomy for LPTV systems is presented.

Remark 11. *If H is strictly proper, then Lemma 9 is not directly applicable since D^H_k is not invertible. For such systems, let $\bar{H} = H\mathcal{D}_{-\rho}$ with ρ the relative degree of H, Then, $u_k = \mathcal{D}_{-\rho}\bar{u}_k = \bar{u}_{k+\rho}$, where $\bar{u}_k$ is the output of the bi-proper system $\bar{F} = \bar{H}^{-1}$ found through Lemma 9. The approach is illustrated in Fig. 2. Note that the results are exact on an infinite interval.*

Remark 12. *Note that in the presented formulation,* (3) *and* (4) *require $(A^{uu}_k - P_{k+1}A^{su}_k)$ to be invertible.*

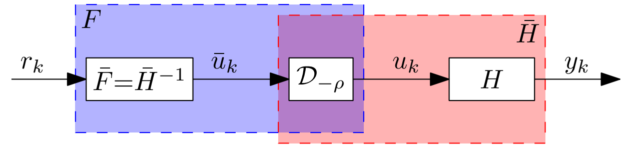


Fig. 2. Time-shifting u_k with the relative degree ρ of H allows to invert the bi-proper system $\bar{H}$. Signal u_k is found by applying a forward shift of ρ samples to the output $\bar{u}_k$ of $\bar{F} = \bar{H}^{-1}$.

B. Dichotomic split

In order to apply Theorem 10, a dichotomic split of the states in stable states x^s_k and unstable states x^u_k has to be found. Suppose such a split is given by a static state transformation matrix T, i.e.,

$$x_k = T\begin{bmatrix} x^s_k \\ x^u_k \end{bmatrix}. \tag{8}$$

Then T provides the dichotomic split of the system (A_k, B_k, C_k, D_k) into (2) for which Theorem 10 provides the bounded stable inversion solution. There are many possible choices to decompose the system. For example, an obvious choice for LTI systems with distinct poles is an eigendecomposition. However, for LTV systems the choice is non-trivial, in particular because the eigenvalues may change over time.

Theorem 13 provides a key result to obtain the dichomatic split, namely the periodicity of the inverse F. This periodicity is exploited to obtain the dichotomic split for LPTV systems through the monodromy matrix as defined in Definition 6, see also [2, sec. 3.1]. Using the monodromy matrix, the main result can be posed: the dichotomic split for LPTV systems in Theorem 14.

Theorem 13. *If H is LPTV, then $F = H^{-1}$ is LPTV.*

Proof. If H is periodic with period $\tau \in \mathbb{N}$, then $A^H_{k+n\tau} = A^H_k$, $B^H_{k+n\tau} = B^H_k$, $C^H_{k+n\tau} = C^H_k$, $D^H_{k+n\tau} = D^H_k$, $\forall n \in \mathbb{N}$. By Lemma 9, A_k, B_k, C_k, D_k are also periodic with period τ and hence F is periodic with period τ. □

Theorem 14 (Dichotomic split for LPTV systems)**.** *Let system F satisfy Assumption 5 and Assumption 7 and let T satisfy*

$$T^{-1}\Psi T = \begin{bmatrix} \Psi^s & 0 \\ 0 & \Psi^u \end{bmatrix}, \tag{9}$$

with Ψ in (1), *$|\lambda_i(\Psi^s)| < 1$, $\forall i$ and $|\lambda_i(\Psi^u)| > 1$, $\forall i$, where λ_i is the i-th eigenvalue. Then the system F after state transformation* (8) *satisfies the dichotomic split in* (2).

Proof. Applying state transformation (8) to F yields:

$$\begin{aligned} \begin{bmatrix} A^{ss}_k & A^{su}_k \\ A^{us}_k & A^{uu}_k \end{bmatrix} &= T^{-1}A_k T, \\ \begin{bmatrix} B^s_k \\ B^u_k \end{bmatrix} &= T^{-1}B_k, \\ \begin{bmatrix} C^s_k & C^u_k \end{bmatrix} &= C_k T, \\ D_k &= D_k. \end{aligned} \tag{10}$$

The transformed system has monodromy matrix

$$\begin{aligned}\tilde{\Psi} &= \tilde{A}_{\tau-1}\tilde{A}_{\tau-2}\dots\tilde{A}_0\\ &= (T^{-1}A_{\tau-1}T)(T^{-1}A_{\tau-2}T)\dots(T^{-1}A_0T)\\ &= T^{-1}A_{\tau-1}A_{\tau-2}\dots A_0T\\ &= T^{-1}\Psi T.\end{aligned}$$

Its characteristic multipliers λ_i are the roots of the characteristic polynomial $p_c(\lambda) = \det[\lambda I - \tilde{\Psi}]$ which are time-invariant since $\tilde{\Psi}$ is constant. By [2, Proposition 3.3], the transformed system is stable over the period τ if and only if $|\lambda_i| < 1$, $\forall i$. Hence, the transformed system satisfies the desired dichotomic split in (2). □

With the dichotomic split of Theorem 14 into (2), Theorem 10 can be applied to find the bounded stable inversion system. A step-by-step procedure for stable inversion of LPTV systems is provided in Procedure 15.

Procedure 15 Stable inversion of LPTV systems.

Given an LPTV system satisfying Assumption 5 and Assumption 7:

1) Calculate monodromy matrix Ψ in (1).
2) Determine a matrix T satisfying (9).
3) Transform the system using state transformation (8) leading to (2).
4) Solve P_k backward in time using (3).
5) Solve g_k backward in time using (4).
6) Solve x_k^s forward in time using (5).
7) Calculate x_k^u using (6).
8) Calculate the bounded output u_k using (7).

Remark 16. *The choice for T is not unique. A typical choice for T to satisfy* (9) *is an ordered eigenvector matrix of Ψ, but also Jordan and Schur decompositions can be exploited. Interestingly, Floquet Theory [2, §1.2] shows that, under some additional conditions, there exists time-varying T_k such that the transformed system has constant state matrix. Such a transformation significantly simplifies Procedure 15. Research related to this will be reported elsewhere.*

C. Special cases

Corollary 17 shows that the regular inversion solution is recovered for stable LPTV systems.

Corollary 17 (Periodically stable system)**.** *If H is stable per period τ, i.e., $|\lambda_i(\Psi)| < 1$, $\forall i$, the stable inversion solution reduces to the direct inverse solution.*

Proof. If $|\lambda_i(\Psi)| < 1$, $\forall i$, all states are considered stable, hence $A_k^{ss} = A_k$, $B_k^s = B_k$, $C_k^s = C_k$ and $A_k^{su}, A_k^{us}, A_k^{uu}, B_k^u, C_k^u$ are nonexistent. Substitution in Theorem 10 shows that P_k, g_k, x_k^u are nonexistent, and consequently the state-space system (5), (6), (7) reduces to

$$\begin{aligned}x_{k+1}^s &= A_kx_k^s + B_kr_k, \quad x_{k_s}^s = 0,\\ u_k &= C_kx_k^s + D_kr_k,\end{aligned}$$

which is the original state-space system F. □

Corollary 18 show that stable inversion for LTI systems is recovered as special case. See also [17] for discrete-time systems and [29] for continuous-time systems.

Corollary 18 (Stable inversion of LTI systems)**.** *Consider a square LTI system (see Definition 3) satisfying Assumption 7 with state-space realization (A, B, C, D), and let T satisfy*

$$T^{-1}AT = \begin{bmatrix}\Psi^s & 0\\ 0 & \Psi^u\end{bmatrix}, \tag{11}$$

where $|\lambda_i(\Psi^s)| < 1$, $\forall i$ and $|\lambda_i(\Psi^u)| > 1$, $\forall i$. Then, for state transformation (8)*, the bounded stable inversion solution is given by*

$$x_{k+1}^s = \Psi^s x_k^s + B^s r_k, \qquad x_{k_s}^s = 0, \tag{12}$$

$$x_k^u = (\Psi^u)^{-1}(x_{k+1}^u - B^u r_k), \quad x_{k_e}^u = 0, \tag{13}$$

$$u_k = C^s x_k^s + C^u x_k^u + D r_k. \tag{14}$$

Proof. Procedure 15 is followed with, by Definition 3, $(A_k, B_k, C_k, D_k) = (A, B, C, D)$:

1) By Definition 6, $\Psi = A$ ($\tau = 1$) and by Assumption 7 $|\lambda_i(A)| \neq 1$, $\forall i$.
2) Substitution of $\Psi = A$ in (9) yields condition (11).
3) (10) reduces to

$$\begin{aligned}\begin{bmatrix}A^{ss} & A^{su}\\ A^{us} & A^{uu}\end{bmatrix} &= \begin{bmatrix}\Psi^s & 0\\ 0 & \Psi^u\end{bmatrix},\\ \begin{bmatrix}B^s\\ B^u\end{bmatrix} &= T^{-1}B,\\ \begin{bmatrix}C^s & C^u\end{bmatrix} &= CT,\\ D &= D.\end{aligned}$$

4) Since $A^{su} = 0$, $A^{us} = 0$, it follows from (3) that $P_k = 0$, $\forall k$.
5) (4) reduces to

$$g_k = -(\Psi^u)^{-1}\left(B^u r_k - g_{k+1}\right), \quad g_{k_e} = 0.$$

6) (5) reduces to (12).
7) x_k^u reduces to (13) since $x_k^u = g_k$ according to (6).
8) (7) becomes (14).

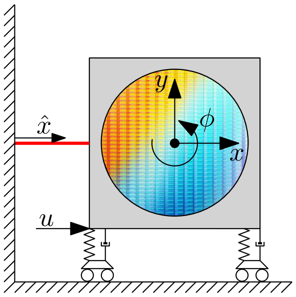
Hence, the LTI solution is recovered. □

Corollary 18 shows that the stable inversion approach for LPTV systems outlined in Procedure 15 simplifies significantly for LTI systems. In particular, the stable and unstable states are completely decoupled ($P_k = 0$, $\forall k$), where the stable state is found by solving (12) forward in time, and the unstable stable state by solving (13) backward in time.

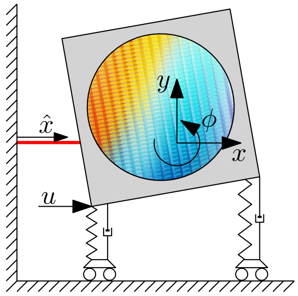
Remark 19. *For LTI systems, the invertibility requirement in Remark 12 is always satisfied since it reduces to Ψ^u being invertible which is satisfied by definition.*

IV. Application to a Wafer Stage

One of the challenges in motion systems is position-dependent behavior, as present in, for example, wafer stages. Wafer stages are key motion systems in wafer scanners used for the production of integrated circuits.

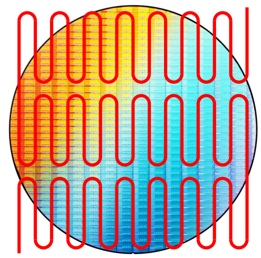


(a) Zero position.

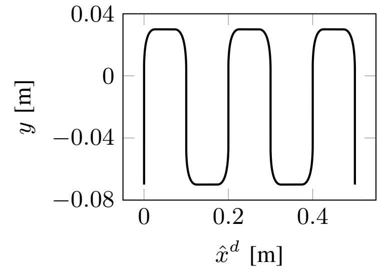


(b) Moved and rotated.

Fig. 3. Top view of wafer stage. The interferometer is fixed on the metrology frame and measures distance $\hat{x}$ to the wafer stage which has degrees of freedom x, y, ϕ. If $\phi \neq 0$, position y effects measurement $\hat{x}$.



(a) Meander pattern over a wafer.



(b) Part of meander pattern constructed by $\hat{x}^d, y$ in Fig. 5(a).

Fig. 4. The wafer is scanned by moving the wafer stage in a meander pattern.

TABLE I
PARAMETER VALUES OF THE WAFER STAGE SYSTEM.

parameter	symbol	value	unit
mass	m	50	kg
inertia	I	2.08	kgm^2
spring constant	c	10^6	N/m
damping constant	d	2500	Ns/m
length	l	0.5	m

A. Wafer stage system

A simplified 2D model of a wafer stage in the horizontal plane is considered as shown in Fig. 3. The stage is actuated by force input u, can translate in x and y direction, and rotate in ϕ direction. The output is the distance $\hat{x}$ between the metrology frame and the wafer stage which is measured through an interferometer located on the metrology frame. The parameters are listed in Table I.

A typical wafer stage movement is a so-called meander pattern as illustrated in Fig. 4(a), see also [26]. The position y is assumed to be prescribed by the periodic movement in Fig. 5(a) and is controlled by a PD controller represented by the spring and damper in Fig. 3. Since for $\phi \neq 0$, position y effects measurement $\hat{x}$, the $\hat{x}$-dynamics are position dependent. The desired trajectory $\hat{x}^d$ for $\hat{x}$ is also shown in Fig. 5(a). The combination of y and $\hat{x}^d$ generate the meander pattern shown in Fig. 4(b).

The *continuous*-time state-space realization (A^c, B^c, C^c, D^c) of the $\hat{x}$-dynamics, linearized around $\phi, \dot{\phi} = 0$, with input u, state $\begin{bmatrix} x & \dot{x} & \phi & \dot{\phi} \end{bmatrix}^\top$, and output $\hat{x}$ is

$$\left[\begin{array}{c|c} A^c & B^c \\ \hline C_k^c & D^c \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{m} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2}\frac{cl^2}{I} & -\frac{1}{2}\frac{dl^2}{I} & \frac{1}{2}\frac{l}{I} \\ \hline 1 & 0 & y_k & 0 & 0 \end{array}\right].$$

System H is the zero-order-hold discretized system

$$H \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C_k & D \end{array}\right] = \left[\begin{array}{c|c} e^{A^c h} & (A^c)^{-1}(A - I)B^c \\ \hline C_k^c & D^c \end{array}\right],$$

with sampling interval $h = 0.001$ s. Note that H is indeed position dependent through C_k. Moreover, for the given parameters, the system is minimum phase if $y_k > 0$, and non-minimum phase for $y_k < 0$. Since y_k is periodic, H is LPTV. Also note that H is strictly proper since $D = 0$. Therefore, the techniques in Remark 11 are used to obtain bi-proper $\bar{H}$. The inverse is given by $\bar{F} = \bar{H}^{-1}$ according to Lemma 9.

B. Results

Forward time simulation of F with reference $r = \hat{x}^d$ and trajectory y in Fig. 5(a) yields the unbounded signal u shown in Fig. 5(b). Although input u is unbounded, the output $\hat{x}$ exactly matches the desired trajectory $\hat{x}^d$ due to the exact inverse, see Fig. 5(a).

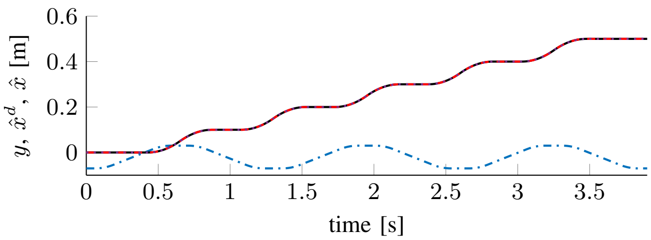
To obtain bounded input u, the stable inversion approach in Theorem 10 is applied. The monodromy matrix Ψ reveals one unstable and three stable states. The stable inversion approach yields bounded u in Fig. 5(b) as desired. For this input, the output position $\hat{x}$ perfectly tracks the desired trajectory $\hat{x}^d$ as shown in Fig. 5(a).

The example demonstrates the application of the stable inversion approach for LPTV systems to a position-dependent wafer stage system.

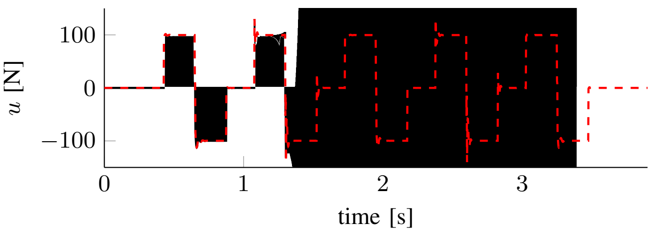
V. CONCLUSIONS

Many systems exhibit LPTV behavior, such as position-dependent systems with periodic tasks and non-equidistantly sampled systems. To exploit the full potential of these systems in, for example, feedforward control, system inverses with bounded output are required. In this paper, the inversion of LPTV systems is investigated and solutions to obtain bounded solutions are developed. The key aspect is the dichotomy of the system in a stable and unstable part. The periodicity of the system and in particular the associated monodromy matrix are exploited to obtain the dichotomy. The approach is demonstrated on a position-dependent wafer stage system example.

The presented stable inversion approach is applicable to LPTV systems under the mild condition of a hyperbolic system. Under additional conditions, Floquet theory may be used to reduce the complexity of the stable inversion approach. Research related to this will be presented elsewhere.



(a) Output $\hat{x}$ with forward simulation (——) and with stable inversion (⋯⋯) perfectly tracks the desired trajectory $\hat{x}^d$ (⋯⋯) for the prescribed periodic position y (- - -).



(b) Forward simulation of F (——) yields unbounded u, whereas stable inversion (- - -) yields bounded u.

Fig. 5. Prescribed position y introduces position-dependence resulting in an LPTV system that is *unstable*. With stable inversion, a bounded solution u resulting in perfect tracking is found.

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REFERENCES

[1] K.J. Åström, P. Hagander, and J. Sternby. Zeros of Sampled Systems. *Automatica*, 20(1):31–38, 1984.
[2] Sergio Bittanti and Patrizio Colaneri. *Periodic Systems - Filtering and Control*. Springer-Verlag, London, Great Britain, 2009.
[3] Frank Boeren, Tom Oomen, and Maarten Steinbuch. Iterative motion feedforward tuning: A data-driven approach based on instrumental variable identification. *Control Engineering Practice*, 37:11–19, 2015.
[4] Joost Bolder and Tom Oomen. Rational Basis Functions in Iterative Learning Control - With Experimental Verification on a Motion System. *Transactions on Control Systems Technology*, 23(2):722–729, 2015.
[5] Douglas A. Bristow, Kira L. Barton, and Andrew G. Alleyne. Iterative Learning Control. In William S. Levine, editor, *The Control Systems Handbook: Control System Advanced Methods*, chapter 36, pages 1–19. CRC Press, 2nd edition, 2010.
[6] J.A. Butterworth, L.Y. Pao, and D.Y. Abramovitch. Analysis and comparison of three discrete-time feedforward model-inverse control techniques for nonminimum-phase systems. *Mechatronics*, 22(5):577–587, 2012.
[7] Degang Chen. An Iterative Solution to Stable Inversion of Nonminimum Phase Systems. In *Proceedings of the 1993 American Control Conference*, pages 2960–2964, San Francisco, California, 1993.
[8] Tongwen Chen and Bruce A. Francis. *Optimal Sampled-Data Control Systems*. Springer, London, Great Britain, 1995.
[9] Garrett M. Clayton, Szuchi Tien, Kam K. Leang, Qingze Zou, and Santosh Devasia. A Review of Feedforward Control Approaches in Nanopositioning for High-Speed SPM. *Journal of Dynamic Systems, Measurement, and Control*, 131(6):061101–061101–19, 2009.
[10] W.A. Coppel. Dichotomies and Reducibility. *Journal of Differential Equations*, 3(4):500–521, 1967.
[11] W.A. Coppel. Dichotomies in Stability Theory. In *Lecture Notes in Mathematics*, volume 629. Springer-Verlag, 1978.
[12] S. Devasia and B. Paden. Exact Output Tracking for Nonlinear Time-Varying Systems. In *Proceedings of 33rd Conference on Decision and Control*, pages 2346–2355, Lake Buena Vista, Florida, 1994.
[13] S. Devasia and B. Paden. Stable Inversion for Nonlinear Nonminimum-Phase Time-Varying Systems. *Transactions on Automatic Control*, 43(2):283–288, 1998.
[14] Santosh Devasia. Output Tracking with Nonhyperbolic and Near Nonhyperbolic Internal Dynamics: Helicopter Hover Control. In *Proceedings of the 1997 American Control Conference*, volume 3, pages 1439–1446, Albuquerque, New Mexico, 1997.
[15] Geir E. Dullerud and Fernando Paganini. *A Course in Robust Control Theory: A Convex Approach*, volume 36. Springer Science+Business Media Inc., 2000.
[16] Hiroshi Fujimoto and Yoichi Hori. High-performance servo systems based on multirate sampling control. *Control Engineering Practice*, 10(7):773–781, 2002.
[17] Koshy George, Michel Verhaegen, and Jacquelien M.A. Scherpen. Stable Inversion of MIMO Linear Discrete Time Non-Minimum Phase Systems. In *Proceedings of the 7th Mediterranean Conference on Control and Automation*, pages 267–281, Haifa, Israel, 1999.
[18] Matthijs Groot Wassink, Marc van de Wal, Carsten Scherer, and Okko Bosgra. LPV control for a wafer stage: beyond the theoretical solution. *Control Engineering Practice*, 13(2):231–245, 2005.
[19] Aristide Halanay and Vlad Ionescu. *Time-Varying Discrete Linear Systems: Input-Output Operators. Riccati Equations. Disturbance Attenuation*, volume 68. Birkhäuser Verlag, 1994.
[20] L.R. Hunt, G. Meyer, and R. Su. Noncausal Inverses for Linear Systems. *Transactions on Automatic Control*, 41(4):608–611, 1996.
[21] Kazys Kazlauskas. Inversion of Linear Periodically Time-Varying Digital Filters. *Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, 41(2):173–175, 1994.
[22] Margreet Kuijper and Jan C. Willems. A behavioral framework for periodically time-varying systems. In *Proceedings of the 36th Conference on Decision and Control*, volume 3, pages 2013–2016, San Diego, California, 1997.
[23] Julián Salt and Pedro Albertos. Model-Based Multirate Controllers Design. *Transactions on Control Systems Technology*, 13(6):988–997, 2005.
[24] Takuya Sogo. On the equivalence between stable inversion for nonminimum phase systems and reciprocal transfer functions defined by the two-sided Laplace transform. *Automatica*, 46(1):122–126, 2010.
[25] Masayoshi Tomizuka. Zero Phase Error Tracking Algorithm for Digital Control. *Journal of Dynamic Systems, Measurement, and Control*, 109(1):65–68, 1987.
[26] Stan H. van der Meulen, Rob L. Tousain, and Okko H. Bosgra. Fixed Structure Feedforward Controller Design Exploiting Iterative Trials: Application to a Wafer Stage and a Desktop Printer. *Journal of Dynamic Systems, Measurement, and Control*, 130(5):051006:1–16, 2008.
[27] Jurgen van Zundert, Joost Bolder, Sjirk Koekebakker, and Tom Oomen. Resource-efficient ILC for LTI/LTV systems through LQ tracking and stable inversion: Enabling large feedforward tasks on a position-dependent printer. *Mechatronics*, 38:76–90, 2016.
[28] Jurgen van Zundert, Tom Oomen, Dip Goswami, and W.P.M.H. Heemels. On the Potential of Lifted Domain Feedforward Controllers with a Periodic Sampling Sequence. In *Proceedings of the 2016 American Control Conference*, pages 4227–4232, Boston, Massachusetts, 2016.
[29] Qingze Zou and Santosh Devasia. Preview-Based Stable-Inversion for Output Tracking of Linear Systems. *Journal of Dynamic Systems, Measurement, and Control*, 121(4):625–630, 1999.